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Research Article

Three New Min-Max Variations of the Hardy-Hilbert Integral Inequality

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Abstract. Building on the work of Li and He in 2007, this article presents three new variations of the Hardy-Hilbert integral inequality using minimum and maximum kernel functions. Detailed proofs are provided.

Keywords. Integral inequalities, Hardy-Hilbert-type integral inequalities, beta function

Mathematics Subject Classification (2020). 26D15

1. Introduction

Integral inequalities inspired by the Hardy-Hilbert integral framework are fundamental tools in modern analysis. They provide precise upper bounds that are essential for solving a variety of mathematical problems in areas such as operator theory, functional analysis, partial differential equations and mathematical physics. Further details can be found in the following references: Chen and Yang [2], and Yang [14, 15]. Classical variants of Hardy-Hilbert integral inequalities are based on a kernel function involving the maximum of two variables, i.e., $\max(x,y)$, where x and y denote the two variables. These inequalities naturally arise in contexts where symmetrisation or duality arguments are used. One of the simplest results of this type is described below and is due to Hardy *et al.* [4]. Let $p>1$, $q=p/(p-1)$ be the Hölder conjugate of p, i.e., $1/p+1/q=1$, and $f,g:(0,+\infty)\mapsto(0,+\infty)$ be two functions. Then, the following inequality holds:

$$\int_{(0,+\infty)}\int_{(0,+\infty)} \frac{1}{\max(x,y)} f(x)g(y)dxdy \le pq \left[\int_{(0,+\infty)} f^p(x)dx\right]^{1/p} \left[\int_{(0,+\infty)} g^q(y)dy\right]^{1/q}, \quad (1.1)$$

provided that the two integrals involved in the upper bound converge. This result is interesting because it clearly demonstrates the interaction between the kernel function $1/\max(x,y)$ and integral norms. This kernel function also ensures symmetry and homogeneity, which are both essential characteristics in numerous applications, including interpolation theory and the boundedness of bilinear operators. Furthermore, the sharp constant factor pq reflects the best possible bound given the assumptions made, emphasizing the extremal nature of the inequality.

This result has been extended by considering more general kernel functions, weighted integrals, and multidimensional analogues. Versions involving the minimum and maximum of variables, logarithmic and fractional power modifications, in particular, provide deeper insight into the structure of functional spaces. Further information on these aspects can be found in the following references: Azar [1], Chesneau [3], Li and He [5], Li *et al*. [6], Saglam *et al*. [7], Sarikaya and Bingol [8], Sulaiman [9–12], and Sun [13].

In this article, we highlight a technical result from [5], which can be viewed as a natural minimum counterpart to eq. (1.1). It is given formally below.

Theorem 1.1 ([5, Theorem 2.1])**.** *Let $p>1$, $q=p/(p-1)$ and $f,g:(1,+\infty)\mapsto(0,+\infty)$ be two functions. Then, we have*

$$\int_{(1,+\infty)}\int_{(1,+\infty)} \frac{1}{\min(x,y)} f(x)g(y)dxdy$$
$$\le \frac{1}{p^{1/p}q^{1/q}} \left[\int_{(1,+\infty)} f^p(x)\left(x^p+\frac{1}{p-1}\right)dx\right]^{1/p} \left[\int_{(1,+\infty)} g^q(y)\left(y^q+\frac{1}{q-1}\right)dy\right]^{1/q},$$

provided that the two integrals involved in the upper bound converge.

Unlike eq. (1.1), the kernel function $1/\min(x,y)$ exhibits distinct behavior that is sensitive to the integrability properties of functions near $+\infty$. Furthermore, Theorem 1.1 involves the weighted integral norms of f and g, where the weight functions are $x^p+1/(p-1)$ and $y^q+1/(q-1)$, respectively. They arise naturally in the analysis and are essential for capturing the balance between decay at $+\infty$ and integrability. The sharp constant factor $1/(p^{1/p}q^{1/q})$ emphasizes the optimality of the inequality under the stated assumptions. Inequalities of this kind are closely related to rearrangement inequalities, monotonicity methods and certain classes of integral operators whose boundedness cannot be captured solely through classical integral theory.

