



Uniqueness of Two Transcendental Meromorphic Functions Whose Differential Polynomials Share the Value 1

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Abstract. In 2008, Tao and Wang [3] proved a uniqueness theorem of meromorphic functions whose some nonlinear differential shares 1 IM with powers of the meromorphic functions. In this paper, we improve their results by introducing another transcendental meromorphic function. We prove uniqueness theorem of differential polynomials of two transcendental meromorphic functions whose some nonlinear differential shares 1 IM with powers of the meromorphic functions, where the degrees of the powers are equal to those of the nonlinear differential polynomials.

Keywords. Differential polynomial, Meromorphic function, Transcendental function, Value sharing, Uniqueness theorem

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1. Introduction and Definitions

The meromorphic functions mentioned throughout the paper will mean meromorphic in the whole complex plane. For any non-constant meromorphic function f , $S(r, f)$ means

$$S(r, f) = o(T(r, f)),$$

possibly outside of a set of finite linear measure. Let f and g be two non constant meromorphic functions defined in the open complex plane $\mathbb{C}$ and let $a \in \mathbb{C} \cup \{\infty\}$. If $f - a$ and $g - a$ have the same zeros CM (Counting Multiplicities) and IM (Ignoring Multiplicities) then we say that f and g share the value 'a' CM and IM respectively. Similarly, f, g share ' ∞ ' CM or IM means that $\frac{1}{f}, \frac{1}{g}$ share '0' CM or IM respectively. For the standard definitions and notations on value distribution theory we refer Hayman [1].

In 2018, Tao and Wang [3] proved the following theorem.

Theorem 1.1 ([3]). Suppose f is a non-constant meromorphic function, k is a positive integer, n is a positive integer and satisfies $2n \geq 3k + 13 + \sqrt{9k^2 + 62k + 129}$, if $f^n(f-1)$ and $f^{n-1}(f-1)f^{(k)}$ share 1 IM, and the zeros of $f-1$ with multiplicity 2 at least, then $f = f^{(k)}$.

Here we mainly extend Theorem 1.1 of Tao and Wang [3]. We consider two differential polynomials $f^p g^q (f-1)(g-1)$ and $f^{p-1} g^{q-1} (f-1)(g-1) f^{(k)} g^{(l)}$ formed with two transcendental meromorphic functions f and g such that they share 1 IM and then study the uniqueness of them.

Let $f(z)$ and $g(z)$ be two meromorphic functions and $a_i, 1 \leq i \leq (p+1)(q+1)$ are small functions of either f or g . Set

$$\begin{aligned} P_{p,q}(f,g) &= \sum_{i=1}^{p+1} \sum_{j=1}^{q+1} a_{(i-1)(q+1)+j} f^{p-i+1} g^{q-j+1} \\ &= a_1 f^p g^q + a_2 f^p g^{q-1} + \cdots + a_{q+1} f^p + a_{q+2} f^{p-1} g^q + a_{q+3} f^{p-1} g^{q-1} + \cdots \\ &\quad + a_{2(q+1)} f^{p-1} + \cdots + a_{p(q+1)+1} g^q + a_{p(q+1)+2} g^{q-1} + \cdots + a_{(p+1)(q+1)}. \end{aligned} \quad (1.1)$$

2. Lemmas

In this section, we present some lemmas which will be needed in the sequel.

Lemma 2.1. Let f and g be two transcendental meromorphic functions with $N(r, f) = S(r, f)$, $N(r, g) = S(r, g)$. If $a_1 \neq 0$ in (1.1), then

$$T(r, P_{p,q}(f, g)) \geq pT(r, f) + qT(r, g) + S(r, f) + S(r, g).$$

Proof. Let $b_i = \frac{a_{i+1}}{a_1}, 1 \leq i \leq (p+1)(q+1) - 1$. Then b_i 's are meromorphic functions satisfying $T(r, b_i) = S(r, f)$ as $r \rightarrow \infty$.

