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Research Article

Certain Composite Image Formulae Involving with Generalized Multiindex Bessel Function and Srivastava Polynomials

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Abstract. In the present paper, we aim to establishing certain composite image formulae involving with Srivastava polynomials and generalized multiindex Bessel function. The obtained results are in the form of hypergeometric function, which are made with the help of Hadamard product. We have derived the known formulas with the help of the main results and also created some new ones.

Keywords. Srivastava polynomials, Generalized multiindex Bessel function, Hypergeometric function, Lavoie-Trottier integral, Oberhettinger integral

Mathematics Subject Classification (2020). 26A33, 33C45, 33C20

1. Introduction

The Srivastava polynomials or general class of polynomials introduced by Srivastava [13] are represented in the following summation form

$$S_N^{\lambda}(x) = \sum_{\kappa=0}^{[N/\lambda]} \frac{(-N)_{\lambda\kappa}}{\kappa!} A_{N,\kappa} x^{\kappa}, \quad N = 0, 1, 2, \dots, \tag{1}$$

where λ is an arbitrary positive integer and the coefficients $A_{N,\kappa}$ $(N, \kappa \geq 0)$ are arbitrary constant of real or complex numbers.

For $A_j, B_j, \delta, c, d \in \mathbb{C}$ $(j = 1, \dots, m)$ the generalized multiindex Bessel function introduced by Nisar *et al.* [7] and is given by the following summation form

$$J_{(B_j)_m,\mu,d}^{(A_j)_m,\delta,c}(x) = \sum_{\kappa=0}^{\infty} \frac{c^{\kappa}(\delta)_{\mu\kappa}}{\prod_{j=1}^{m} \Gamma\left(A_j\kappa + B_j + \frac{d+1}{2}\right)} \frac{x^{\kappa}}{\kappa!}, \quad m \in N, \tag{2}$$

where $\sum_{j=1}^{m} Re(A_j) > \max\{0, Re(\mu) - 1\}$, $\mu > 0$, $Re(B_j) > 0$, $Re(\delta) > 0$ and here $(\delta)_v$ denotes the Pochhammer symbol defined for $\delta, v \in \mathbb{C}$ in terms of the Gamma function by

$$(\delta)_v = \frac{\Gamma(\delta + v)}{\Gamma(\delta)} = \begin{cases} 1, & v = 0, \delta \in \mathbb{C}\setminus\{0\}, \\ \delta(\delta+1), \dots, (\delta+n-1), & v = n \in \mathbb{N}, \delta \in \mathbb{C}. \end{cases} \tag{3}$$

2. Preliminaries and Definitions

For $A_j, B_j, \delta \in \mathbb{C}$, $\mu > 0$, $Re(\delta) > 0$, Choi and Agarwal [4] introduced the multiindex Bessel function in the following summation form, provided that $\sum_{j=1}^{m} Re(A_j) > \max\{0; Re(\mu) - 1\}$ and $Re(B_j) > -1$,

$$J_{(B_j)_m,\mu}^{(A_j)_m,\delta}(x) = \sum_{\kappa=0}^{\infty} \frac{(\delta)_{\mu\kappa}}{\prod_{j=1}^{m} \Gamma(A_j\kappa + B_j + 1)} \frac{(-x)^{\kappa}}{\kappa!}, \quad m \in \mathbb{N}. \tag{4}$$

For our present investigation, we recall the following integral formulae which are given by Lavoie-Trottier [4] and Oberhettinger [8] in equations (5) and (6) respectively,

$$\int_0^1 \xi^{\epsilon-1}(1-\xi)^{2\epsilon-1}\left(1-\frac{\xi}{3}\right)^{2v-1}\left(1-\frac{\xi}{4}\right)^{\epsilon-1} d\xi = \left(\frac{4}{9}\right)^{v} \frac{\Gamma(v)\Gamma(\epsilon)}{\Gamma(v+\epsilon)}, \tag{5}$$

where $Re(v) > 0$ and $Re(\epsilon) > 0$,

$$\int_0^{\infty} \xi^{v-1}\left(\xi + C + \sqrt{\xi^2 + 2C\xi}\,\right)^{-\varepsilon} d\xi = \frac{2\varepsilon}{C^{\varepsilon}}\left(\frac{C}{2}\right)^{v} \frac{\Gamma(2v)\Gamma(\epsilon - v)}{\Gamma(1+v+\epsilon)}, \tag{6}$$

where $Re(\epsilon) > Re(v) > 0$.

Let $f(x) = \sum_{\kappa=0}^{\infty} C_\kappa x^\kappa$ and $g(x) = \sum_{\kappa=0}^{\infty} D_\kappa x^\kappa$ are two analytic functions with their radii of convergence R_f and R_g respectively, then their Hadamard product [9], [15] is given by the following power series

$$f * g(x) = g * f(x) = \sum_{\kappa=0}^{\infty} C_\kappa D_\kappa x^\kappa, \quad |x| < R, \tag{7}$$

where $R_c \geq R_f \cdot R_g$ is the radius of convergence of the composite series.

Now the generalized hypergeometric function [3] in terms of Pochhammer symbol is expressed as follows with p and q,

$${}_pF_q\left[\begin{matrix}\vartheta_1, \dots, \vartheta_p; \\ \zeta_1, \dots, \zeta_q;\end{matrix}\ x\right] = \sum_{\kappa=0}^{\infty} \frac{(\vartheta_1)_\kappa, \dots, (\vartheta_p)_\kappa}{(\zeta_1)_\kappa, \dots, (\zeta_q)_\kappa} \frac{x^\kappa}{\kappa!}, \tag{8}$$

where $x, \vartheta_1, \dots, \vartheta_p; \zeta_1, \dots, \zeta_q \in \mathbb{C}$ with $\zeta_1, \dots, \zeta_q$ are positive integers.

For $x \in \mathbb{C}, \vartheta_j, \beta_j \in \mathbb{C}$ and $a_j, b_j \in \mathbb{R}$ the definition of Fox-Wright function ${}_p\psi_q$ (see [16]) is defined as below:

$$ {}_p\psi_q\left[\begin{matrix}(a_i,\vartheta_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\,\middle|\,x\right]=\sum_{\kappa=0}^{\infty}\frac{\Gamma(a_1+\vartheta_1\kappa),\ldots,\Gamma(a_p+\vartheta_p\kappa)}{\Gamma(b_1+\beta_1\kappa),\ldots,\Gamma(b_q+\beta_q\kappa)}\frac{x^\kappa}{\kappa!}, \tag{9} $$

where $\vartheta_i, \beta_j \neq 0$; $i = 1,\ldots,p$; $j = 1,\ldots,q$ and for all values of the x under the condition:

$$ 1+\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\vartheta_i>0. \tag{10} $$