In what follows, we propose three new minimum-maximum (min-max) variations of the inequality stated in Theorem 1.1. Each variation is based on a distinct kernel function introducing refined structural properties to the inequality. These kernel functions are given by

$$k_1(x,y)=\frac{1}{\min(x/y,y/x)},$$
$$k_2(x,y)=\frac{1}{\min(x,y)\max(x/y,y/x)}$$

and

$$k_3(x,y) = \left[\max\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\alpha},$$

respectively, where α denotes an adjustable parameter. These modifications produce new Hardy-Hilbert-type integral inequalities, thereby extending the existing theory. They also open up potential avenues for applications in weighted norm inequalities, symmetric and quasi-symmetric function spaces, and the analysis of bilinear forms.

The remainder of the article is as follows: Sections 2, 3 and 4 are devoted to the first, second and third min-max variations. A conclusion is given in Section 5.

2. First Variation

The theorem below presents our main result concerning the first variation.

Theorem 2.1. *Let $p > 1$, $q = p/(p-1)$ and $f, g : (1,+\infty) \mapsto (0,+\infty)$ be two functions. Then, we have*

$$\begin{aligned}
&\int_{(1,+\infty)}\int_{(1,+\infty)} \frac{1}{\min(x/y, y/x)} f(x)g(y)dxdy \\
&\leq \frac{1}{2p^{1/p}q^{1/q}} \left[\int_{(1,+\infty)} f^p(x)x\left(x^{2p} + \frac{1}{p-1}\right)dx\right]^{1/p} \left[\int_{(1,+\infty)} g^q(y)y\left(y^{2q} + \frac{1}{q-1}\right)dy\right]^{1/q},
\end{aligned}$$

provided that the two integrals involved in the upper bound converge.

Proof. By a suitable decomposition of the integrand and the application of the Hölder integral inequality using $1/p + 1/q = 1$, we have

$$\begin{aligned}
&\int_{(1,+\infty)}\int_{(1,+\infty)} \frac{1}{\min(x/y, y/x)} f(x)g(y)dxdy \\
&= \int_{(1,+\infty)}\int_{(1,+\infty)} \frac{1}{[\min(x/y, y/x)]^{1/p}} \left(\frac{x}{y}\right)^2 f(x) \frac{1}{[\min(x/y, y/x)]^{1/q}} \left(\frac{y}{x}\right)^2 g(y)dxdy \\
&\leq A^{1/p}B^{1/q}, \qquad (2.1)
\end{aligned}$$

where

$$A = \int_{(1,+\infty)}\int_{(1,+\infty)} \frac{1}{\min(x/y, y/x)} \left(\frac{x}{y}\right)^{2p} f^p(x)dxdy$$

and

$$B = \int_{(1,+\infty)}\int_{(1,+\infty)} \frac{1}{\min(x/y, y/x)} \left(\frac{y}{x}\right)^{2q} g^q(y)dxdy.$$

Let us investigate the exact expressions for A and B in turn.

For A, by the Fubini-Tonelli integral theorem, we can write

$$A = \int_{(1,+\infty)} xf^p(x)\left[\int_{(1,+\infty)} \frac{1}{\min(x/y, y/x)} \left(\frac{x}{y}\right)^{2p} \frac{1}{x}dy\right]dx.$$