Further

$$\begin{aligned} P_{p,q}(f, g) &= a_1 f^p g^q \left[\left(1 + \frac{b_1}{g} + \cdots + \frac{b_q}{g^q} \right) + \left(\frac{b_{q+1}}{f} + \frac{b_{q+2}}{fg} + \cdots + \frac{b_{2q+1}}{fg^q} \right) + \cdots \right. \\ &\quad \left. + \left(\frac{b_{p(q+1)}}{f^p} + \frac{b_{p(q+1)+1}}{f^p g} + \cdots + \frac{b_{(p+1)(q+1)-1}}{f^p g^q} \right) \right]. \end{aligned}$$

For sufficiently large r , set

$$A(z) = \max_{1 \leq i \leq (p+1)(q+1)-1} \left\{ \max_{|z|=r} |b_i|^{\frac{1}{i}} \right\}.$$

Let $E_1 = \{\theta : 0 \leq \theta \leq 2\pi, |f^s(re^{i\theta})g^t(re^{i\theta})| \geq [2A(re^{i\theta})]^{s(q+1)+t}; 0 \leq s \leq p \text{ and } 0 \leq t \leq q\}$, $E_2 = [0, 2\pi] \setminus E_1$.

On E_1 , we have

$$\begin{aligned}
 |P_{p,q}(f,g)| &\geq |a_1| |f|^p |g|^q \left[\left(1 - \frac{|b_1|}{|g|} - \dots - \frac{|b_q|}{|g|^q} \right) - \left(\frac{|b_{q+1}|}{|f|} + \frac{|b_{q+2}|}{|f||g|} + \dots + \frac{|b_{2q+1}|}{|f||g|^q} \right) - \dots \right. \\
 &\quad \left. - \left(\frac{|b_{p(q+1)}|}{|f|^p} + \frac{|b_{p(q+1)+1}|}{|f|^p |g|} + \dots + \frac{|b_{(p+1)(q+1)-1}|}{|f|^p |g|^q} \right) \right] \\
 &\geq |a_1| |f|^p |g|^q \left[\left(1 - \frac{A(z)}{2A(z)} - \dots - \frac{[A(z)]^q}{[2A(z)]^q} \right) - \left(\frac{[A(z)]^{q+1}}{[2A(z)]^{q+1}} + \dots + \frac{[A(z)]^{2q+1}}{[2A(z)]^{2q+1}} \right) - \dots \right. \\
 &\quad \left. - \left(\frac{[A(z)]^{p(q+1)}}{[2A(z)]^{p(q+1)}} + \frac{[A(z)]^{p(q+1)+1}}{[2A(z)]^{p(q+1)+1}} + \dots + \frac{[A(z)]^{(p+1)(q+1)-1}}{[2A(z)]^{(p+1)(q+1)-1}} \right) \right] \\
 &= |a_1| |f|^p |g|^q \left[\left(1 - \frac{1}{2} - \dots - \frac{1}{2^q} \right) - \left(\frac{1}{2^{q+1}} + \dots + \frac{1}{2^{2q+1}} \right) - \dots \right. \\
 &\quad \left. - \left(\frac{1}{2^{p(q+1)}} + \dots + \frac{1}{2^{(p+1)(q+1)-1}} \right) \right] \\
 &= \frac{|a_1| |f|^p |g|^q}{2^{(p+1)(q+1)-1}}.
 \end{aligned}$$

Now

$$\begin{aligned}
 pm(r, f) + qm(r, g) &= m(r, f^p g^q) \\
 &= \frac{1}{2\pi} \int_{E_1} \log^+ |f^p(z) g^q(z)| d\theta + \frac{1}{2\pi} \int_{E_2} \log^+ |f^p(z) g^q(z)| d\theta \\
 &\leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left[\frac{2^{(p+1)(q+1)-1}}{|a_1|} |P_{p,q}(f, g)| \right] d\theta \\
 &\quad + \frac{1}{2\pi} \int_0^{2\pi} \log^+ [2A(z)]^{p(q+1)+q} d\theta \\
 &= m(r, P_{p,q}(f, g)) + S(r, f) + S(r, g) \quad (\text{since } a_i \text{ are small} \\
 &\quad \text{functions of either } f \text{ or } g). \quad (2.1)
 \end{aligned}$$

