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Research Article

Mean Inequalities for Derivatives of the Generalised Exponential Integral Function

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Abstract. In this paper, among other things, we establish arithmetic, geometric and harmonic mean inequalities for derivatives of the generalised exponential integral function. The methods of proof rely heavily on monotonicity properties of certain functions associated with the generalised exponential integral function. The results obtained generalise some existing results in the literature.

Keywords. Exponential integral function, Incomplete gamma function, Inequality

Mathematics Subject Classification (2020). 26D07, 26D20, 33Bxx

1. Introduction

The Euler's integral of second kind, which is commonly referred to as the gamma function, is defined as

$$\Gamma(u) = \int_0^\infty t^{u-1} e^{-t}\, dt,$$

for $u > 0$ and satisfies the basic properties

$$\Gamma(u+1) = u\Gamma(u),$$

$$\Gamma(s+1) = s!, \quad s \in \mathbb{N}_0,$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N} = \{1,2,3,\ldots\}$. It is frequently chanced upon in the theory of special functions. This is largely due to its strong relationship with other special functions. Its applications span across almost every scientific discipline. The upper incomplete gamma function is defined as

$$\Gamma(u,y) = \int_y^\infty t^{u-1}e^{-t}\,dt, \tag{1.1}$$

for $u > 0$, whilst the lower incomplete gamma function is defined as

$$\gamma(u,y) = \int_0^y t^{u-1}e^{-t}\,dt, \tag{1.2}$$

for $u > 0$. It is easily observed that

$$\gamma(u,y) + \Gamma(u,y) = \Gamma(u). \tag{1.3}$$

The incomplete gamma functions are non-trivial generalisations of the ordinary gamma function. A lot of their special cases have applications in areas such as statistics, probability theory, mathematical physics and engineering. They were first considered by Prym [15] and as result, they are also called Prym's functions. For the purpose of this study, we are interested in the upper incomplete gamma function. It satisfies the following properties among others [7, 13]

$$\Gamma(u+1,y) = u\Gamma(u,y) + y^u e^{-y}, \tag{1.4}$$

$$\frac{\partial}{\partial y}\Gamma(u,y) = -y^{u-1}e^{-y}, \tag{1.5}$$

$$\Gamma(m,y) = (m-1)!e^{-y}\sum_{s=0}^{m-1}\frac{y^s}{s!}, \quad m \in \mathbb{N}. \tag{1.6}$$

Closely connected to the upper incomplete gamma function is the classical exponential integral function which is defined as [1, p. 228],

$$\begin{aligned} E(y) &= \int_y^\infty \frac{e^{-t}}{t}\,dt && (1.7)\\ &= \int_1^\infty \frac{e^{-yt}}{t}\,dt && (1.8)\\ &= \Gamma(0,y), \quad \text{for } y > 0. && (1.9) \end{aligned}$$

A generalised version of the exponential integral function is defined as [6, 13],

$$\begin{aligned} E_k(y) &= y^{k-1}\int_y^\infty \frac{e^{-t}}{t^k}\,dt && (1.10)\\ &= \int_1^\infty \frac{e^{-yt}}{t^k}\,dt && (1.11)\\ &= y^{k-1}\Gamma(1-k,y), && (1.12) \end{aligned}$$

for $y > 0$, where $k \in (-\infty,\infty)$. The particular case $E_1(z) = E(z)$ is otherwise known as the *Theis well function* [2].

Derivatives of the generalised exponential integral function are given as [6]

$$E_k^{(r)}(y) = (-1)^r \int_1^{\infty} t^{r-k} e^{-yt}\, dt \tag{1.13}$$

$$= (-1)^r E_{k-r}(y) \tag{1.14}$$

$$= (-1)^r y^{k-(r+1)} \Gamma(1-k+r, y). \tag{1.15}$$

The exponential integral function has many applications in areas such as transient groundwater flow, hydrological problems, mathematical physics, engineering, quantum mechanics, and applied mathematics. Due to this, it has been studied along different directions. Interested readers may refer to Barry *et al*. [2], Bhandari and Bissu [3], Chiccoli *et al*. [5], Lin *et al*. [9], Salem [16], Sroysang [17], Sulaiman [18], Zenku *et al*. [21] and the references in there.