Let us now focus on the central integral. Performing the change of variables $u = y/x$ and using standard primitives together with $p > 1$, we get

$$\int_{(1,+\infty)} \frac{1}{\min(x/y, y/x)} \left(\frac{x}{y}\right)^{2p} \frac{1}{x}dy = \int_{(1/x,+\infty)} \frac{1}{\min(u, 1/u)u^{2p}}du$$

$$
\begin{aligned}
&= \int_{(1/x,1)} \frac{1}{\min(u,1/u)u^{2p}} du + \int_{(1,+\infty)} \frac{1}{\min(u,1/u)u^{2p}} du \\
&= \int_{(1/x,1)} \frac{1}{u \times u^{2p}} du + \int_{(1,+\infty)} \frac{1}{1/u \times u^{2p}} du \\
&= \int_{(1/x,1)} u^{-2p-1} du + \int_{(1,+\infty)} u^{-2p+1} du \\
&= \left[-\frac{1}{2p} u^{-2p} \right]_{(1/x,1)} + \left[-\frac{1}{2(p-1)} u^{-2(p-1)} \right]_{(1,+\infty)} \\
&= \frac{1}{2p}(x^{2p}-1) + \frac{1}{2(p-1)} \\
&= \frac{1}{2p}\left(x^{2p} + \frac{1}{p-1} \right).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
A &= \int_{(1,+\infty)} x f^p(x) \frac{1}{2p}\left(x^{2p} + \frac{1}{p-1} \right) dx \\
&= \frac{1}{2p} \int_{(1,+\infty)} f^p(x) x \left(x^{2p} + \frac{1}{p-1} \right) dx. \qquad (2.2)
\end{aligned}
$$

For B, we proceed in a similar way. By the Fubini-Tonelli integral theorem, we can write

$$
B = \int_{(1,+\infty)} y g^q(y) \left[\int_{(1,+\infty)} \frac{1}{\min(x/y, y/x)} \left(\frac{y}{x} \right)^{2q} \frac{1}{y} dx \right] dy.
$$

Let us now focus on the central integral. Performing the change of variables $v = x/y$ and using standard primitives together with $q > 1$, we get

$$
\begin{aligned}
\int_{(1,+\infty)} \frac{1}{\min(x/y, y/x)} \left(\frac{y}{x} \right)^{2q} \frac{1}{y} dx &= \int_{(1/y,+\infty)} \frac{1}{\min(v,1/v)v^{2q}} dv \\
&= \int_{(1/y,1)} \frac{1}{\min(v,1/v)v^{2q}} dv + \int_{(1,+\infty)} \frac{1}{\min(v,1/v)v^{2q}} dv \\
&= \int_{(1/y,1)} \frac{1}{v \times v^{2q}} dv + \int_{(1,+\infty)} \frac{1}{1/v \times v^{2q}} dv \\
&= \int_{(1/y,1)} v^{-2q-1} dv + \int_{(1,+\infty)} v^{-2q+1} dv \\
&= \left[-\frac{1}{2q} v^{-2q} \right]_{(1/y,1)} + \left[-\frac{1}{2(q-1)} v^{-2(q-1)} \right]_{(1,+\infty)} \\
&= \frac{1}{2q}(y^{2q}-1) + \frac{1}{2(q-1)} \\
&= \frac{1}{2q}\left(y^{2q} + \frac{1}{q-1} \right).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
B &= \int_{(1,+\infty)} y g^q(y) \frac{1}{2q}\left(y^{2q} + \frac{1}{q-1} \right) dy \\
&= \frac{1}{2q} \int_{(1,+\infty)} g^q(y) y \left(y^{2q} + \frac{1}{q-1} \right) dy. \qquad (2.3)
\end{aligned}
$$

Using eqs. (2.1), (2.2) and (2.3), and $1/p+1/q=1$, we get

$$\begin{aligned}&\int_{(1,+\infty)}\int_{(1,+\infty)}\frac{1}{\min(x/y,y/x)}f(x)g(y)dxdy\\&\leq\left[\frac{1}{2p}\int_{(1,+\infty)}f^p(x)x\left(x^{2p}+\frac{1}{p-1}\right)dx\right]^{1/p}\left[\frac{1}{2q}\int_{(1,+\infty)}g^q(y)y\left(y^{2q}+\frac{1}{q-1}\right)dy\right]^{1/q}\\&=\frac{1}{2p^{1/p}q^{1/q}}\left[\int_{(1,+\infty)}f^p(x)x\left(x^{2p}+\frac{1}{p-1}\right)dx\right]^{1/p}\left[\int_{(1,+\infty)}g^q(y)y\left(y^{2q}+\frac{1}{q-1}\right)dy\right]^{1/q}.\end{aligned}$$

