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Research Article

The Criteria for Oscillations of Second Order Linear Functional and Difference Equations With Delay

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Abstract. We establish some conditions for all the solutions of the oscillatory theory of second-order linear differential equations with delay, $u''(t)+r(t)u(\tau(t))=0$, for $t\geq t_0$. Here $r\in([t_0,\infty),R^+)$, $\tau\in([t_0,\infty),R)$, where r and τ are continuous functions of non-negative real number and $\tau(t)$ is an increasing function and it is also delay operator then $\tau(t)\leq t$, for $t_0\leq t$, and $\lim_{t\to\infty}\tau(t)=\infty$. The second order functional equations of oscillatory theory is $u(h(t))=A(t)u(t)+B(t)u(h(h(t)))$, for $t\geq t_0$, where A and B are continuous funtions and $A,B\in([t_0,\infty),R)$, $h\in([t_0,\infty),R^+)$ are continuous real valued functions and u is also a real valued function with unknown variables. The second-order linear difference equation of oscillatory theory is $\delta^2u(m)+r(m)u(\tau(m))=0$, where $\delta^2=\delta(\delta),\delta u(m)=u(m+1)-u(m)$, $\lim_{n\to\infty}\tau(m)=+\infty$, $\tau:Z^+\to Z^+$, $r:Z^+\to R^+$, $\tau(m)\leq m-1$.

Keywords. Oscillation criteria, Non-oscillatory, Second order differential equation, Delay differential equations, Functional equation

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1. Introduction

A significant area of applied theory in differential equations is the study of oscillatory processes in the natural, technological, and social sciences. Oscillations are everywhere in the world around us. Proving the existence or non-existence of oscillatory periodic, almost-periodic, etc. is one of the fundamental issues with the traditional Theory of Oscillations. Few deal with arbitrary even-order equations, self-adjoint fourth-order equations, or systems of first or second order equations. 'The oscillation deals with the quantification of the amount that a sequence or function tends to move between extremes. There are several related notions: oscillation of a sequence of real numbers, oscillation of a real-valued function at a point, and oscillation of a function on an interval (or open set).

Definition 1.1 ([12])**.** A nontrivial function $y(x) \in C([x_0,\infty),R)$ which is not eventually zero for x large enough is said to be oscillatory if it has arbitrarily significant zeros.
Otherwise, $y(x)$ is called non-oscillatory, i.e., $y(x)$ is non-oscillatory if there exists $x_1 \geq x_0$ such that $y(x) \neq 0$, for $x \geq x_1$. In other words, a non-oscillatory solution must be eventually positive or eventually negative.

Definition 1.2 ([5])**.** Let $t_0 \in R^+$ and $a_0 = \inf\{\tau(t) : t \geq t_0\}$. A continuous function $x : [a_0,+\infty) \to R$ is said to be a proper solution of the equation (1) on the interval $[t_0,+\infty)$, if it is absolutely continuous together with its first derivative on every compact interval in $[t_0,+\infty)$, satisfies the equality (1) almost everywhere in $[t_0,+\infty)$, and $\sup\{|x(s)| : s \geq t\} > 0$, for $t \geq t_0$.

Definition 1.3 ([19])**.** A proper solution x of the equation (1) is said to be oscillatory, if it has a sequence of zeros tending to infinity, otherwise it is non-oscillatory.

We establish some conditions for all the solutions of the oscillatory theory of second-order ordinary differential equations with delay,

$$u''(t) + r(t)u(\tau(t)) = 0, \quad \text{for } t \geq t_0, \tag{I}$$

Here $r,\tau \in ([t_0,\infty),R^+)$, where r and τ are continuous functions of non-negative real numbers and $\tau(t)$ is delay operator as well as an increasing function and the delay operator $\tau(t) \leq t$, for $t_0 \leq t$, and $\lim_{t\to\infty} \tau(t) = \infty$, we studied many references, see, [1,2,4,5,7–10,12–23,33].