3. Main Results

Theorem 3.1. *Let $\xi > 0$, $\nu,\varepsilon \in \mathbb{C}$ be such that $Re(\nu) > 0$, $Re(\epsilon) > 0$ and the conditions given in* (1) *and* (2) *are satisfied, then for the product of Srivastava polynomials $S_N^\lambda(\cdot)$ and generalized multiindex Bessel function $J(\cdot)$, the following integral formula holds true*

$$ \begin{aligned} &\int_0^1 \xi^{\nu-1}(1-\xi)^{2\varepsilon-1}\left(1-\frac{\xi}{3}\right)^{2\nu-1}\left(1-\frac{\xi}{4}\right)^{\varepsilon-1}S_N^\lambda(x)J_{(B_j)_m,\mu,d}^{(A_j)_m,\delta,c}(x)d\xi\\ &=\left(\frac{4}{9}\right)^{\nu}\frac{\Gamma(\upsilon)\Gamma(\epsilon)}{\Gamma(\upsilon+\epsilon)}\sum_{\kappa=0}^{[N/\lambda]}\frac{(-N)_{\lambda\kappa}}{\kappa!}\bar{A}_{N,\kappa}\theta^\kappa{}_2F_1\left[\begin{matrix}\varepsilon+\kappa,1;\\ \nu+\varepsilon+\kappa;\end{matrix}\;\theta\right]*J_{(B_j)_m,\mu,d}^{(A_j)_m,\delta,c}(\theta), \end{aligned} \tag{11} $$

where $X = \theta(1-\xi)^2\left(1-\frac{\xi}{4}\right)$ and $\bar{A}_{N,\kappa} = A_{N,\kappa}\frac{(\varepsilon)_\kappa}{(\nu+\varepsilon)_\kappa}$.

Proof. First, we refer to the left-hand side of equation (11) as the sign I_1 then making the use of equation (1) and (2) in equation (11), we have

$$ \begin{aligned} I_1 \equiv &\int_0^1 \xi^{\nu-1}(1-\xi)^{2\varepsilon-1}\left(1-\frac{\xi}{3}\right)^{2\nu-1}\left(1-\frac{\xi}{4}\right)^{\varepsilon-1}\sum_{\kappa=0}^{[N/\lambda]}\frac{(-N)_{\lambda\kappa}}{\kappa!}A_{N,\kappa}\theta^\kappa(1-\xi)^{2\kappa}\left(1-\frac{\xi}{4}\right)^{\kappa}\\ &\times\sum_{s=0}^{\infty}\frac{c^s(\delta)_{\mu s}}{\prod_{j=1}^{m}\Gamma\left(A_js+B_j+\frac{d+1}{2}\right)}\frac{\theta^s}{s!}(1-\xi)^{2s}\left(1-\frac{\xi}{4}\right)^{s}d\xi. \end{aligned} $$

After adjust the order of integration and summation under the theorem's condition,

$$ \begin{aligned} I_1 \equiv &\sum_{\kappa=0}^{[N/\lambda]}\frac{(-N)_{\lambda\kappa}}{\kappa!}A_{N,\kappa}\theta^\kappa\sum_{s=0}^{\infty}\frac{c^s(\delta)_{\mu s}}{\prod_{j=1}^{m}\Gamma\left(A_js+B_j+\frac{d+1}{2}\right)}\frac{\theta^s}{s!}\\ &\times\int_0^1 \xi^{\nu-1}(1-\xi)^{2(\varepsilon+\kappa+s)-1}\left(1-\frac{\xi}{3}\right)^{2\nu-1}\left(1-\frac{\xi}{4}\right)^{(\varepsilon+\kappa+s)-1}d\xi. \end{aligned} $$

By using equation (5) and after some simplification, we get

$$ I_1 \equiv \left(\frac{4}{9}\right)^{\nu}\frac{\Gamma(\upsilon)\Gamma(\epsilon)}{\Gamma(\upsilon+\epsilon)}\sum_{\kappa=0}^{[N/\lambda]}\frac{(-N)_{\lambda\kappa}}{\kappa!}\frac{(\varepsilon)_\kappa A_{N,\kappa}}{(\nu+\varepsilon)_\kappa}\theta^\kappa\sum_{s=0}^{\infty}\frac{c^s(\delta)_{\mu s}}{\prod_{j=1}^{m}\Gamma\left(A_js+B_j+\frac{d+1}{2}\right)}\frac{(\varepsilon+\kappa)_s}{(\nu+\varepsilon+\kappa)_s}\frac{\theta^s}{s!}. $$

Now apply Hadamard product (7) and using the notation $\bar{A}_{N,\kappa} = A_{N,\kappa}\frac{(\varepsilon)_\kappa}{(\nu+\varepsilon)_\kappa}$,

$$ I_1 = \left(\frac{4}{9}\right)^{\nu}\frac{\Gamma(\upsilon)\Gamma(\epsilon)}{\Gamma(\upsilon+\epsilon)}\sum_{\kappa=0}^{[N/\lambda]}\frac{(-N)_{\lambda\kappa}}{\kappa!}\bar{A}_{N,\kappa}\,\theta^\kappa\,{}_2F_1\left[\begin{matrix}\varepsilon+\kappa,1;\\ \nu+\varepsilon+\kappa;\end{matrix}\;\theta\right]*J_{(B_j)_m,\mu,d}^{(A_j)_m,\delta,c}(\theta). $$

□

Theorem 3.2. *Let $\xi>0$, $\nu,\varepsilon\in\mathbb{C}$ be such that $Re(\nu)>0$, $Re(\epsilon)>0$ and the conditions given in* (1) *and* (2) *are satisfied, then for the product of Srivastava polynomials $S_N^\lambda(\cdot)$ and generalized multiindex Bessel function $J(\cdot)$, the following integral formula holds true*

$$\int_0^1 \xi^{\nu-1}(1-\xi)^{2\varepsilon-1}\left(1-\frac{\xi}{3}\right)^{2\nu-1}\left(1-\frac{\xi}{4}\right)^{\varepsilon-1} S_N^\lambda(Y)\ \ J_{(B_j)_m,\mu,d}^{(A_j)_m,\delta,c}(Y)\ d\xi$$
$$=\left(\frac{4}{9}\right)^\nu \frac{\Gamma(\upsilon)\Gamma(\epsilon)}{\Gamma(\upsilon+\epsilon)}\sum_{\kappa=0}^{[N/\lambda]}\frac{(-N)_{\lambda\kappa}}{\kappa!}\bar{A}_{N,\kappa}\left(\frac{4\theta}{9}\right)^\kappa {}_2F_1\left[\begin{matrix}\nu+\kappa,1; \\ \nu+\varepsilon+\kappa;\end{matrix}\ \frac{4\theta}{9}\right] * J_{(B_j)_m,\mu,d}^{(A_j)_m,\delta,c}\left(\frac{4\theta}{9}\right), \tag{12}$$

where $Y=\theta\xi\left(1-\frac{\xi}{3}\right)^2$ *and* $\bar{A}_{N,\kappa}=A_{N,\kappa}\frac{(\nu)_\kappa}{(\nu+\varepsilon)_\kappa}$.