This completes the proof of Theorem 2.1. □

The obtained result is similar to that in Theorem 1.1, with notable changes to both the kernel function and the structure of the weighted norms. Specifically, the kernel function $1/\min(x/y,y/x)$ reflects a form of scale sensitivity, which is manifested in weights involving x^{2p} and y^{2q}, as well as x and y factors in the integrals. These additional factors amplify the effect of large values of x and y, thereby emphasizing different integrability conditions compared to Theorem 1.1. Moreover, the sharp constant factor $1/(2p^{1/p}q^{1/q})$ reveals the increased complexity of the kernel function. The structure of the inequality suggests connections with weighted Hardy-type integral inequalities and opens potential avenues for exploring symmetry-invariant forms in functional analysis.

As an aside, we can observe that the kernel function under consideration can be expressed in the following maximum form:

$$\frac{1}{\min(x/y,y/x)}=\max\left(\frac{x}{y},\frac{y}{x}\right). \tag{2.4}$$

This remark leads to a more general variation in Section 4 depending on one adjustable parameter.

3. Second Variation

The theorem below presents our main result concerning the second variation.

Theorem 3.1. *Let $p>1$, $q=p/(p-1)$ and $f,g:(1,+\infty)\mapsto(0,+\infty)$ be two functions. Then, we have*

$$\begin{aligned}&\int_{(1,+\infty)}\int_{(1,+\infty)}\frac{1}{\min(x,y)\max(x/y,y/x)}f(x)g(y)dxdy\\&\leq\frac{1}{(p-1)^{1/p}(q-1)^{1/q}}\left[\int_{(1,+\infty)}f^p(x)\left(x^{p-1}-\frac{1}{p}\right)dx\right]^{1/p}\left[\int_{(1,+\infty)}g^q(y)\left(y^{q-1}-\frac{1}{q}\right)dy\right]^{1/q},\end{aligned}$$

provided that the two integrals involved in the upper bound converge.

Proof. By a suitable decomposition of the integrand and the application of the Hölder integral inequality using $1/p+1/q=1$, we have

$$\begin{aligned}&\int_{(1,+\infty)}\int_{(1,+\infty)}\frac{1}{\min(x,y)\max(x/y,y/x)}f(x)g(y)dxdy\\&=\int_{(1,+\infty)}\int_{(1,+\infty)}\frac{1}{[\min(x,y)\max(x/y,y/x)]^{1/p}}\left(\frac{x}{y}\right)f(x)\frac{1}{[\min(x,y)\max(x/y,y/x)]^{1/q}}\left(\frac{y}{x}\right)g(y)dxdy\\&\leq C^{1/p}D^{1/q},\end{aligned} \tag{3.1}$$

where

$$C=\int_{(1,+\infty)}\int_{(1,+\infty)}\frac{1}{\min(x,y)\max(x/y,y/x)}\left(\frac{x}{y}\right)^p f^p(x)dxdy$$

and

$$D=\int_{(1,+\infty)}\int_{(1,+\infty)}\frac{1}{\min(x,y)\max(x/y,y/x)}\left(\frac{y}{x}\right)^q g^q(y)dxdy.$$

Let us investigate the exact expressions for C and D in turn.