Again since $N(r, f) = S(r, f)$, $N(r, g) = S(r, g)$ and a_i are small functions of either f or g , thus

$$N(r, P_{p,q}(f, g)) = S(r, f) + S(r, g). \quad (2.2)$$

Therefore, from (2.1) and (2.2), we have

$$\begin{aligned}
 T(r, P_{p,q}(f, g)) &= m(r, P_{p,q}(f, g)) + N(r, P_{p,q}(f, g)) \\
 &\geq pm(r, f) + qm(r, g) + S(r, f) + S(r, g) \\
 &= pT(r, f) + qT(r, g) + S(r, f) + S(r, g).
 \end{aligned}$$

□

Lemma 2.2 ([2]). Let f be a non-constant meromorphic function with zero point of $f - 1$ is at least 2, then

$$\bar{N}\left(r, \frac{1}{f-1}\right) \leq \bar{N}\left(r, \frac{f}{f'}\right) \leq \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + S(r, f).$$

Lemma 2.3 ([3]). Suppose F and G are non-constant meromorphic functions and satisfy $\bar{N}(r, F) + \bar{N}(r, \frac{1}{F}) = S(r, F)$, $\bar{N}(r, G) + \bar{N}(r, \frac{1}{G}) = S(r, G)$. If F and G share a non-zero constant a IM, then $F = G$ or $FG = 1$.

3. Main Theorems

Theorem 3.1. Let f and g be two transcendental meromorphic functions with $N(r, f) = S(r, f)$, $N(r, g) = S(r, g)$. Let $F = f^p g^q (f - 1)(g - 1)$ and $G = f^{p-1} g^{q-1} (f - 1)(g - 1) f^{(k)} g^{(l)}$, where k, l, p and q (at least one of p or $q \geq 2$) are four non-negative integers. If F and G share 1 IM, then $S(r, F) = S(r, G) = S(r, f) + S(r, g)$.

Proof. According to Lemma 2.1, we obtain

$$T(r, F) \geq (p + 1)T(r, f) + (q + 1)T(r, g) + S(r, f) + S(r, g). \quad (3.1)$$

Also,

$$\begin{aligned} T(r, F) &\leq T(r, f^p) + T(r, g^q) + T(r, f - 1) + T(r, g - 1) \\ &\leq (p + 1)T(r, f) + (q + 1)T(r, g) + O(1). \end{aligned} \quad (3.2)$$

From (3.1) and (3.2), we have

$$S(r, f) + S(r, g) = O(1). \quad (3.3)$$

Hence

$$T(r, F) = (p + 1)T(r, f) + (q + 1)T(r, g) + O(1). \quad (3.4)$$

From (3.3) and (3.4), we obtain

$$\frac{S(r, f) + S(r, g)}{T(r, F)} = \frac{O(1)}{(p + 1)T(r, f) + (q + 1)T(r, g) + O(1)}$$

i.e.,

$$S(r, f) + S(r, g) = o(T(r, F)) = S(r, F). \quad (3.5)$$

Further using second basic theorem, since F and G share 1 IM, we obtain

$$\begin{aligned} T(r, F) &\leq \bar{N}\left(r, \frac{1}{F-1}\right) + \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}(r, F) + S(r, F) \\ &\leq \bar{N}\left(r, \frac{1}{G-1}\right) + \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f-1}\right) + \bar{N}\left(r, \frac{1}{g}\right) + \bar{N}\left(r, \frac{1}{g-1}\right) \\ &\quad + \bar{N}(r, f) + \bar{N}(r, f-1) + \bar{N}(r, g) + \bar{N}(r, g-1) + S(r, f) + S(r, g) \\ &\leq T(r, G) + 2T(r, f) + 2T(r, g) + S(r, f) + S(r, g). \end{aligned}$$