Proof. First, we refer to the left-hand side of equation (12) as the sign I_2 then making the use of equation (1) and (2) in equation (12), we have

$$I_2\equiv\int_0^1 \xi^{\nu-1}(1-\xi)^{2\varepsilon-1}\left(1-\frac{\xi}{3}\right)^{2\nu-1}\left(1-\frac{\xi}{4}\right)^{\varepsilon-1}\sum_{\kappa=0}^{[N/\lambda]}\frac{(-N)_{\lambda\kappa}}{\kappa!}A_{N,\kappa}\ \theta^\kappa\xi^\kappa\left(1-\frac{\xi}{3}\right)^{2\kappa}$$
$$\times\sum_{s=0}^{\infty}\frac{c^s(\delta)_{\mu s}}{\prod_{j=1}^m\Gamma\left(A_j\ s+B_j+\frac{d+1}{2}\right)}\frac{\theta^s\xi^s}{s!}\left(1-\frac{\xi}{3}\right)^{2s}d\xi.$$

After changing the order of integration and summation under the condition of the theorem

$$I_2\equiv\sum_{\kappa=0}^{[N/\lambda]}\frac{(-N)_{\lambda\kappa}}{\kappa!}A_{N,\kappa}\left(\frac{4\theta}{9}\right)^\kappa\sum_{s=0}^{\infty}\frac{c^s(\delta)_{\mu s}}{\prod_{j=1}^m\Gamma\left(A_js+B_j+\frac{d+1}{2}\right)}\frac{\theta^s}{s!}\left(\frac{4}{9}\right)^s$$
$$\times\int_0^1 \xi^{(\nu+\kappa+s)-1}(1-\xi)^{2\varepsilon-1}\left(1-\frac{\xi}{3}\right)^{2(\nu+\kappa+s)-1}\left(1-\frac{\xi}{4}\right)^{\varepsilon-1}d\xi.$$

By using equation (5) and after some simplification, we get

$$I_2\equiv\left(\frac{4}{9}\right)^\nu\frac{\Gamma(\upsilon)\Gamma(\epsilon)}{\Gamma(\upsilon+\epsilon)}\sum_{\kappa=0}^{[N/\lambda]}\frac{(-N)_{\lambda\kappa}}{\kappa!}\frac{(\nu)_\kappa}{(\nu+\varepsilon)_\kappa}A_{N,\kappa}\left(\frac{4\theta}{9}\right)^\kappa$$
$$\times\sum_{s=0}^{\infty}\frac{c^s(\delta)_{\mu s}}{\prod_{j=1}^m\Gamma\left(A_j\ s+B_j+\frac{d+1}{2}\right)}\frac{(\nu+\kappa)_s}{(\nu+\varepsilon+\kappa)_s}\frac{\theta^s}{s!}\left(\frac{4}{9}\right)^s.$$

Now apply Hadamard product (7) and using the notation

$$\bar{A}_{N,\kappa}=A_{N,\kappa}\frac{(\nu)_\kappa}{(\nu+\varepsilon)_\kappa}$$
$$=\left(\frac{4}{9}\right)^\nu\frac{\Gamma(\upsilon)\Gamma(\epsilon)}{\Gamma(\upsilon+\epsilon)}\sum_{\kappa=0}^{[N/\lambda]}\frac{(-N)_{\lambda\kappa}}{\kappa!}\bar{A}_{N,\kappa}\left(\frac{4\theta}{9}\right)^\kappa {}_2F_1\left[\begin{matrix}\nu+\kappa,1; \\ \nu+\varepsilon+\kappa;\end{matrix}\ \frac{4\theta}{9}\right] * J_{(B_j)_m,\mu,d}^{(A_j)_m,\delta,c}\left(\frac{4\theta}{9}\right).$$ □

Theorem 3.3. *Let $\xi>0$, $\nu,\varepsilon\in\mathbb{C}$ be such that $Re(\epsilon)>Re(\nu)>0$ and the conditions given in* (1) *and* (2) *are satisfied, then for the product of Srivastava polynomials $S_N^\lambda(\cdot)$ and generalized multiindex Bessel function $J(\cdot)$, the following integral formula holds true*

$$\int_0^\infty \xi^{\nu-1}\left(\xi+C+\sqrt{\xi^2+2C\xi}\right)^{-\varepsilon} S_N^\lambda(Z)J_{(B_j)_m,\mu,d}^{(A_j)_m,\delta,c}(Z)\ d\xi$$
$$=\frac{2\varepsilon}{C^\varepsilon}\left(\frac{C}{2}\right)^\nu\frac{\Gamma(2\upsilon)\Gamma(\epsilon-\upsilon)}{\Gamma(1+\upsilon+\epsilon)}\sum_{\kappa=0}^{[N/\lambda]}\frac{(-N)_{\lambda\kappa}}{\kappa!}\bar{A}_{N,\kappa}\left(\frac{\theta}{C}\right)^\kappa$$

$$\times {}_3F_2\left[\begin{matrix}\varepsilon+\kappa+1, \varepsilon-\nu+\kappa, 1; \\ \varepsilon+\kappa,\ \nu+\varepsilon+\kappa+1;\end{matrix}\ \frac{\theta}{C}\right] * J_{(B_j)_m,\mu,d}^{(A_j)_m,\delta,c}\left(\frac{\theta}{C}\right), \tag{13}$$

where $Z=\frac{\theta}{\xi+C+\sqrt{\xi^2+2C\xi}}$ *and* $\bar{A}_{N,\kappa}=A_{N,\kappa}\frac{(\varepsilon+1)_\kappa}{(\varepsilon)_\kappa}\frac{(\varepsilon-\nu)_\kappa}{(1+\nu+\varepsilon)_\kappa}$.

Proof. First, we refer to the left-hand side of equation (13) as the sign I_3 then making the use of equation (1) and (2) in equation (13), we have

$$I_3 \equiv \int_0^\infty \frac{\xi^{\nu-1}}{(\xi+C+\sqrt{\xi^2+2C\xi})^\varepsilon}\sum_{\kappa=0}^{[N/\lambda]}\frac{(-N)_{\lambda\kappa}}{\kappa!}A_{N,\kappa}\,\theta^\kappa(\xi+C+\sqrt{\xi^2+2C\xi})^{-\kappa}$$
$$\times\sum_{s=0}^\infty \frac{c^s(\delta)_{\mu s}}{\prod_{j=1}^m \Gamma\left(A_j\,s+B_j+\frac{d+1}{2}\right)}\frac{\theta^s}{s!}(\xi+C+\sqrt{\xi^2+2C\xi})^{-s}\,d\xi.$$

After adjust the order of integration and summation under the theorem's condition,

$$I_3 \equiv \sum_{\kappa=0}^{[N/\lambda]}\frac{(-N)_{\lambda\kappa}}{\kappa!}A_{N,\kappa}\,\theta^\kappa\sum_{s=0}^\infty \frac{c^s(\delta)_{\mu s}}{\prod_{j=1}^m \Gamma\left(A_j\,s+B_j+\frac{d+1}{2}\right)}\frac{\theta^s}{s!}\int_0^\infty \frac{\xi^{\nu-1}d\xi}{(\xi+C+\sqrt{\xi^2+2C\xi})^{\varepsilon+\kappa+s}}.$$