For C, by the Fubini-Tonelli integral theorem, we can write

$$C=\int_{(1,+\infty)}f^p(x)\left[\int_{(1,+\infty)}\frac{1}{\min(1,y/x)\max(x/y,y/x)}\left(\frac{x}{y}\right)^p\frac{1}{x}dy\right]dx.$$

Let us now focus on the central integral. Performing the change of variables $u=y/x$ and using standard primitives together with $p>1$, we get

$$\begin{aligned}
&\int_{(1,+\infty)}\frac{1}{\min(1,y/x)\max(x/y,y/x)}\left(\frac{x}{y}\right)^p\frac{1}{x}dy\\
&=\int_{(1/x,+\infty)}\frac{1}{\min(1,u)\max(u,1/u)u^p}du\\
&=\int_{(1/x,1)}\frac{1}{\min(1,u)\max(u,1/u)u^p}du+\int_{(1,+\infty)}\frac{1}{\min(1,u)\max(u,1/u)u^p}du\\
&=\int_{(1/x,1)}\frac{1}{u\times(1/u)\times u^p}du+\int_{(1,+\infty)}\frac{1}{1\times u\times u^p}du\\
&=\int_{(1/x,1)}u^{-p}du+\int_{(1,+\infty)}u^{-(p+1)}du\\
&=\left[-\frac{1}{p-1}u^{-p+1}\right]_{(1/x,1)}+\left[-\frac{1}{p}u^{-p}\right]_{(1,+\infty)}\\
&=\frac{1}{p-1}(x^{p-1}-1)+\frac{1}{p}\\
&=\frac{1}{p-1}\left(x^{p-1}-\frac{1}{p}\right).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
C&=\int_{(1,+\infty)}f^p(x)\frac{1}{p-1}\left(x^{p-1}-\frac{1}{p}\right)dx\\
&=\frac{1}{p-1}\int_{(1,+\infty)}f^p(x)\left(x^{p-1}-\frac{1}{p}\right)dx. \qquad (3.2)
\end{aligned}$$

For D, we proceed in a similar way. By the Fubini-Tonelli integral theorem, we can write

$$D=\int_{(1,+\infty)}g^q(y)\left[\int_{(1,+\infty)}\frac{1}{\min(1,x/y)\max(x/y,y/x)}\left(\frac{y}{x}\right)^q\frac{1}{y}dx\right]dy.$$

Let us now focus on the central integral. Performing the change of variables $v=x/y$ and using standard primitives together with $q>1$, we get

$$\begin{aligned}
&\int_{(1,+\infty)}\frac{1}{\min(1,x/y)\max(x/y,y/x)}\left(\frac{y}{x}\right)^q\frac{1}{y}dx\\
&=\int_{(1/y,+\infty)}\frac{1}{\min(1,v)\max(v,1/v)v^q}dv
\end{aligned}$$

$$
\begin{aligned}
&= \int_{(1/y,1)} \frac{1}{\min(1,v)\max(v,1/v)v^q}dv + \int_{(1,+\infty)} \frac{1}{\min(1,v)\max(v,1/v)v^q}dv \\
&= \int_{(1/y,1)} \frac{1}{v\times(1/v)\times v^q}dv + \int_{(1,+\infty)} \frac{1}{1\times v\times v^p}dv \\
&= \int_{(1/y,1)} v^{-q}dv + \int_{(1,+\infty)} v^{-(q+1)}dv \\
&= \left[-\frac{1}{q-1}v^{-q+1}\right]_{(1/y,1)} + \left[-\frac{1}{q}v^{-q}\right]_{(1,+\infty)} \\
&= \frac{1}{q-1}(y^{q-1}-1)+\frac{1}{q} \\
&= \frac{1}{q-1}\left(y^{q-1}-\frac{1}{q}\right).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
D &= \int_{(1,+\infty)} g^q(y)\frac{1}{q-1}\left(y^{q-1}-\frac{1}{q}\right)dy \\
&= \frac{1}{q-1}\int_{(1,+\infty)} g^q(y)\left(y^{q-1}-\frac{1}{q}\right)dy. \qquad (3.3)
\end{aligned}
$$