By solution (1) we understand that the delay function $\tau(t)$ is continuously differentiable and is defined on $[\tau(t_1),\infty)$, for $t_1 \geq t_0$. Therefore, equation (I) holds for $t \geq t_1$, and such types of solutions are said to be oscillatory if they contain arbitrarily large zeros, and otherwise, it is said to be non-oscillatory.

The second-order functional equations of oscillatory theory is

$$u(h(t)) = A(t)u(t) + B(t)u(h(h(t))), \quad \text{for } t \geq t_0. \tag{II}$$

Here A and B are continuous funtions and $A,B \in ([t_0,\infty),R)$; $h \in ([t_0,\infty),R)$ is continuous real valued functions and u is also a real function with unknown variables.

The second-order linear difference equation of oscillatory theory is

$$\delta^2 u(m) + r(m)u(\tau(m)) = 0, \tag{III}$$

where $\delta^2 = \delta(\delta)$, $\delta u(m) = u(m+1) - u(m)$, $\lim_{n\to\infty} \tau(m) = +\infty$, $\tau : Z^+ \to Z^+$, $r : Z^+ \to R^+$, $\tau(m) \leq m-1$.

By solution (I), we understand that the sequences of $u(m)$ are defined for $\min\{\tau(m) : m \geq 0\} \leq m$ and satisfy (III) for every $m \geq 0$. A solution u of (III) is known as oscillatory if the term u of the solutions is neither sometimes negative nor sometimes positive. If not, the solution is known as non-oscillatory.

From (II) we determine that $h(t) \neq t$, for every $t \geq t_0$, $\lim_{t\to\infty} h(t) = \infty$, $t \to \infty$ and h^2 denotes the second order iterative function h, that is,

$$h^0(t) = t,\ h^2(t) = hh(t), \quad \text{for all } t \geq t_0 \tag{1}$$

It has been developed over the last ten years. We studied some previous investigations, see [3, 6, 11, 24–32, 34]. Here, we propose that the oscillations of all solutions of the equations (I), (II), and (III), that is

$$G^* = \limsup_{t\to\infty} \int_{\tau(s)}^{t} r(s)\tau(s)ds \le 1, \quad G_* = \liminf_{t\to\infty} \int_{\tau(t)}^{t} r(s)\tau(s)ds \le 1, \tag{2}$$

for equation (I),

$$0 < \limsup_{t\to\infty}\{B(t)A(h(t))\} < 1, \quad 0 < \liminf_{t\to\infty}\{B(t)A(h(t))\} \le \frac{1}{4}, \tag{3}$$

for equation (II).

2. The Oscillation Criteria for Equation (I)

In this section, we study the delay differential equation (I). In this case, the second-order ordinary differential equations hold for $\tau(t) \equiv t$. Many investigations are made by several people, i.e., Kneser [15]. Essential contributions are made by Hartman [11]. The following well-known oscillation criteria were specifically obtained by Hille [12] in 1948.

Let us consider

$$\limsup_{t\to\infty} t \int_{\tau(t)}^{\infty} r(s)ds > 1 \tag{4}$$

or

$$\liminf_{t\to\infty} t \int_{\tau(t)}^{\infty} r(s)ds > \frac{1}{4} \tag{5}$$

Assumed criteria are satisfied, and the integral is divergent. Then equation (I) is oscillatory with $\tau(t) \equiv t$. Earlier conditions of oscillatory results can be found in many investigations for the delay differential equation (I), that is

$$\int_{t}^{\infty} r(t)dt = \infty.$$

The direction of generalisation of Hille's criteria shows that, if $\tau(t) \ge \lambda t$, for $t \ge 0$ with $0 < \lambda \le 1$, then the condition is

$$\liminf_{t\to\infty} r(s)ds > \frac{1}{4\lambda} \tag{6}$$

is enough for the oscillation of (I). The generalised condition of the equation is

$$\liminf_{t\to\infty} t \int_{t}^{+\infty} \frac{\tau(s)}{s} r(s)ds > 1. \tag{7}$$

There are no additional restrictions required on τ. We obtained general criteria to improve the last conditions, see [33].