By using equation (6) and after some simplification, we get

$$I_3 \equiv \frac{2\varepsilon}{C^\varepsilon}\left(\frac{C}{2}\right)^\nu\frac{\Gamma(2\upsilon)\Gamma(\epsilon-\upsilon)}{\Gamma(1+\upsilon+\epsilon)}\sum_{\kappa=0}^{[N/\lambda]}\frac{(-N)_{\lambda\kappa}}{\kappa!}\frac{(\varepsilon+1)_\kappa}{(\varepsilon)_\kappa}\frac{(\varepsilon-\nu)_\kappa}{(1+\nu+\varepsilon)_\kappa}A_{N,\kappa}\left(\frac{\theta}{C}\right)^\kappa$$
$$\times\sum_{s=0}^\infty \frac{c^s(\delta)_{\mu s}}{\prod_{j=1}^m \Gamma\left(A_j\,s+B_j+\frac{d+1}{2}\right)}\frac{(1+\varepsilon+\kappa)_s(\varepsilon-\nu+\kappa)_s}{(\varepsilon+\kappa)_s(1+\nu+\varepsilon+\kappa)_s}\frac{1}{s!}\left(\frac{\theta}{C}\right)^s.$$

Now apply Hadamard product (7) and using the notation

$$\bar{A}_{N,\kappa}=A_{N,\kappa}\frac{(\varepsilon+1)_\kappa}{(\varepsilon)_\kappa}\frac{(\varepsilon-\nu)_\kappa}{(1+\nu+\varepsilon)_\kappa}$$

$$I_3 \equiv \frac{2\varepsilon}{C^\varepsilon}\left(\frac{C}{2}\right)^\nu\frac{\Gamma(2\upsilon)\Gamma(\epsilon-\upsilon)}{\Gamma(1+\upsilon+\epsilon)}\sum_{\kappa=0}^{[N/\lambda]}\frac{(-N)_{\lambda\kappa}}{\kappa!}\bar{A}_{N,\kappa}\left(\frac{\theta}{C}\right)^\kappa$$
$$\times {}_3F_2\left[\begin{matrix}\varepsilon+\kappa+1, \varepsilon-\nu+\kappa, 1; \\ \varepsilon+\kappa,\ \nu+\varepsilon+\kappa+1;\end{matrix}\ \frac{\theta}{C}\right] * J_{(B_j)_m,\mu,d}^{(A_j)_m,\delta,c}\left(\frac{\theta}{C}\right).$$

□

Theorem 3.4. *Let* $\xi>0$, $\nu,\varepsilon\in\mathbb{C}$ *be such that* $Re(\epsilon)>Re(\nu)>0$ *and the conditions given in* (1) *and* (2) *are satisfied, then for the product of Srivastava polynomials* $S_N^\lambda(\cdot)$ *and generalized multiindex Bessel function* $J(\cdot)$, *the following integral formula holds true*

$$\int_0^\infty \xi^{\nu-1}(\xi+C+\sqrt{\xi^2+2C\xi})^{-\varepsilon}S_N^\lambda(W)J_{(B_j)_m,\mu,d}^{(A_j)_m,\delta,c}(W)\,d\xi$$
$$=\frac{2\varepsilon}{C^\varepsilon}\left(\frac{C}{2}\right)^\nu\frac{\Gamma(2\upsilon)\Gamma(\epsilon-\upsilon)}{\Gamma(1+\upsilon+\epsilon)}\sum_{\kappa=0}^{[N/\lambda]}\frac{(-N)_{\lambda\kappa}}{\kappa!}\bar{A}_{N,\kappa}\left(\frac{\theta}{2}\right)^\kappa$$
$$\times {}_4F_3\left[\begin{matrix}\varepsilon+\kappa+1, \nu+\kappa, \nu+\kappa+\frac{1}{2}, 1; \\ \varepsilon+\kappa,\ \frac{\nu+\varepsilon+2\kappa+1}{2},\ \frac{\nu+\varepsilon+2\kappa+1}{2};\end{matrix}\ \frac{\theta}{2}\right] * J_{(B_j)_m,\mu,d}^{(A_j)_m,\delta,c}\left(\frac{\theta}{2}\right), \tag{14}$$

where $W=\frac{\theta\xi}{\xi+C+\sqrt{\xi^2+2C\xi}}$ *and* $\bar{A}_{N,\kappa}=A_{N,\kappa}\frac{(\varepsilon+1)_\kappa}{(\varepsilon)_\kappa}\frac{(\nu)_\kappa}{\left(\frac{\nu+\varepsilon+1}{2}\right)_\kappa}\frac{\left(\nu+\frac{1}{2}\right)_\kappa}{\left(\frac{\nu+\varepsilon+2}{2}\right)_\kappa}$.

Proof. First, we refer to the left-hand side of equation (14) as the sign I_4 then making the use of equation (1) and (2) in equation (14), we have

$$I_4 \equiv \int_0^\infty \frac{\xi^{\nu-1}}{\left(\xi+C+\sqrt{\xi^2+2C\xi}\right)^{\varepsilon}} \sum_{\kappa=0}^{[N/\lambda]} \frac{(-N)_{\lambda\kappa}}{\kappa!} A_{N,\kappa}\,(\theta\xi)^{\kappa}\left(\xi+C+\sqrt{\xi^2+2C\xi}\right)^{-\kappa} \times \sum_{s=0}^{\infty} \frac{c^s(\delta)_{\mu s}}{\prod_{j=1}^{m}\Gamma\left(A_j\,s+B_j+\frac{d+1}{2}\right)} \frac{(\theta\xi)^s}{s!}\left(\xi+C+\sqrt{\xi^2+2C\xi}\right)^{-s} d\xi.$$

After changing the order of integration and summation under the condition of the theorem

$$I_4 \equiv \sum_{\kappa=0}^{[N/\lambda]} \frac{(-N)_{\lambda\kappa}}{\kappa!} A_{N,\kappa}\,\theta^{\kappa} \sum_{s=0}^{\infty} \frac{c^s(\delta)_{\mu s}}{\prod_{j=1}^{m}\Gamma\left(A_j\,s+B_j+\frac{d+1}{2}\right)} \frac{\theta^s}{s!} \int_0^\infty \frac{\xi^{\nu+\kappa+s-1}d\xi}{\left(\xi+C+\sqrt{\xi^2+2C\xi}\right)^{\varepsilon+\kappa+s}}.$$