It follows from eqs. (3.1), (3.2) and (3.3), that

$$
\begin{aligned}
&\int_{(1,+\infty)}\int_{(1,+\infty)} \frac{1}{\min(x,y)\max(x/y,y/x)}f(x)g(y)dxdy \\
&\le \left[\frac{1}{p-1}\int_{(1,+\infty)} f^p(x)\left(x^{p-1}-\frac{1}{p}\right)dx\right]^{1/p}\left[\frac{1}{q-1}\int_{(1,+\infty)} g^q(y)\left(y^{q-1}-\frac{1}{q}\right)dy\right]^{1/q} \\
&= \frac{1}{(p-1)^{1/p}(q-1)^{1/q}}\left[\int_{(1,+\infty)} f^p(x)\left(x^{p-1}-\frac{1}{p}\right)dx\right]^{1/p}\left[\int_{(1,+\infty)} g^q(y)\left(y^{q-1}-\frac{1}{q}\right)dy\right]^{1/q}.
\end{aligned}
$$

This completes the proof of Theorem 3.1. □

We can observe that the kernel function under consideration can be expressed as follows:

$$
\frac{1}{\min(x,y)\max(x/y,y/x)} = \frac{\min(x/y,y/x)}{\min(x,y)} = \frac{1}{\max(x,y)}.
$$

In light of this, Theorem 3.1 mainly differs from eq. (1.1) in terms of the domain of integration, i.e., $(1,+\infty)^2$, and on the weight functions involved in the upper bound. These weight functions now involve expressions of the form $x^{p-1}-1/p$ and $y^{q-1}-1/q$. Unlike simpler polynomial weights, these expressions introduce a more refined penalization depending on the growth of x and y.

In this sense, Theorem 3.1 can be viewed as a hybrid variation of eq. (1.1) and Theorem 2.1.

4. Third Variation

The theorem below presents our main result concerning the third and last variation.

Theorem 4.1. *Let $p>1$, $q=p/(p-1)$, $\alpha,\beta>0$ such that $\beta\min(p,q)>\alpha+1$, and $f,g:(1,+\infty)\mapsto(0,+\infty)$ be two functions. Then, we have*

$$
\int_{(1,+\infty)}\int_{(1,+\infty)} \left[\max\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\alpha} f(x)g(y)dxdy
$$

$$\leq \frac{1}{(\alpha+\beta p-1)^{1/p}(\alpha+\beta q-1)^{1/q}}\left[\int_{(1,+\infty)} f^p(x)x\left(x^{\alpha+\beta p-1}+\frac{2\alpha}{\beta p-\alpha-1}\right)dx\right]^{1/p}$$
$$\times\left[\int_{(1,+\infty)} g^q(y)y\left(y^{\alpha+\beta q-1}+\frac{2\alpha}{\beta q-\alpha-1}\right)dy\right]^{1/q},$$

provided that the two integrals involved in the upper bound converge.

Proof. By a suitable decomposition of the integrand using $1=(x/y)^\beta(y/x)^\beta$ and the application of the Hölder integral inequality using $1/p+1/q=1$, we have

$$\int_{(1,+\infty)}\int_{(1,+\infty)}\left[\max\left(\frac{x}{y},\frac{y}{x}\right)\right]^\alpha f(x)g(y)dxdy$$
$$=\int_{(1,+\infty)}\int_{(1,+\infty)}\left[\max\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\alpha/p}\left(\frac{x}{y}\right)^\beta f(x)\left[\max\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\alpha/q}\left(\frac{y}{x}\right)^\beta g(y)dxdy$$
$$\leq E^{1/p}F^{1/q}, \tag{4.1}$$

where

$$E=\int_{(1,+\infty)}\int_{(1,+\infty)}\left[\max\left(\frac{x}{y},\frac{y}{x}\right)\right]^\alpha\left(\frac{x}{y}\right)^{\beta p} f^p(x)dxdy$$

and

$$F=\int_{(1,+\infty)}\int_{(1,+\infty)}\left[\max\left(\frac{x}{y},\frac{y}{x}\right)\right]^\alpha\left(\frac{y}{x}\right)^{\beta q} g^q(y)dxdy.$$

Let us investigate the exact expressions for E and F in turn.