Nantomah [11] established the following mean inequalities,

$$E(y) + E\left(\frac{1}{y}\right) \geq 2\Gamma(0,1), \quad y > 0, \tag{1.16}$$

$$E(y)E\left(\frac{1}{y}\right) \leq \Gamma(0,1)^2, \qquad y > 0, \tag{1.17}$$

$$\frac{2E(y)E(1/y)}{E(y)+E(1/y)} \leq \Gamma(0,1), \quad y > 0. \tag{1.18}$$

Later on, Nantomah [12] extended (1.18) to the generalised exponential integral function by proving that

$$\frac{2E_k(y)E_k(1/y)}{E_k(y)+E_k(1/y)} \leq \Gamma(1-k,1), \quad y > 0,\ k \in \mathbb{N}. \tag{1.19}$$

Motivated by the works [11] and [12], the goal of this paper is to, among other things, extend (1.16), (1.17) and (1.18) to derivatives of the generalised exponential integral function. Results of this nature, concerning other special functions can be found in the works of Bouali *et al*. [4], Li and Qi [8], Matejícka [10], Yildirim [19], Yin *et al*. [20]. We present our results in the next section.

2. Results and Discussion

We begin with the following result which is known in the literature as l'Hospital rule for monotonicy.

Lemma 2.1 ([14]). *Let $-\infty \leq \alpha < \beta \leq \infty$ and f and g be continuous functions that are differentiable on (α,β), with $f(\alpha+) = g(\alpha+) = 0$ or $f(\beta-) = g(\beta-) = 0$. Suppose that $g(y)$ and $g'(y)$ are nonzero for all $y \in (\alpha,\beta)$. If $\frac{f'(y)}{g'(y)}$ is decreasing (or increasing) on (α,β), then $\frac{f(y)}{g(y)}$ is also decreasing (or increasing) on (α,β), respectively.*

Lemma 2.2. *For $m \in \mathbb{N}$, the assertion*

$$\Gamma(m,y) - \Gamma(m+1,y) < 0 \tag{2.1}$$

is true, that is, for a positive integers m, the function $\Gamma(m,y)$ is increasing in terms of m.

Proof. By applying (1.6), we have

$$\begin{aligned}\Gamma(m,y)-\Gamma(m+1,y) &= (m-1)!e^{-y}\sum_{s=0}^{m-1}\frac{y^s}{s!}-m!e^{-y}\sum_{s=0}^{m}\frac{y^s}{s!}\\ &= (m-1)!e^{-y}\sum_{s=0}^{m-1}\frac{y^s}{s!}-m!e^{-y}\left[\sum_{s=0}^{m-1}\frac{y^s}{s!}+\frac{y^m}{m!}\right]\\ &= (m-1)!e^{-y}\sum_{s=0}^{m-1}\frac{y^s}{s!}-m!e^{-y}\sum_{s=0}^{m-1}\frac{y^s}{s!}-y^m e^{-y}\\ &= (1-m)(m-1)!e^{-y}\sum_{s=0}^{m-1}\frac{y^s}{s!}-y^m e^{-y}\\ &< 0.\end{aligned}$$

□

Lemma 2.3. *Let $y>0$, $r\in\mathbb{N}_0$ and $k\in\mathbb{N}$ such that $r-k+1\geq 0$. Then, the function*

$$P(y)=yE_k^{(r+1)}(y) \tag{2.2}$$

is increasing if r is even and is decreasing if r is odd.

Proof. By differentiating and applying (1.14), we have

$$\begin{aligned}P'(y) &= E_k^{(r+1)}(y)+yE_k^{(r+2)}(y)\\ &= (-1)^{r+1}E_{k-(r+1)}(y)+y(-1)^{r+2}E_{k-(r+2)}(y)\\ &:= \phi(y).\end{aligned}$$

Suppose that r is even. Then with the aid of identity (1.15) and Lemma 2.2, we arrive at

$$\begin{aligned}\phi(y) &= -E_{k-(r+1)}(y)+yE_{k-(r+2)}(y)\\ &= y^{k-r-2}[\Gamma(r-k+3,y)-\Gamma(r-k+2,y)]\\ &> 0.\end{aligned}$$

By the same procedure, if r is odd, then we arrive at $\phi(y)<0$. □

Theorem 2.1. *Let $y>0$, $r\in\mathbb{N}_0$ and $k\in\mathbb{N}$ such that $r\geq k$. Then*

$$E_k^{(r)}(y)+E_k^{(r)}\left(\frac{1}{y}\right)\geq 2\Gamma(r-k+1,1) \tag{2.3}$$

holds if r is even, and

$$E_k^{(r)}(y)+E_k^{(r)}\left(\frac{1}{y}\right)\leq -2\Gamma(r-k+1,1) \tag{2.4}$$

holds if r is odd. In both cases, equality occurs if $y=1$.