In this paper, by using previous knowledge, we provided different types of oscillation criteria and solved that equation (I) is oscillatory if

$$P = \limsup_{t\to\infty} \int_{\tau(t)}^{t} r(s)\tau(s)ds > 1 \tag{I$_1$}$$

or

$$Q = \liminf_{t\to\infty} \int_{\tau(t)}^{t} r(s)\tau(s)ds > \frac{1}{e}. \tag{I$_2$}$$

Here (I_1) and (I_2) are similar to oscillation conditions, see [34],

$$\alpha = \limsup_{t\to\infty} \int_{\tau(t)}^{t} r(s)ds > 1 \tag{J_1}$$

or

$$\beta = \liminf_{t\to\infty} \int_{\tau(t)}^{t} r(s)ds > \frac{1}{e}. \tag{J_2}$$

Thus, the first order linear delay differential equation is

$$u'(t) + r(t)u(\tau(t)) = 0. \tag{8}$$

The conditions (4), (5) are contract oscillation for ordinary differential equations. The conditions for (I_1), (I_2) works only in delay equations. We can view that the above conditions (4),(5) and (I_1), (I_2) will differ. In contrast, the first-order linear difference equations, here, the oscillatory nature is only caused by the delay equations.

Without any delay, equation (I) is oscillating, i.e, $\tau(t) \equiv t$. The existence of the second-order nature of the equations and delay equations of the criteria are (4), (5), and (I_1), (I_2).

We derived some theorems by [22].

Theorem 2.1 ([22]). *Let us consider*

$$\limsup_{t\to\infty} \left\{ \int_{\tau(t)}^{t} r(s)\tau(s)ds + \tau(t)\int_{t}^{+\infty} r(s)ds + \frac{1}{\tau(t)}\int_{0}^{\tau(t)} sr(s)\tau(s)ds \right\} > 1 \tag{9}$$

then equation (I) *is oscillatory.*

Theorem 2.2 ([22]). *Let us consider*

$$\limsup_{t\to\infty} \tau(t)\int_{t}^{+\infty} r(s)ds + \liminf_{t\to\infty} \frac{1}{t}\int_{0}^{t} r(s)s\tau(s)ds > 1 \tag{10}$$

or

$$\liminf_{t\to\infty} \tau(t)\int_{t}^{+\infty} r(s)ds + \limsup_{t\to\infty} \frac{1}{t}\int_{0}^{t} r(s)s\tau(s)ds > 1 \tag{11}$$

then equation (I) *is oscillatory.*

Theorem 2.3 ([22]). *Let us consider*

$$\limsup_{t\to\infty} \tau(t)\int_{t}^{+\infty} r(s)ds > 1 \tag{12}$$

or

$$\limsup_{t\to\infty} \frac{1}{t}\int_{0}^{t} r(s)ds > 1 \tag{13}$$

then equation (I) *is oscillatory.*

Theorem 2.4 ([22]). *Let us consider*

$$\liminf_{t\to\infty} \tau(t)\int_{t}^{+\infty} r(s)ds > 1 \tag{14}$$

or

$$\liminf_{t\to\infty} \frac{1}{t}\int_{0}^{t} r(s)ds > 1 \tag{15}$$

then equation (I) *is oscillatory.*

Theorem 2.5 ([22]). *Let us consider* $\tau(t) \equiv t$ *and*

$$\limsup_{t\to\infty}\left\{t\int_t^{+\infty} r(s)ds + \frac{1}{t}\int_0^t s^2 r(s)ds\right\} > 1. \tag{16}$$

Then equation (I) *is oscillatory. The assumed condition is*

$$\int^{+\infty} \tau(s)r(s)ds = +\infty \tag{17}$$

is satisfied.

The equation (17) is necessary for equation (I) to be oscillatory. The oscillatory results which exist for equation (I) by reducing it to a first-order equation. The applicable conditions can only delay equations and are not appropriate for ordinary equations.