Thus, using (3.3) and (3.4), we have from above

$$T(r, G) \geq (p - 1)T(r, f) + (q - 1)T(r, g) + O(1). \quad (3.6)$$

Also, from (3.3) and (3.6), we obtain

$$\frac{S(r, f) + S(r, g)}{T(r, G)} \leq \frac{O(1)}{(p - 1)T(r, f) + (q - 1)T(r, g) + O(1)}$$

and since at least one of $p, q \geq 2$, thus

$$S(r, f) + S(r, g) = o(T(r, G)) = S(r, G). \quad (3.7)$$

Hence (3.5), (3.7) give the required result. $\square$

Theorem 3.2. Let f and g be two transcendental meromorphic functions with $N(r, f) = S(r, f)$, $N(r, g) = S(r, g)$. Let $F = f^p g^q (f - 1)(g - 1)$ and $G = f^{p-1} g^{q-1} (f - 1)(g - 1) f^{(k)} g^{(l)}$, where k, l, p

and q (at least one of p or $q \geq 2$), $p > 3k + 1$, $q > 3l + 1$ are four non-negative integers. If F and G share 1 IM and the zeros of $f - 1$ and $g - 1$ are with multiplicity 2 at least, then $fg = f^{(k)}g^{(l)}$.

Proof. Let $H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right)$.

Case 1. $H \neq 0$. Then $F \neq G$. Set $U = \frac{F'}{F-1} - \frac{G'}{G-1}$.

Subcase 1.1. $U = 0$.

Then

$$F = CG + 1 - C, \quad C \neq 0 \text{ is a constant.} \quad (3.8)$$

If $C = 1$, then $F = G$, a contradiction. Therefore, $C \neq 1$.

Thus, from (3.8) we have

$$\frac{1}{G} = \frac{C}{F - (1 - C)}$$

i.e.,

$$\bar{N}\left(r, \frac{1}{G}\right) = \bar{N}\left(r, \frac{1}{F - (1 - C)}\right). \quad (3.9)$$

Now if $\bar{N}\left(r, \frac{1}{f}\right) \neq S(r, f)$, $\exists z_0 \in \mathbb{C}$ such that $f(z_0) = 0$.

Putting $z = z_0$ in (3.8), we get $C = 1$, a contradiction.

Thus,

$$\bar{N}\left(r, \frac{1}{f}\right) = S(r, f).$$

Similarly, we get

$$\bar{N}\left(r, \frac{1}{g}\right) = S(r, g), \quad \bar{N}\left(r, \frac{1}{f-1}\right) = S(r, f), \quad \bar{N}\left(r, \frac{1}{g-1}\right) = S(r, g).$$

Hence

$$\begin{aligned} \bar{N}\left(r, \frac{1}{G}\right) &\leq \bar{N}\left(r, \frac{1}{f^{p-1}}\right) + \bar{N}\left(r, \frac{1}{g^{q-1}}\right) + \bar{N}\left(r, \frac{1}{f-1}\right) + \bar{N}\left(r, \frac{1}{g-1}\right) \\ &\quad + \bar{N}\left(r, \frac{f}{f^{(k)}}\right) + \bar{N}\left(r, \frac{g}{g^{(l)}}\right) + O(1) \\ &\leq \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{g}\right) + S(r, f) + S(r, g) + T\left(r, \frac{f^{(k)}}{f}\right) + T\left(r, \frac{g^{(l)}}{g}\right) + O(1) \\ &\leq S(r, f) + S(r, g) + T\left(r, \frac{f^{(k)}}{f}\right) + T\left(r, \frac{g^{(l)}}{g}\right) + O(1) \\ &= m\left(r, \frac{f^{(k)}}{f}\right) + N\left(r, \frac{f^{(k)}}{f}\right) + m\left(r, \frac{g^{(l)}}{g}\right) + N\left(r, \frac{g^{(l)}}{g}\right) + O(1) \quad (\text{using (3.3)}) \\ &\leq k\bar{N}(r, f) + k\bar{N}\left(r, \frac{1}{f}\right) + l\bar{N}(r, g) + l\bar{N}\left(r, \frac{1}{g}\right) + O(1) \\ &\quad (\text{since } m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f) \text{ and } m\left(r, \frac{g^{(l)}}{g}\right) = S(r, g)). \end{aligned}$$