By using equation (6) and after some simplification, we get

$$I_4 \equiv \frac{2\varepsilon}{C^{\varepsilon}}\left(\frac{C}{2}\right)^{\nu} \frac{\Gamma(2\upsilon)\Gamma(\epsilon-\upsilon)}{\Gamma(1+\upsilon+\epsilon)} \sum_{\kappa=0}^{[N/\lambda]} \frac{(-N)_{\lambda\kappa}}{\kappa!} \frac{(\varepsilon+1)_{\kappa}}{(\varepsilon)_{\kappa}} \frac{(\nu)_{\kappa}}{\left(\frac{\nu+\varepsilon+1}{2}\right)_{\kappa}} \frac{\left(\nu+\frac{1}{2}\right)_{\kappa}}{\left(\frac{\nu+\varepsilon+2}{2}\right)_{\kappa}} A_{N,\kappa}\left(\frac{\theta}{2}\right)^{\kappa} \times \sum_{s=0}^{\infty} \frac{c^s(\delta)_{\mu s}}{\prod_{j=1}^{m}\Gamma\left(A_j\,s+B_j+\frac{d+1}{2}\right)} \frac{\theta^s}{s!} \frac{(\varepsilon+\kappa+1)_s\,(\nu+\kappa)_s\left(\nu+\kappa+\frac{1}{2}\right)_s}{(\varepsilon+\kappa)_s\left(\frac{\nu+\varepsilon+2\kappa+1}{2}\right)_s\left(\frac{\nu+\varepsilon+2\kappa+2}{2}\right)_s}\left(\frac{\theta}{2}\right)^s.$$

Now apply Hadamard product (7) and using the notation

$$\bar{A}_{N,\kappa} = A_{N,\kappa} \frac{(\varepsilon+1)_{\kappa}}{(\varepsilon)_{\kappa}} \frac{(\nu)_{\kappa}}{\left(\frac{\nu+\varepsilon+1}{2}\right)_{\kappa}} \frac{\left(\nu+\frac{1}{2}\right)_{\kappa}}{\left(\frac{\nu+\varepsilon+2}{2}\right)_{\kappa}},$$

$$I_4 \equiv \frac{2\varepsilon}{C^{\varepsilon}}\left(\frac{C}{2}\right)^{\nu} \frac{\Gamma(2\upsilon)\Gamma(\epsilon-\upsilon)}{\Gamma(1+\upsilon+\epsilon)} \sum_{\kappa=0}^{[N/\lambda]} \frac{(-N)_{\lambda\kappa}}{\kappa!} \bar{A}_{N,\kappa}\left(\frac{\theta}{2}\right)^{\kappa} \times {}_4F_3\left[\begin{matrix}\varepsilon+\kappa+1,\nu+\kappa,\nu+\kappa+\frac{1}{2},1; \\ \varepsilon+\kappa,\frac{\nu+\varepsilon+2\kappa+1}{2},\frac{\nu+\varepsilon+2\kappa+2}{2};\end{matrix}\ \frac{\theta}{2}\right] * J_{(B_j)_m,\mu,d}^{(A_j)_m,\delta,c}\left(\frac{\theta}{2}\right).$$

□

4. Special Cases I

In this section, we will derive the following known formulas with the help of the main results of the present paper:

Corollary 4.1. *If we take $c=-1$ and $d=1$ in Theorem 3.1 and also use ordinary product instead of Hadamard product, then we get the following integral formula*

$$\int_0^1 \xi^{\nu-1}(1-\xi)^{2\varepsilon-1}\left(1-\frac{\xi}{3}\right)^{2\nu-1}\left(1-\frac{\xi}{4}\right)^{\varepsilon-1} S_N^{\lambda}\left[\theta(1-\xi)^2\left(1-\frac{\xi}{4}\right)\right] J_{(B_j)_m,\mu}^{(A_j)_m,\delta}\left[\theta(1-\xi)^2\left(1-\frac{\xi}{4}\right)\right] d\xi$$
$$= \left(\frac{2}{3}\right)^{2\nu} \frac{\Gamma(\upsilon)}{\Gamma(\delta)} \sum_{\kappa=0}^{[N/\lambda]} \frac{(-N)_{\lambda\kappa}}{\kappa!} A_{N,\kappa}\theta^{\kappa}\ {}_2\psi_{m+1}\left[\begin{matrix}(\varepsilon+\kappa,1),(\delta,\mu); \\ (\nu+\varepsilon+\kappa,1),(B_1+1,A_1),\ldots,(B_m+1,A_m);\end{matrix}\ -\theta\right] \tag{15}$$

which result obtain by Suthar et al. ([11, eq. (2.8), p. 1410]).

Corollary 4.2. *If we take $c=-1$ and $d=1$ in Theorem 3.2 and also use ordinary product instead of Hadamard product, then we have*

$$\int_0^1 \xi^{\nu-1}(1-\xi)^{2\varepsilon-1}\left(1-\frac{\xi}{3}\right)^{2\nu-1}\left(1-\frac{\xi}{4}\right)^{\varepsilon-1} S_N^{\lambda}\left[\theta\xi\left(1-\frac{\xi}{3}\right)^2\right] J_{(B_j)_m,\mu}^{(A_j)_m,\delta}\left[\theta\xi\left(1-\frac{\xi}{3}\right)^2\right] d\xi$$
$$=\left(\frac{2}{3}\right)^{2\nu}\frac{\Gamma(\varepsilon)}{\Gamma(\delta)}\sum_{\kappa=0}^{[N/\lambda]}\frac{(-N)_{\lambda\kappa}}{\kappa!}A_{N,\kappa}\left(\frac{4\theta}{9}\right)^{\kappa}$$
$$\times\ {}_2\Psi_{m+1}\left[\begin{matrix}(\nu+\kappa,1),(\delta,\mu);\\ (\nu+\varepsilon+\kappa,1),(B_1+1,A_1),\ldots,(B_m+1,A_m);\end{matrix}\ -\frac{4\theta}{9}\right] \tag{16}$$

which result obtain by Suthar et al. ([11, eq. (2.10), p. 1411]).

Corollary 4.3. *If we take $c=-1$ and $d=1$ in Theorem 3.3 and also use ordinary product instead of Hadamard product, then we obtain*

$$\int_0^\infty \xi^{\nu-1}\left(\xi+C+\sqrt{\xi^2+2C\xi}\right)^{-\varepsilon} S_N^{\lambda}\left[\frac{\theta}{\xi+C+\sqrt{\xi^2+2C\xi}}\right] J_{(B_j)_m,\mu}^{(A_j)_m,\delta}\left[\frac{\theta}{\xi+C+\sqrt{\xi^2+2C\xi}}\right] d\xi$$
$$=\frac{2^{1-\nu}}{C^{\varepsilon-\nu}}\frac{\Gamma(2\upsilon)}{\Gamma(\delta)}\sum_{\kappa=0}^{[N/\lambda]}\frac{(-N)_{\lambda\kappa}}{\kappa!}A_{N,\kappa}\left(\frac{\theta}{C}\right)^{\kappa}$$
$$\times\ {}_3\Psi_{m+2}\left[\begin{matrix}(\varepsilon+\kappa+1,1),(\varepsilon-\nu+\kappa+1,1),(\delta,\mu);\\ (\nu+\varepsilon+\kappa+1,1),(\varepsilon+\kappa,1),(B_1+1,A_1),\ldots,(B_m+1,A_m);\end{matrix}\ -\frac{\theta}{C}\right] \tag{17}$$

which result obtain by Suthar et al. ([11, eq. (2.5), p. 1409]).