For E, by the Fubini-Tonelli integral theorem, we can write

$$E=\int_{(1,+\infty)} xf^p(x)\left\{\int_{(1,+\infty)}\left[\max\left(\frac{x}{y},\frac{y}{x}\right)\right]^\alpha\left(\frac{x}{y}\right)^{\beta p}\frac{1}{x}dy\right\}dx.$$

Let us now focus on the central integral. Performing the change of variables $u=y/x$ and using standard primitives together with $\beta p>\alpha+1$, we get

$$\int_{(1,+\infty)}\left[\max\left(\frac{x}{y},\frac{y}{x}\right)\right]^\alpha\left(\frac{x}{y}\right)^{\beta p}\frac{1}{x}dy$$
$$=\int_{(1/x,+\infty)}\left[\max\left(\frac{1}{u},u\right)\right]^\alpha\frac{1}{u^{\beta p}}du$$
$$=\int_{(1/x,1)}\left[\max\left(\frac{1}{u},u\right)\right]^\alpha\frac{1}{u^{\beta p}}du+\int_{(1,+\infty)}\left[\max\left(\frac{1}{u},u\right)\right]^\alpha\frac{1}{u^{\beta p}}du$$
$$=\int_{(1/x,1)}\frac{1}{u^\alpha}\frac{1}{u^{\beta p}}du+\int_{(1,+\infty)}u^\alpha\frac{1}{u^{\beta p}}du$$
$$=\int_{(1/x,1)}u^{-\alpha-\beta p}du+\int_{(1,+\infty)}u^{\alpha-\beta p}du$$
$$=\left[\frac{1}{-\alpha-\beta p+1}u^{-\alpha-\beta p+1}\right]_{(1/x,1)}+\left[\frac{1}{\alpha-\beta p+1}u^{\alpha-\beta p+1}\right]_{(1,+\infty)}$$
$$=\frac{1}{\alpha+\beta p-1}(x^{\alpha+\beta p-1}-1)-\frac{1}{\alpha-\beta p+1}$$
$$=\frac{1}{\alpha+\beta p-1}\left(x^{\alpha+\beta p-1}+\frac{2\alpha}{\beta p-\alpha-1}\right).$$

Therefore, we have

$$E = \int_{(1,+\infty)} x f^p(x) \frac{1}{\alpha+\beta p-1}\left(x^{\alpha+\beta p-1} + \frac{2\alpha}{\beta p-\alpha-1}\right)dx$$
$$= \frac{1}{\alpha+\beta p-1}\int_{(1,+\infty)} f^p(x)x\left(x^{\alpha+\beta p-1} + \frac{2\alpha}{\beta p-\alpha-1}\right)dx. \tag{4.2}$$

For F, we proceed in a similar way. By the Fubini-Tonelli integral theorem, we can write

$$F = \int_{(1,+\infty)} y g^q(y)\left\{\int_{(1,+\infty)} \left[\max\left(\frac{x}{y},\frac{y}{x}\right)\right]^\alpha \left(\frac{y}{x}\right)^{\beta q}\frac{1}{y}dx\right\}dy.$$