Proof. The cases for $y=1$ are easy to establish. For this reason, we shall prove the results for only $y\in(0,1)\cup(1,\infty)$. Let $\mathcal{A}(y)=E_k^{(r)}(y)+E_k^{(r)}(1/y)$ for $y\in(0,1)\cup(1,\infty)$, $r\in\mathbb{N}_0$ and $k\in\mathbb{N}$. Since $\mathcal{A}(y)$ remains the same when y is replaced with $1/y$, it suffices to prove the results (2.3) and (2.4) for $y\in(1,\infty)$. Now suppose that $y\in(1,\infty)$. Upon differentiating, we obtain

$$\begin{aligned}y\mathcal{A}'(y) &= yE_k^{(r+1)}(y)-\frac{1}{y}E_k^{(r+1)}\left(\frac{1}{y}\right)\\ &:= h(y).\end{aligned}$$

At this stage, suppose that r is even. Then, the increasing property of the function $P(y)$ in Lemma 2.3 implies that, for $y > 1/y$, we have $P(y) > P(1/y)$. Thus, $h(y) > 0$ and accordingly, $\mathcal{A}(y)$ is increasing.

Therefore, for $y \in (1,\infty)$, we have

$$\mathcal{A}(y) > \lim_{y\to 1^+} \mathcal{A}(y) = 2E_k^{(r)}(1) = 2\Gamma(r-k+1,1),$$

which gives rise to (2.3). Likewise, suppose that r is odd. Then, the decreasing property of the function $P(y)$ in Lemma 2.3 implies that, for $y > 1/y$, we have $P(y) < P(1/y)$. Thus, $h(y) < 0$ and accordingly, $\mathcal{A}(y)$ is decreasing. Therefore, for $y \in (1,\infty)$, we have

$$\mathcal{A}(y) < \lim_{y\to 1^+} \mathcal{A}(y) = -2E_k^{(r)}(1) = -2\Gamma(r-k+1,1),$$

which gives rise to (2.4). □

Lemma 2.4. *Let $y > 0$, $r \in \mathbb{N}_0$ and $k \in \mathbb{N}$ such that $r \ge k$. Then, the function*

$$B(y) = y\frac{E_k^{(r+1)}(y)}{E_k^{(r)}(y)} \tag{2.5}$$

is decreasing.

Proof. With the aid of identity (1.15), we have

$$-B(y) = -\frac{yE_k^{(r+1)}(y)}{E_k^{(r)}(y)} = \frac{\Gamma(r-k+2,y)}{\Gamma(r-k+1,y)} = \frac{f_1(y)}{g_1(y)},$$

where $f_1(y) = \Gamma(r-k+2,y)$, $g_1(y) = \Gamma(r-k+1,y)$ and $f_1(\infty) = g_1(\infty) = 0$. Then

$$\frac{f_1'(y)}{g_1'(y)} = \frac{-e^{-y}y^{r-k+1}}{-e^{-y}y^{r-k}} = y$$

and

$$\left(\frac{f_1'(y)}{g_1'(y)}\right)' = 1 > 0.$$

Hence, $\frac{f_1'(y)}{g_1'(y)}$ is increasing and by Lemma 2.1, $-B(y)$ is increasing. Therefore, $B(y)$ is decreasing. □

Theorem 2.2. *Let $y > 0$, $r \in \mathbb{N}_0$ and $k \in \mathbb{N}$ such that $r \ge k$. Then*

$$E_k^{(r)}(y)E_k^{(r)}\left(\frac{1}{y}\right) \le \Gamma(r-k+1,1)^2. \tag{2.6}$$

Equality occurs if $y = 1$.