Thus, using (3.9),

$$\bar{N}\left(r, \frac{1}{F - (1 - C)}\right) = O(1).$$

Now using Lemma 2.1 and (3.3), we get

$$\begin{aligned}
 (p+1)T(r, f) + (q+1)T(r, g) + O(1) &\leq T(r, F) \\
 &\leq \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{F - (1-C)}\right) + \bar{N}(r, F) + O(1), \\
 &\quad \text{(by second basic theorem)} \\
 &= O(1), \text{ a contradiction.}
 \end{aligned}$$

Subcase 1.2. $U \neq 0$.

Firstly, suppose that f and g have no common zero.

Let z_0 is a zero of f with the multiplicity r and z_1 is a zero of g with the multiplicity s ($r, s \neq 0$). Then, z_0 is a zero of $\frac{F'}{F-1}$ with the multiplicity $pr-1$, a zero of $\frac{G'}{G-1}$ with the multiplicity $(p-1)r + r - k - 1 = pr - k - 1$ and z_1 is a zero of $\frac{F'}{F-1}$ with the multiplicity $qs-1$, a zero of $\frac{G'}{G-1}$ with the multiplicity $(q-1)s + s - l - 1 = qs - l - 1$. Thus, U has zero at z_0 with multiplicity $p - k - 1$ and has zero at z_1 with multiplicity $q - l - 1$. Noting that $m(r, U) = O(1)$, we have

$$\begin{aligned}
 (p-k-1)\bar{N}\left(r, \frac{1}{f}\right) + (q-l-1)\bar{N}\left(r, \frac{1}{g}\right) &\leq \bar{N}\left(r, \frac{1}{U}\right) + O(1) \\
 &\leq T(r, U) + O(1) \\
 &\leq N(r, U) + O(1) \\
 &\leq N\left(r, \frac{(F-1)'}{F-1}\right) + N\left(r, \frac{(G-1)'}{G-1}\right) + O(1) \\
 &= \bar{N}(r, F-1) + \bar{N}\left(r, \frac{1}{F-1}\right) + \bar{N}(r, G-1) \\
 &\quad + \bar{N}\left(r, \frac{1}{G-1}\right) + O(1). \tag{3.10}
 \end{aligned}$$

Now where $\frac{1}{F-1}$ and $\frac{1}{G-1}$ have poles, there $\frac{1}{\frac{F}{F-1}}$ also has poles. Thus,

$$\bar{N}\left(r, \frac{1}{F-1}\right) \leq \bar{N}\left(r, \frac{1}{\frac{F}{F-1}}\right) \quad \text{and} \quad \bar{N}\left(r, \frac{1}{G-1}\right) \leq \bar{N}\left(r, \frac{1}{\frac{G}{G-1}}\right). \tag{3.11}$$

Thus, from (3.10) and (3.11) and since $\bar{N}(r, F) = O(1) = \bar{N}(r, G)$, we get

$$\begin{aligned}
 (p-k-1)\bar{N}\left(r, \frac{1}{f}\right) + (q-l-1)\bar{N}\left(r, \frac{1}{g}\right) &\leq 2\bar{N}\left(r, \frac{1}{\frac{F}{F-1}}\right) + O(1) \\
 &\leq 2T\left(r, \frac{f^{(k)}g^{(l)}}{fg}\right) + O(1) \\
 &= 2m\left(r, \frac{f^{(k)}g^{(l)}}{fg}\right) + 2N\left(r, \frac{f^{(k)}g^{(l)}}{fg}\right) + O(1) \\
 &\leq 2\left[k\bar{N}(r, f) + k\bar{N}\left(r, \frac{1}{f}\right) + l\bar{N}(r, g) + l\bar{N}\left(r, \frac{1}{g}\right)\right] + O(1) \\
 &\quad \text{(since } m\left(r, \frac{f^{(k)}g^{(l)}}{fg}\right) = O(1))
 \end{aligned}$$

$$= 2k\bar{N}\left(r, \frac{1}{f}\right) + 2l\bar{N}\left(r, \frac{1}{g}\right) + O(1)$$

(since $N(r, f) = S(r, f), N(r, g) = S(r, g)$).