Let us now focus on the central integral. Performing the change of variables $v = x/y$ and using standard primitives together with $\beta q > \alpha+1$, we get

$$\int_{(1,+\infty)} \left[\max\left(\frac{x}{y},\frac{y}{x}\right)\right]^\alpha \left(\frac{y}{x}\right)^{\beta q}\frac{1}{y}dx$$
$$= \int_{(1/y,+\infty)} \left[\max\left(v,\frac{1}{v}\right)\right]^\alpha \frac{1}{v^{\beta q}}dv$$
$$= \int_{(1/y,1)} \left[\max\left(v,\frac{1}{v}\right)\right]^\alpha \frac{1}{v^{\beta q}}dv + \int_{(1,+\infty)} \left[\max\left(v,\frac{1}{v}\right)\right]^\alpha \frac{1}{v^{\beta q}}dv$$
$$= \int_{(1/y,1)} \frac{1}{v^\alpha}\frac{1}{v^{\beta q}}dv + \int_{(1,+\infty)} v^\alpha \frac{1}{v^{\beta q}}dv$$
$$= \int_{(1/y,1)} v^{-\alpha-\beta q}dv + \int_{(1,+\infty)} v^{\alpha-\beta q}dv$$
$$= \left[\frac{1}{-\alpha-\beta q+1}v^{-\alpha-\beta q+1}\right]_{(1/y,1)} + \left[\frac{1}{\alpha-\beta q+1}v^{\alpha-\beta q+1}\right]_{(1,+\infty)}$$
$$= \frac{1}{\alpha+\beta q-1}(y^{\alpha+\beta q-1}-1) - \frac{1}{\alpha-\beta q+1}$$
$$= \frac{1}{\alpha+\beta q-1}\left(y^{\alpha+\beta q-1} + \frac{2\alpha}{\beta q-\alpha-1}\right).$$

Therefore, we have

$$F = \int_{(1,+\infty)} y g^q(y) \frac{1}{\alpha+\beta q-1}\left(y^{\alpha+\beta q-1} + \frac{2\alpha}{\beta q-\alpha-1}\right)dy$$
$$= \frac{1}{\alpha+\beta q-1}\int_{(1,+\infty)} g^q(y)y\left(y^{\alpha+\beta q-1} + \frac{2\alpha}{\beta q-\alpha-1}\right)dy. \tag{4.3}$$

Using eqs. (4.1), (4.2) and (4.3), we get

$$\int_{(1,+\infty)}\int_{(1,+\infty)} \left[\max\left(\frac{x}{y},\frac{y}{x}\right)\right]^\alpha f(x)g(y)dxdy$$
$$\le \left[\frac{1}{\alpha+\beta p-1}\int_{(1,+\infty)} f^p(x)x\left(x^{\alpha+\beta p-1} + \frac{2\alpha}{\beta p-\alpha-1}\right)dx\right]^{1/p}$$
$$\times \left[\frac{1}{\alpha+\beta q-1}\int_{(1,+\infty)} g^q(y)y\left(y^{\alpha+\beta q-1} + \frac{2\alpha}{\beta q-\alpha-1}\right)dy\right]^{1/q}$$
$$= \frac{1}{(\alpha+\beta p-1)^{1/p}(\alpha+\beta q-1)^{1/q}}\left[\int_{(1,+\infty)} f^p(x)x\left(x^{\alpha+\beta p-1} + \frac{2\alpha}{\beta p-\alpha-1}\right)dx\right]^{1/p}$$

$$\times \left[\int_{(1,+\infty)} g^q(y) y \left(y^{\alpha+\beta q-1} + \frac{2\alpha}{\beta q-\alpha-1} \right) dy \right]^{1/q}.$$

This completes the proof of Theorem 4.1. □

If we take $\alpha = 1$ and $\beta = 2$, Theorem 4.1 reduces to Theorem 2.1, taking into account the remark in eq. (2.4). Thus, Theorem 4.1 can be viewed as a one-parameter generalization with respect to α, β being another adjustable parameter independent of the kernel function.

5. Conclusion

This article expands upon a fundamental inequality by introducing new variations that advance our understanding of min-max kernel functions and their integral behavior. The proposed extensions reveal the subtle interactions between kernel function structures and weighted norms, suggesting the existence of richer analytic frameworks. These developments pave the way for future exploration in related function spaces and operator theory.

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Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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