Corollary 4.4. *If we take $c=-1$ and $d=1$ in Theorem 3.4 and also use ordinary product instead of Hadamard product, then we attain*

$$\int_0^\infty \xi^{\nu-1}\left(\xi+C+\sqrt{\xi^2+2C\xi}\right)^{-\varepsilon} S_N^{\lambda}\left[\frac{\theta\xi}{\xi+C+\sqrt{\xi^2+2C\xi}}\right] J_{(B_j)_m,\mu,d}^{(A_j)_m,\delta,c}\left[\frac{\theta\xi}{\xi+C+\sqrt{\xi^2+2C\xi}}\right] d\xi$$
$$=\frac{2^{1-\nu}}{C^{\varepsilon-\nu}}\frac{\Gamma(2\upsilon)}{\Gamma(\delta)}\sum_{\kappa=0}^{[N/\lambda]}\frac{(-N)_{\lambda\kappa}}{\kappa!}A_{N,\kappa}\left(\frac{\theta}{2}\right)^{\kappa}$$
$$\times\ {}_3\Psi_{m+2}\left[\begin{matrix}(2\varepsilon+2\kappa,2),(\varepsilon+\kappa+1,1),(\delta,\mu);\\ (\nu+\varepsilon+2\kappa+1,1),(\varepsilon+\kappa,1),(B_1+1,A_1),\ldots,(B_m+1,A_m);\end{matrix}\ -\frac{\theta}{2}\right] \tag{18}$$

which result obtain by Suthar et al. ([11, eq. (2.7), p. 1410]).

Corollary 4.5. *If we take $N=0$ and replace $\nu\to\nu+\varepsilon$ in Theorem 3.1 and also use ordinary product instead of Hadamard product, then we get the following integral formula*

$$\int_0^1 \xi^{(\upsilon+\varepsilon)-1}(1-\xi)^{2\varepsilon-1}\left(1-\frac{\xi}{3}\right)^{2(\upsilon+\varepsilon)-1}\left(1-\frac{\xi}{4}\right)^{\varepsilon-1} J_{(B_j)_m,\mu,d}^{(A_j)_m,\delta,c}\left[\theta(1-\xi)^2\left(1-\frac{\xi}{4}\right)\right] d\xi$$
$$=\left(\frac{2}{3}\right)^{2(\upsilon+\varepsilon)}\frac{\Gamma(\upsilon+\varepsilon)}{\Gamma(\delta)}\ {}_2\Psi_{m+1}\left[\begin{matrix}(\varepsilon,1),(\delta,\mu);\\ (2\varepsilon+\ \nu,1),\left(B_j+\frac{d+1}{2},A_j\right)\ (j=1,\ldots,m);\end{matrix}\ c\theta\right] \tag{19}$$

which result obtain by Nisar et al. ([7, Theorem 7, p. 164]).

Corollary 4.6. *If we take $N = 0$ and replace $\varepsilon \to \nu + \varepsilon$ in Theorem 3.2 and also use ordinary product instead of Hadamard product, then we get the following integral formula*

$$\int_0^1 \xi^{\nu-1}(1-\xi)^{2(\upsilon+\varepsilon)-1}\left(1-\frac{\xi}{3}\right)^{2\nu-1}\left(1-\frac{\xi}{4}\right)^{(\upsilon+\varepsilon)-1} J_{(B_j)_m,\mu,d}^{(A_j)_m,\delta,c}\left[\theta\xi\left(1-\frac{\xi}{3}\right)^2\right] d\xi$$
$$= \left(\frac{2}{3}\right)^{2\nu} \frac{\Gamma(\upsilon+\varepsilon)}{\Gamma(\delta)}\ {}_2\Psi_{m+1}\left[\begin{matrix}(\nu,1),(\delta,\mu);\\ (2\nu+\varepsilon,1),\left(B_j+\frac{d+1}{2},A_j\right)\ (j=1,\dots,m);\end{matrix}\ \frac{4c\theta}{9}\right] \tag{20}$$

which result obtain by Nisar et al. ([7, Theorem 8, p. 165]).

Corollary 4.7. *If we take $N = 0$ in Theorem 3.3 and also use ordinary product instead of Hadamard product, then we obtain*

$$\int_0^\infty \xi^{\nu-1}\left(\xi+C+\sqrt{\xi^2+2C\xi}\right)^{-\varepsilon} J_{(B_j)_m,\mu,d}^{(A_j)_m,\delta,c}\left[\frac{\theta}{\xi+C+\sqrt{\xi^2+2C\xi}}\right] d\xi$$
$$= \frac{2^{1-\nu}}{C^{\varepsilon-\nu}} \frac{\Gamma(2\upsilon)}{\Gamma(\delta)}\ {}_3\Psi_{m+2}\left[\begin{matrix}(\varepsilon+1,1),(\varepsilon-\nu,1),(\delta,\mu);\\ (\nu+\varepsilon+1,1),(\varepsilon,1),\left(B_j+\frac{d+1}{2},A_j\right)\ (j=1,\dots,m);\end{matrix}\ \frac{c\theta}{C}\right] \tag{21}$$

which result obtain by Nisar et al. ([7, Theorem 5, p. 162]).

Corollary 4.8. *If we take $N = 0$ in Theorem 3.3 and also use ordinary product instead of Hadamard product, then we attain*

$$\int_0^\infty \xi^{\nu-1}\left(\xi+C+\sqrt{\xi^2+2C\xi}\right)^{-\varepsilon} J_{(B_j)_m,\mu,d}^{(A_j)_m,\delta,c}\left[\frac{\theta\xi}{\xi+C+\sqrt{\xi^2+2C\xi}}\right] d\xi$$
$$= \frac{2^{1-\nu}}{C^{\varepsilon-\nu}} \frac{\Gamma(\varepsilon-\nu)}{\Gamma(\delta)}\ {}_3\Psi_{m+2}\left[\begin{matrix}(2\nu,2),(\varepsilon+1,1),(\delta,\mu);\\ (\nu+\varepsilon+1,2),(\varepsilon,1),\left(B_j+\frac{d+1}{2},A_j\right)\ (j=1,\dots,m);\end{matrix}\ \frac{c\theta}{2}\right] \tag{22}$$

which result obtain by Nisar et al. ([7, Theorem 6, p. 163]).

5. Special Cases II

In this section, we derive some new integral formulas as special cases of our main results obtained in the previous section. We therefore obtain the following results for the generalized multiindex Bessel function involved with the classical Hermite and Laguarre polynomials.