Hence

$$\begin{aligned} & (p - 3k - 1)\bar{N}\left(r, \frac{1}{f}\right) + (q - 3l - 1)\bar{N}\left(r, \frac{1}{g}\right) = O(1) \\ \Rightarrow \quad & \bar{N}\left(r, \frac{1}{g}\right) = S(r, g) \quad \text{and} \quad \bar{N}\left(r, \frac{1}{f}\right) = S(r, f), \end{aligned} \quad (3.12)$$

because $p > 3k + 1, q > 3l + 1$.

Secondly, suppose that f and g has atleast one common zero.

Let z_0 is a zero of f with the multiplicity r and a zero of g with the multiplicity s ($r, s \neq 0$). Then z_0 is a zero of $\frac{F'}{F-1}$ with the multiplicity $pr + qs - 1$, a zero of $\frac{G'}{G-1}$ with the multiplicity $(p-1)r + (q-1)s + r - k + s - l - 1 = pr - k + qs - l - 1$. So z_0 is a zero of U with multiplicity atleast $p - k + q - l - 1$. Noting that $m(r, U) = O(1)$, we have

$$\begin{aligned} & (p - k + q - 3l - 1)\bar{N}\left(r, \frac{1}{f}\right) + 2l\bar{N}\left(r, \frac{1}{g}\right) \leq 2k\bar{N}\left(r, \frac{1}{f}\right) + 2l\bar{N}\left(r, \frac{1}{g}\right) + O(1) \\ \Rightarrow \quad & (p + q - 3k - 3l - 1)\bar{N}\left(r, \frac{1}{f}\right) = O(1) \\ \Rightarrow \quad & \bar{N}\left(r, \frac{1}{f}\right) = O(1) \end{aligned} \quad (3.13)$$

because $p > 3k + 1, q > 3l + 1$.

We can also write

$$\begin{aligned} & 2k\bar{N}\left(r, \frac{1}{f}\right) + (p - 3k + q - l - 1)\bar{N}\left(r, \frac{1}{g}\right) \leq 2k\bar{N}\left(r, \frac{1}{f}\right) + 2l\bar{N}\left(r, \frac{1}{g}\right) + O(1) \\ \Rightarrow \quad & (p + q - 3k - 3l - 1)\bar{N}\left(r, \frac{1}{g}\right) = O(1) \\ \Rightarrow \quad & \bar{N}\left(r, \frac{1}{g}\right) = O(1) \end{aligned} \quad (3.14)$$

because $p > 3k + 1, q > 3l + 1$.

In all cases, from second basic theorem, we get

$$\begin{aligned} T(r, F) & \leq \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{F-1}\right) + \bar{N}(r, F) + S(r, F) \\ & \leq \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{g}\right) + \bar{N}\left(r, \frac{1}{f-1}\right) + \bar{N}\left(r, \frac{1}{g-1}\right) + \bar{N}\left(r, \frac{1}{\frac{G}{F}-1}\right) + O(1) \\ & \hspace{15em} \text{(using (3.11) and since } \bar{N}(r, F) = O(1)) \\ & \leq (k+2)\bar{N}\left(r, \frac{1}{f}\right) + (l+2)\bar{N}\left(r, \frac{1}{g}\right) + O(1) \quad \text{(by Lemma 2.2 and since} \\ & \hspace{15em} N(r, f) = S(r, f), N(r, g) = S(r, g)) \\ & = O(1) \quad \text{(using (3.13), (3.14)),} \end{aligned}$$

a contradiction.

Case 2. $H = 0$.