Theorem 2.6 ([20]). *Let equation* (17) *be satisfied, and let us assume that,*

$$u'(t) + r(t)\left(\int_t^{\tau(t)} r(\mu)\mu\tau(\mu)d\mu\right)u(\tau(t)) + r(t)\tau(t)u(\tau(t)) \le 0 \tag{18}$$

is the first-order differential equation with an inequality, and after a long delay, we find negative solutions for equation (I)*, which is oscillatory.*

In the inequality, the first-order linear differential equation with oscillatory delay is

$$x'(t) + f(t)x(\Delta(t)) \le 0. \tag{19}$$

Concluding the comparison theorem, which derives the oscillation (19).

Thus, the equation is

$$y'(t) + g(t)y(\sigma(t)) = 0. \tag{20}$$

Let $f, g, \Delta, \sigma : R \to R$ are always continuous function and integrable, $t \ge (\Delta(t), \sigma(t))$, for every $t \in R$, $f(t) \ge g(t)$, $\sigma(t) \ge \Delta(t)$, for every $t \in R$, and $\Delta(t) \to \infty$ and $\sigma(t) \to \infty$ as $t \to \infty$.

From equations (19) and (20), we observe that the two equations are oscillatory and the inequality (19) has eventually negative solutions.

By using the condition (18) and both criteria together, that is, (J_1) and (J_2), we obtain the following.

Theorem 2.7 ([20]). *Let us consider*

$$Z = \limsup_{t\to\infty}\left(\int_{\tau(t)}^t \tau(s)r(s)ds + \int_{\tau(t)}^t\left(\int_0^{\tau(s)} \mu\tau(\mu)r(\mu)d\mu\right)r(s)ds\right) > 1 \tag{I_3}$$

or

$$z = \liminf_{t\to\infty}\left(\int_{\tau(t)}^t \tau(s)r(s)ds + \int_{\tau(t)}^t\left(\int_0^{\tau(s)} \mu\tau(\mu)r(\mu)d\mu\right)r(s)ds\right) > \frac{1}{e}. \tag{I_4}$$

Equation (I) *is oscillatory.*

The condition (18) is shows that, equation (17) is satisfied, or else $z = Z = 0$. Consider $\tau(t) \to \infty$, the relation (I_3) and (I_4) implies that the similar relations with no change for all $T \ge 0$.

Example 2.1. Let $(\rho, 1/e]$, $1/e - < \xi < 1$, $\tau(t) = \lambda t$, and $r(t) = \frac{\xi(1-\xi)}{\lambda^\xi t^2}$, where $\lambda = e^{1/(\xi-1)}$. Then (I_4) is satisfied and otherwise (I) has a nonoscilltory solution, namely $x(t) = t^\xi$. Therefore, the constant

$c = -\frac{-\xi(\xi-1)}{\lambda^{\xi}}$, we noted that in ($I_4$) is constant with limit sign and equals to $\lambda c|\ln\lambda|(1+\lambda c) = (\xi/e)$, $(1+(\xi(1-\xi)/e) > \xi/e > 1/e - \rho$.

3. The Oscillatory Criteria for Equation (II)

Theoretically, in this criteria, the functional equation (II), the solutions of this equation oscillates, if

$$\alpha = \limsup_{t\to\infty}\{A(h(t))B(t)\} > 1 \tag{I_1'}$$

or

$$\beta = \liminf_{t\to\infty}\{A(h(t))B(t)\} > \frac{1}{4} \tag{I_2'}$$

and, we also studied that the improved condition (I_1') is

$$\limsup_{t\to\infty}\left\{A(h(t))B(t) + \sum_{i=0}^{n}\prod_{i=0}^{j} B(h^{i+1}(t))A(h^{i+2}(t))\right\} > 1 \tag{21}$$

where $n \ge 0$ is an integer.