For Hermite Polynomials. If we set $\lambda = 2$ and $A_{N,\kappa} = (-1)^\kappa$, then general class of polynomials $S_N^\lambda(x)$ reduce to Hermite polynomials such that, ([14, eq. (1.4), p. 158]),

$$S_N^2(x) \to x^{\frac{N}{2}} H_N\left(\frac{1}{2\sqrt{x}}\right), \tag{23}$$

where $H_N(\cdot)$ denotes the classical Hermite polynomial ([12, eq. (5.5.4), p. 106]) and defined by

$$H_N(x) = \sum_{\kappa=0}^{[N/2]} (-1)^\kappa \frac{N!}{\kappa!(N-2\kappa)!}(2x)^{N-2\kappa}. \tag{24}$$

Corollary 5.1. *Let the conditions of Theorem 3.1 be satisfied and $\lambda = 2$ and $A_{N,\kappa} = (-1)^\kappa$, then the following formula holds:*

$$\int_0^1 \xi^{\nu-1}(1-\xi)^{2\varepsilon+N-1}\left(1-\frac{\xi}{3}\right)^{2\nu-1}\left(1-\frac{\xi}{4}\right)^{\varepsilon+N/2-1} \theta^{\frac{N}{2}}\, H_N\left(\frac{1}{2\sqrt{X}}\right) J_{(B_j)_m,\mu,d}^{(A_j)_m,\delta,c}[X]\, d\xi$$

$$= \left(\frac{4}{9}\right)^{\nu} \frac{\Gamma(\upsilon)\Gamma(\epsilon)}{\Gamma(\upsilon+\epsilon)} \sum_{\kappa=0}^{[N/2]} \frac{(-N)_{2\kappa}}{\kappa!} \frac{(-1)^{\kappa}(\varepsilon)_{\kappa}\theta^{\kappa}}{(\nu+\varepsilon)_{\kappa}} \, {}_2F_1\left[\begin{matrix} \varepsilon+\kappa, 1; \\ \nu+\varepsilon+\kappa; \end{matrix} \; \theta\right] * J^{(A_j)_m,\delta,c}_{(B_j)_m,\mu,d}(\theta), \tag{25}$$

where $X = \theta(1-\xi)^2\left(1-\frac{\xi}{4}\right)$.

Corollary 5.2. *Let the conditions of Theorem* 3.2 *be satisfied and* $\lambda = 2$ *and* $A_{N,\kappa} = (-1)^{\kappa}$, *then the following formula holds:*

$$\int_0^1 \xi^{\nu+N/2-1}(1-\xi)^{2\varepsilon-1}\left(1-\frac{\xi}{3}\right)^{2\nu+N-1}\left(1-\frac{\xi}{4}\right)^{\varepsilon-1} \theta^{\frac{N}{2}} H_N\left(\frac{1}{2\sqrt{Y}}\right) J^{(A_j)_m,\delta,c}_{(B_j)_m,\mu,d}[Y]\, d\xi$$

$$= \left(\frac{4}{9}\right)^{\nu} \frac{\Gamma(\upsilon)\Gamma(\epsilon)}{\Gamma(\upsilon+\epsilon)} \sum_{\kappa=0}^{[N/2]} \frac{(-N)_{2\kappa}}{\kappa!} \frac{(-1)^{\kappa}(\nu)_{\kappa}}{(\nu+\varepsilon)_{\kappa}} \left(\frac{4\theta}{9}\right)^{\kappa} {}_2F_1\left[\begin{matrix} \nu+\kappa, 1; \\ \nu+\varepsilon+\kappa; \end{matrix} \; \frac{4\theta}{9}\right] * J^{(A_j)_m,\delta,c}_{(B_j)_m,\mu,d}\left(\frac{4\theta}{9}\right), \tag{26}$$

where $Y = \theta\xi\left(1-\frac{\xi}{3}\right)^2$.

Corollary 5.3. *Let the conditions of Theorem* 3.3 *be satisfied and* $\lambda = 2$ *and* $A_{N,\kappa} = (-1)^{\kappa}$, *then the following formula holds:*

$$\int_0^{\infty} \frac{\xi^{\nu-1}}{\left(\xi+C+\sqrt{\xi^2+2C\xi}\right)^{\varepsilon+N/2}} \theta^{\frac{N}{2}} H_N\left(\frac{1}{2\sqrt{Z}}\right) J^{(A_j)_m,\delta,c}_{(B_j)_m,\mu,d}[Z]\, d\xi$$

$$= \frac{2\varepsilon}{C^{\varepsilon}}\left(\frac{C}{2}\right)^{\nu} \frac{\Gamma(2\upsilon)\Gamma(\epsilon-\upsilon)}{\Gamma(1+\upsilon+\epsilon)} \sum_{\kappa=0}^{[N/2]} \frac{(-N)_{2\kappa}}{\kappa!} \frac{(-1)^{\kappa}(\varepsilon+1)_{\kappa}(\varepsilon-\nu)_{\kappa}}{(\varepsilon)_{\kappa}(1+\nu+\varepsilon)_{\kappa}} \left(\frac{\theta}{C}\right)^{\kappa}$$

$$\times {}_3F_2\left[\begin{matrix} \varepsilon+\kappa+1, \varepsilon-\nu+\kappa, 1; \\ \varepsilon+\kappa, \ \nu+\varepsilon+\kappa+1; \end{matrix} \; \frac{\theta}{C}\right] * J^{(A_j)_m,\delta,c}_{(B_j)_m,\mu,d}\left(\frac{\theta}{C}\right), \tag{27}$$

where $Z = \frac{\theta}{\xi+C+\sqrt{\xi^2+2C\xi}}$.

Corollary 5.4. *Let the conditions of Theorem* 3.4 *be satisfied and* $\lambda = 2$ *and* $A_{N,\kappa} = (-1)^{\kappa}$, *then the following formula holds:*

$$\int_0^{\infty} \frac{\xi^{\nu+N/2-1}}{\left(\xi+C+\sqrt{\xi^2+2C\xi}\right)^{\varepsilon+N/2}} \theta^{\frac{N}{2}} H_N\left(\frac{1}{2\sqrt{W}}\right) J^{(A_j)_m,\delta,c}_{(B_j)_m,\mu,d}[W]\, d\xi$$

$$= \frac{2\varepsilon}{C^{\varepsilon}}\left(\frac{C}{2}\right)^{\nu} \frac{\Gamma(2\upsilon)\Gamma(\epsilon-\upsilon)}{\Gamma(1+\upsilon+\epsilon)} \sum_{\kappa=0}^{[N/2]} \frac{(-N)_{2\kappa}}{\kappa!} \frac{(-1)^{\kappa}(\varepsilon+1)_{\kappa}(\nu)_{\kappa}\left(\nu+\frac{1}{2}\right)_{\kappa}}{(\varepsilon)_{\kappa}\left(\frac{\nu+\varepsilon+1}{2}\right)_{\kappa}\left(\frac{\nu+\varepsilon+2}{2}\right)_{\kappa}} \left(\frac{\theta}{2}\right)^{\kappa}$$

$$\times {}_4F_3\left[\begin{matrix} \varepsilon+\kappa+1, \nu+\kappa, \nu+\kappa+\frac{1}{2}, 1; \\ \varepsilon+\kappa, \frac{\nu+\varepsilon+2\kappa+1}{2}, \frac{\nu+\varepsilon+2\kappa+2}{2}; \end{matrix} \; \frac{\theta}{2}\right] * J^{(A_j)_m,\delta,c}_{(B_j)_m,\mu,d}\left(\frac{\theta}{2}\right), \tag{28}$$

where $W = \frac{\theta\xi}{\xi+C+\sqrt{\xi^2+2C\xi}}$.

Remark 5.1. For particular $d = -c = 1$ above four results reduce to known results given by Suthar *et al*. ([11, eq. (3.7)-eq. (3.10)]).