Proof. The case for $y = 1$ is easy to establish. Due to this, we shall prove the results for only $y \in (0,1)\cup(1,\infty)$. Let $\mathcal{G}(y) = E_k^{(r)}(y)E_k^{(r)}(1/y)$ for $y \in (0,1)\cup(1,\infty)$, $r \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Since $\mathcal{G}(y)$ remains the same when y is replaced with $1/y$, it suffices to prove the result (2.6) for $y \in (1,\infty)$. Now let $y \in (1,\infty)$. After differentiating and rearranging, we obtain

$$y\frac{\mathcal{G}'(y)}{\mathcal{G}(y)} = y\frac{E_k^{(r+1)}(y)}{E_k^{(r)}(y)} - \frac{1}{y}\frac{E_k^{(r+1)}(1/y)}{E_k^{(r)}(1/y)} := \alpha(y).$$

The decreasing property of the function $B(y)$ in Lemma 2.4 implies that, for $y > 1/y$, we have $B(y) < B(1/y)$. Thus, $\alpha(y) < 0$ and consequently, $\mathcal{G}(y)$ is decreasing.
Therefore, for $y \in (1,\infty)$, we have

$$\mathcal{G}(y) < \lim_{y\to 1^+} \mathcal{G}(y) = (E_k^{(r)}(1))^2 = \Gamma(r-k+1,1)^2,$$

which gives rise to (2.6). □

Corollary 2.3. *Let $y > 0$, $r \in \mathbb{N}_0$ and $k \in \mathbb{N}$ such that $r \geq k$. Then, the limits*

$$\lim_{y\to 0^+} y\frac{E_k^{(r+1)}(y)}{E_k^{(r)}(y)} = -(r-k+1), \tag{2.7}$$

$$\lim_{y\to 0^+} \frac{E_k^{(r)}(y)E_k^{(r+2)}(y)}{(E_k^{(r+1)}(y))^2} = \frac{r-k+2}{r-k+1}, \tag{2.8}$$

are valid.

Proof. By using identity (1.15), we have

$$\begin{aligned}\lim_{y\to 0^+} y\frac{E_k^{(r+1)}(y)}{E_k^{(r)}(y)} &= -\lim_{y\to 0^+}\frac{\Gamma(r-k+2,y)}{\Gamma(r-k+1,y)} = -\frac{\Gamma(r-k+2)}{\Gamma(r-k+1)}\\ &= -(r-k+1).\end{aligned}$$

Similarly,

$$\begin{aligned}\lim_{y\to 0^+} \frac{E_k^{(r)}(y)E_k^{(r+2)}(y)}{(E_k^{(r+1)}(y))^2} &= \lim_{y\to 0^+}\frac{\Gamma(r-k+1,y)\Gamma(r-k+3,y)}{(\Gamma(r-k+2,y))^2}\\ &= \frac{\Gamma(r-k+1)\Gamma(r-k+3)}{(\Gamma(r-k+2))^2}\\ &= \frac{(r-k)!(r-k+2)!}{((r-k+1)!)^2} = \frac{r-k+2}{r-k+1}.\end{aligned}$$

□

Lemma 2.5. *Let $y > 0$, $r \in \mathbb{N}_0$ and $k \in \mathbb{N}$ such that $r \geq k$. Then, the function*

$$T(y) = \frac{yE_k^{(r+1)}(y)}{\left(E_k^{(r)}(y)\right)^2} \tag{2.9}$$

is decreasing if r is even and increasing if r is odd.

Proof. A direct computation yields

$$\begin{aligned}T'(y) &= \left(y\frac{E_k^{(r+1)}(y)}{E_k^{(r)}(y)}\right)'\frac{1}{E_k^{(r)}(y)} + \left(y\frac{E_k^{(r+1)}(y)}{E_k^{(r)}(y)}\right) - \frac{E_k^{(r+1)}(y)}{(E_k^{(r)}(y))^2}\\ &= \frac{1}{E_k^{(r)}(y)}\left[\left(y\frac{E_k^{(r+1)}(y)}{E_k^{(r)}(y)}\right)' - y\left(\frac{E_k^{(r+1)}(y)}{E_k^{(r)}(y)}\right)^2\right].\end{aligned}$$

In view of Lemma 2.4, the expression in the square brackets is negative. Also, $E_k^{(r)}(y) > 0$ if r is even and $E_k^{(r)}(y) < 0$ if r is odd. Therefore, $T'(y) < 0$ if r is even and $T'(y) > 0$ if r is odd. □

Theorem 2.4. *Let $y>0$, $r\in\mathbb{N}_0$ and $k\in\mathbb{N}$ such that $r\geq k$. Then*

$$\frac{2E_k^{(r)}(y)E_k^{(r)}(1/y)}{E_k^{(r)}(y)+E_k^{(r)}(1/y)}\leq \Gamma(r-k+1,1) \tag{2.10}$$

holds if r is even, and

$$\frac{2E_k^{(r)}(y)E_k^{(r)}(1/y)}{E_k^{(r)}(y)+E_k^{(r)}(1/y)}\geq -\Gamma(r-k+1,1) \tag{2.11}$$

holds if r is odd. In both cases, equality occurs if $y=1$.