Remark 3.1. We notice that the right side of condition (I_2') remains constant and equation (I_2') cannot be replaced by the weaker condition, i.e.,

$$\beta = \liminf_{t\to\infty}\{A(h(t))B(t)\} > \frac{1-\eta}{4}, \quad \text{for all } \eta\in(0,1]. \tag{22}$$

The condition (I_1') extended to a higher order linear functional equation with coefficients of variables and constant delay. We cannot replace 1/4 with a small value in (I_2)' and the lower bound under the same condition is concerned, see [28]. Thus

$$A(h(t))B(t) \le \frac{1}{4}, \quad \text{for } t. \tag{I_3}$$

The above equation implies that (II) has a non-oscillatory solution.
Here, the limit does not exist for equation (I_3), i.e., $\lim_{t\to\infty}\{A(h(t))B(t)\}$, and it is clear that there is a gap between the two conditions (I_1') and (I_2'). Here, the interesting problem is to fill the gap between the conditions,
If $0 \le \beta \le 1/4$, from the condition (21), we obtain that every solution of (II) is oscillating,

$$\alpha > \frac{2_{\beta-1}}{\beta-1}. \tag{23}$$

Example 3.1. Assume that the functional equation is

$$2u(h(s)) = (1+W)[s]^n u(s) + (1-W)[s]^{-2n} u(h(h(s))),$$

where $n\in R$, $h(s) = [s]^2$, $t\in R^+$ and $W\in(0,1)$. The condition (22) is satisfied when $\eta\in(0,1]$ for all $W > \sqrt{n}$, $W\in(0,\sqrt{n})$. Then,

$$\begin{aligned}\liminf_{t\to\infty}\{A(h(t))B(t)\} &= \liminf_{t\to\infty}\frac{1-W}{2}[s]^{-2n} + \frac{1+W}{2}[s]^{2n} = \liminf_{t\to\infty}\frac{(1-W)(1+W)}{4}\\ &= \frac{1-W^2}{4} > \frac{1-\eta}{4}.\end{aligned}$$

Therefore, the equation has a non-oscillatory solution for $[s]^n$.

Theorem 3.1. *Let us consider* $0 \le \beta \le 1/4$ ([29]),

$$\limsup_{t\to\infty}\left\{\bar{\beta}A(h(t))B(t)+\sum_{i=0}^{n}\frac{\beta}{i}\prod_{i=0}^{i}(h^{i+1}(t))A(h^{i+2}(t))\right\}>1, \tag{24}$$

where $n \ge 0$ *some integer and* $\bar{\beta}=\frac{1}{(1+\sqrt{1-4\alpha})/2}$. *Then, all solutions of* (II) *is oscillate.*

Corollary 3.1. *Let us consider* $0 \le \beta \le 1/4$ ([29]),

$$\left(1+\frac{1+\sqrt{1-4\beta}}{2}\right)^2<\alpha \tag{25}$$

solutions of (II) *is oscillates.*

Remark 3.2. We notice that as $\beta \to 0$ and condition (24) reduces to condition (21) and conditions (25) and (23) reduces to condition (I_1'). It is clear that $0<\beta\le 1/4$ because

$$1>\frac{2\beta-1}{\beta-1}\left(1+\frac{1+\sqrt{1-4\beta}}{2}\right)^2. \tag{26}$$

It's significant to note that when the $\beta \to 1/4$ condition (25) becomes

$$\alpha>1/4. \tag{27}$$

Equation (27) is not modified, because a small value can not change the lower bound of 1/4.

Example 3.2. Let us assume, the functional equation

$$y(h(t))=y(t)+\left(\frac{1}{4}+q\cos^2\right)y(h(t)-2\sin^2(h(t))), \tag{28}$$

where $h(t)=(t-2\sin^2 t)$, $A(t)=1$, $B(t)=\frac{1}{4}+q\cos^2 t$, q is a constant and which is greater than 0. Now,

$$\left.\begin{aligned}\beta&=\liminf_{t\to\infty}\left(\frac{1}{4}+q\cos^2 t\right)=\frac{1}{4},\\ \alpha&=\limsup_{t\to\infty}\left(\frac{1}{4}+q\cos^2 t\right)=\frac{1}{4}+q=\frac{1}{4}+\frac{3}{4}.\end{aligned}\right\} \tag{29}$$

Therefore, all the solutions are oscillating for Corollary 3.1 and the criteria (I_1') are fulfilled for $q>3/4$ (which is a smaller number) only.