For Laguerre Polynomials. If we set $\lambda = 1$ and $A_{N,\kappa} = \frac{1}{(1)_{\kappa}}$ then general class of polynomials $S_N^{\lambda}(x)$ reduce to associated Laguerre polynomials such that, ([14, eq. (1.8), p. 159]),

$$S_N^1(x) = L_N^{(a)}(x) \xrightarrow{a=0} L_N(x), \tag{29}$$

where $L_N(x)$ denotes the classical Laguerre polynomial ([12, eq. (5.5.4), p. 106]) and defined by

$$L_N(x) = \sum_{\kappa=0}^{N} \binom{N}{\kappa} \frac{(-1)^\kappa}{\kappa!} x^\kappa. \tag{30}$$

Corollary 5.5. *Let the conditions of Theorem* 3.1 *be satisfied and* $\lambda = 1$ *and* $A_{N,\kappa} = \frac{1}{(1)_\kappa}$, *then the following formula holds:*

$$\int_0^1 \xi^{\nu-1}(1-\xi)^{2\varepsilon-1}\left(1-\frac{\xi}{3}\right)^{2\nu-1}\left(1-\frac{\xi}{4}\right)^{\varepsilon-1} L_N(x)\, J^{(A_j)_m,\delta,c}_{(B_j)_m,\mu,d}[X]\, d\xi$$
$$= \left(\frac{4}{9}\right)^\nu \frac{\Gamma(\upsilon)\Gamma(\epsilon)}{\Gamma(\upsilon+\epsilon)} \sum_{\kappa=0}^{N} \frac{(-N)_\kappa}{\kappa!} \frac{(\varepsilon)_\kappa \theta^\kappa}{(1)_\kappa(\nu+\varepsilon)_\kappa}\, {}_2F_1\left[\begin{matrix}\varepsilon+\kappa, 1; \\ \nu+\varepsilon+\kappa;\end{matrix}\ \theta\right] * J^{(A_j)_m,\delta,c}_{(B_j)_m,\mu,d}(\theta), \tag{31}$$

where $X = \theta(1-\xi)^2\left(1-\frac{\xi}{4}\right)$.

Corollary 5.6. *Let the conditions of Theorem* 3.2 *be satisfied and* $\lambda = 1$ *and* $A_{N,\kappa} = \frac{1}{(1)_\kappa}$, *then the following formula holds:*

$$\int_0^1 \xi^{\nu-1}(1-\xi)^{2\varepsilon-1}\left(1-\frac{\xi}{3}\right)^{2\nu-1}\left(1-\frac{\xi}{4}\right)^{\varepsilon-1} L_N(Y)\ J^{(A_j)_m,\delta,c}_{(B_j)_m,\mu,d}[Y]\, d\xi$$
$$= \left(\frac{4}{9}\right)^\nu \frac{\Gamma(\upsilon)\Gamma(\epsilon)}{\Gamma(\upsilon+\epsilon)} \sum_{\kappa=0}^{N} \frac{(-N)_\kappa}{\kappa!} \frac{(\nu)_\kappa}{(1)_\kappa(\nu+\varepsilon)_\kappa} \left(\frac{4\theta}{9}\right)^\kappa {}_2F_1\left[\begin{matrix}\nu+\kappa, 1; \\ \nu+\varepsilon+\kappa;\end{matrix}\ \frac{4\theta}{9}\right] * J^{(A_j)_m,\delta,c}_{(B_j)_m,\mu,d}\left(\frac{4\theta}{9}\right), \tag{32}$$

where $Y = \theta\xi\left(1-\frac{\xi}{3}\right)^2$.

Corollary 5.7. *Let the conditions of Theorem* 3.3 *be satisfied and* $\lambda = 1$ *and* $A_{N,\kappa} = \frac{1}{(1)_\kappa}$, *then the following formula holds:*

$$\int_0^\infty \frac{\xi^{\nu-1}}{\left(\xi+C+\sqrt{\xi^2+2C\xi}\right)^\varepsilon} L_N(Z)\, J^{(A_j)_m,\delta,c}_{(B_j)_m,\mu,d}[Z]\, d\xi$$
$$= \frac{2\varepsilon}{C^\varepsilon}\left(\frac{C}{2}\right)^\nu \frac{\Gamma(2\upsilon)\Gamma(\epsilon-\upsilon)}{\Gamma(1+\upsilon+\epsilon)} \sum_{\kappa=0}^{N} \frac{(-N)_\kappa}{\kappa!} \frac{(\varepsilon+1)_\kappa(\varepsilon-\nu)_\kappa}{(1)_\kappa(\varepsilon)_\kappa(1+\nu+\varepsilon)_\kappa} \left(\frac{\theta}{C}\right)^\kappa$$
$$\times {}_3F_2\left[\begin{matrix}\varepsilon+\kappa+1, \varepsilon-\nu+\kappa, 1; \\ \varepsilon+\kappa,\ \nu+\varepsilon+\kappa+1;\end{matrix}\ \frac{\theta}{C}\right] * J^{(A_j)_m,\delta,c}_{(B_j)_m,\mu,d}\left(\frac{\theta}{C}\right), \tag{33}$$

where $Z = \frac{\theta}{\xi+C+\sqrt{\xi^2+2C\xi}}$.

Corollary 5.8. *Let the conditions of Theorem* 3.4 *be satisfied and* $\lambda = 1$ *and* $A_{N,\kappa} = \frac{1}{(1)_\kappa}$, *then the following formula holds:*

$$\int_0^\infty \frac{\xi^{\nu-1}}{\left(\xi+C+\sqrt{\xi^2+2C\xi}\right)^\varepsilon} L_N(W)\, J^{(A_j)_m,\delta,c}_{(B_j)_m,\mu,d}[W]\, d\xi$$
$$= \frac{2\varepsilon}{C^\varepsilon}\left(\frac{C}{2}\right)^\nu \frac{\Gamma(2\upsilon)\Gamma(\epsilon-\upsilon)}{\Gamma(1+\upsilon+\epsilon)} \sum_{\kappa=0}^{N} \frac{(-N)_\kappa}{\kappa!} \frac{(\varepsilon+1)_\kappa(\nu)_\kappa\left(\nu+\frac{1}{2}\right)_\kappa}{(1)_\kappa(\varepsilon)_\kappa\left(\frac{\nu+\varepsilon+1}{2}\right)_\kappa\left(\frac{\nu+\varepsilon+2}{2}\right)_\kappa} \left(\frac{\theta}{2}\right)^\kappa$$
$$\times {}_4F_3\left[\begin{matrix}\varepsilon+\kappa+1, \nu+\kappa, \nu+\kappa+\frac{1}{2}, 1; \\ \varepsilon+\kappa, \frac{\nu+\varepsilon+2\kappa+1}{2}, \frac{\nu+\varepsilon+2\kappa+2}{2};\end{matrix}\ \frac{\theta}{2}\right] * J^{(A_j)_m,\delta,c}_{(B_j)_m,\mu,d}\left(\frac{\theta}{2}\right), \tag{34}$$

where $W = \frac{\theta\xi}{\xi+C+\sqrt{\xi^2+2C\xi}}$.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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