Proof. The cases for $y=1$ are easy to establish. For this reason, we shall prove the results for only $y\in(0,1)\cup(1,\infty)$. Let $\mathcal{H}(y)=\frac{2E_k^{(r)}(y)E_k^{(r)}(1/y)}{E_k^{(r)}(y)+E_k^{(r)}(1/y)}$, for $y\in(0,1)\cup(1,\infty)$, $r\in\mathbb{N}_0$ and $k\in\mathbb{N}$. Since $\mathcal{H}(y)$ remains the same when y is replaced with $1/y$, it suffices to prove the results (2.10) and (2.11) for $y\in(1,\infty)$. Now suppose that $y\in(1,\infty)$. After differentiating and rearranging, we arrive at

$$y\left[\frac{1}{E_k^{(r)}(y)}+\frac{1}{E_k^{(r)}(1/y)}\right]\frac{\mathcal{H}'(y)}{\mathcal{H}(y)}=y\frac{E_k^{(r+1)}(y)}{(E_k^{(r)}(y))^2}-\frac{1}{y}\frac{E_k^{(r+1)}(1/y)}{(E_k^{(r)}(1/y))^2}:=D(y).$$

At this stage, suppose that r is even. Then, the decreasing property of the function $T(y)$ in Lemma 2.5 implies that, for $y>1/y$, we have $T(y)<T(1/y)$. Thus, $D(y)<0$ and consequently, $\mathcal{H}(y)$ is decreasing. Therefore, for $y\in(1,\infty)$, we have

$$\mathcal{H}(y)<\lim_{y\to1^+}\mathcal{H}(y)=E_k^{(r)}(1)=\Gamma(r-k+1,1),$$

which gives (2.10).

Similarly, suppose that r is odd. Then the increasing property of the function $T(y)$ in Lemma 2.5 implies that, for $y>1/y$, we have $T(y)>T(1/y)$. Thus, $D(y)>0$ and consequently, $\mathcal{H}(y)$ is increasing. Therefore, for $y\in(1,\infty)$, we have

$$\mathcal{H}(y)>\lim_{y\to1^+}\mathcal{H}(y)=E_k^{(r)}(1)=-\Gamma(r-k+1,1),$$

which gives (2.11). □

3. Some Remarks

Remark 3.1. Lemma 2.3 implies that

$$E_k^{(r+1)}(y)+yE_k^{(r+2)}(y)>0 \tag{3.1}$$

if r is even and

$$E_k^{(r+1)}(y)+yE_k^{(r+2)}(y)<0 \tag{3.2}$$

if r is odd.

Remark 3.2. The decreasing property of the function $B(y)$ in Lemma 2.4 implies that

$$y\frac{E_k^{(r+1)}(y)}{E_k^{(r)}(y)}<-(r-k+1) \tag{3.3}$$

for $y>0$, $r\in\mathbb{N}_0$ and $k\in\mathbb{N}$ such that $r\geq k$.

Remark 3.3. Lemma 2.5 implies that

$$\frac{E_k^{(r)}(y)E_k^{(r+1)}(y)+y[E_k^{(r)}(y)E_k^{(r+2)}(y)-2(E_k^{(r+1)}(y))^2]}{[E_k^{(r)}(y)]^3}<(>)0 \tag{3.4}$$

if r is even (odd), respectively. This also implies that

$$E_k^{(r)}(y)E_k^{(r+1)}(y)+y[E_k^{(r)}(y)E_k^{(r+2)}(y)-2(E_k^{(r+1)}(y))^2]<0, \tag{3.5}$$

for all $r\in\mathbb{N}_0$.

Remark 3.4. If $r=0$ in Theorem 2.4, then we obtain (1.19) as a particular case. Also, if $r=0$ and $k=1$ in Theorem 2.1, Theorem 2.2 and Theorem 2.4, then, we respectively obtain (1.16), (1.17) and (1.18). Therefore, this paper is a generalisation of the papers [11] and [12].

4. Conclusion

We have established arithmetic, geometric and harmonic mean inequalities for derivatives of the generalised exponential integral function. The results obtained serve as generalisations of some known results in the literature. Also, along the way, we established some limit properties involving certain ratios of the function. The findings further broaden the areas of application of the function.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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