Then

$$\frac{1}{F-1} = \frac{A}{G-1} + B. \quad (3.15)$$

Subcase 2.1. Suppose $\bar{N}(r, \frac{1}{f}) = S(r, f)$ and $\bar{N}(r, \frac{1}{g}) = S(r, g)$. Then, $\bar{N}(r, F) + \bar{N}(r, \frac{1}{F}) = S(r, F)$ and $\bar{N}(r, G) + \bar{N}(r, \frac{1}{G}) = S(r, G)$. Thus, by Lemma 2.3, we get $F = G$ or $FG = 1$.

First suppose that $F = G$. Then $fg = f^{(k)}g^{(l)}$, conclusion established.

Next suppose that $FG = 1$. Then, we get

$$f^{2p}g^{2q}(f-1)^2(g-1)^2 = \frac{fg}{f^{(k)}g^{(l)}}.$$

So,

$$T(r, f^{2p}g^{2q}(f-1)^2(g-1)^2) + O(1) \leq T\left(r, \frac{f^{(k)}}{f}\right) + T\left(r, \frac{g^{(l)}}{g}\right)$$

$$\Rightarrow (2p+2)T(r, f) + (2q+2)T(r, g) \leq kT(r, f) + lT(r, g) + O(1) \quad (\text{using Lemma 2.1})$$

$$\Rightarrow (2p+2-k)T(r, f) + (2q+2-l)T(r, g) \leq O(1),$$

a contradiction since $p > 3k+1$, $q > 3l+1$.

Subcase 2.2. Suppose $\bar{N}(r, \frac{1}{f}) \neq S(r, f)$ or $\bar{N}(r, \frac{1}{g}) \neq S(r, g)$. Without loss of generality let $\bar{N}(r, \frac{1}{f}) \neq S(r, f)$. Then, $\exists z_0 \in \mathbb{C}$ such that $f(z_0) = 0$.

Putting $z = z_0$ in (3.15), we get $B = A - 1$.

First suppose that $A = 1$. Then $B = 0$ and thus $F = G$, i.e., $fg = f^{(k)}g^{(l)}$, conclusion established.

Next suppose that $A \neq 1$. Then, we get

$$\frac{G}{F} = \frac{1}{A-1} \frac{1}{F - \frac{A}{A-1}}. \quad (3.16)$$

Now from Lemma 2.1, we have

$$\begin{aligned} & (p+1)T(r, f) + (q+1)T(r, g) \\ & \leq T(r, F) + O(1) \\ & \leq \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{F - \frac{A}{A-1}}\right) + \bar{N}(r, F) + O(1) \quad (\text{by second basic theorem}) \\ & \leq \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f-1}\right) + \bar{N}\left(r, \frac{1}{g}\right) + \bar{N}\left(r, \frac{1}{g-1}\right) + \bar{N}\left(r, \frac{G}{F}\right) + O(1) \quad (\text{using (3.16)}) \\ & \leq 2T(r, f) + 2T(r, g) + k\bar{N}\left(r, \frac{1}{f}\right) + l\bar{N}\left(r, \frac{1}{g}\right) + O(1) \quad (\text{using Lemma 2.2}) \\ & \leq (2+k)T(r, f) + (2+l)T(r, g) + O(1). \end{aligned}$$

Therefore,

$$(p-1-k)T(r, f) + (q-1-l)T(r, g) = O(1),$$

a contradiction.

Hence the theorem. □

Example 3.1. Let $f = e^z$, $g = e^{-z}$, $k = 1$, $l = 0$, $p = 4$, $q = 3$. Here f and g are two transcendental meromorphic functions with $N(r, f) = S(r, f)$ and $N(r, g) = S(r, g)$. Also, k, l, p, q satisfy all the conditions of Theorem 3.2.

Now $F = -(e^z - 1)^2$ and $G = -(e^{-z} - 1)^2$. Thus, F and G share 1 IM.

Hence our chosen functions and integers satisfies all the conditions of Theorem 3.2 except the zeros of $f - 1$ and $g - 1$ are with multiplicity 2 at least.

But $fg = 1 = f^{(k)}g^{(l)}$. Hence, the conclusion established.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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