4. The Oscillatory Criteria for Equation (III)

In this section, we studied the second order linear differential equation (III), here $\delta u(m)=u(m+1)-u(m)$, $\delta^2=\delta\circ\delta$, $\tau:N\to N$, $r:N\to R^+$, $\tau(m)\le m-1$, and $\lim\limits_{n\to\infty}\tau(m)=+\infty$.

All the solutions of (III) are oscillatory (see [31]), if

$$\limsup_{m\to\infty}\left\{\sum_{j=\tau(m)}^{m}r(t)[\tau(j)-2]-[2-\tau(m)]\sum_{j=m+1}^{\infty}r(j)\right\}>1. \tag{I_1''}$$

The second order delay difference equation with the constant variable

$$\delta^2u(m)+r(m)u(m-z)=0,\quad z\ge 1 \tag{$\mathrm{III_c}$}$$

All solutions are oscillatory. Now

$$\liminf_{m\to\infty}\sum_{j=m-z}^{m-1} r(j)(j-z) > \left[2\left(\frac{1}{\frac{z+1}{1}}\right)^{z+1}\right], \tag{30}$$

and show that the following conditions are

$$1 < \limsup_{m\to\infty}\left\{\sum_{j=m-z}^{m} r(j)(1+z-(m-j)) + \left[\left[\sum_{j=m-1}^{m-k-1} r(j)(1-z)^2\right] - (2+z-m)\right]\sum_{j=m+1}^{\infty} r(j)\right\}, \quad \text{for } m_0 < m_1 \tag{31}$$

or

$$\liminf_{m\to\infty}\sum_{j=m-z}^{m} r(j)(j-z-1) > \left(\frac{1}{\frac{z+1}{1}}\right)^{z+1}. \tag{I_2''}$$

We notice that the condition (30) is comparable to the discrete form of the condition (I_2).

The behaviour of the solutions of equation (III) is oscillatory, and we determine that equation (III) is a delay equation with a constant variable.

Theorem 4.1. *Assume that*

$$\left.\begin{aligned} &\inf\left\{\frac{1}{1-\gamma}\liminf_{m\to\infty}\frac{m}{\gamma}\sum_{j=1}^{m} jr(j)\tau^{\gamma}(j)/\gamma\in(0,1)\right\} > 1,\\ &\liminf_{m\to\infty}\frac{1}{m}\sum_{j=1}^{m} jr(j)\tau^{\gamma}(j) > 0,\end{aligned}\right\} \tag{32}$$

every solution of (III) *is oscillation.*

Theorem 4.2. *Assume that* $\gamma > 0$*, then*

$$\liminf_{m\to\infty}\frac{1}{m}\sum_{j=1}^{m} j^2 r(j) > \max\left\{\frac{1}{\gamma}\gamma(1-\gamma)/\gamma\in[0,1]\right\}. \tag{33}$$

Then all the solutions of the equation

$$\delta^2 u(m) + r(m)u(\lambda m) = 0, \quad m \ge \max\left\{1, \frac{1}{\gamma}\right\},\ m\in N \tag{34}$$

is oscillate.

Theorem 4.3. *Let* m_0 *be an integer and*

$$\liminf_{m\to\infty}\frac{1}{m}\sum_{j=1}^{m} j^2 r(j) > \frac{1}{4}. \tag{35}$$

all the solutions of the equation

$$\delta^2 u(m) + r(m)u(m-m_0) = 0, \quad m \ge \max\{1, m_0+1\},\ m\in N \tag{36}$$

is oscillate.

Theorem 4.4. *Let*

$$\liminf_{m\to\infty}\frac{\tau(m)}{m} = \gamma\in[0,\infty), \tag{37}$$

$$\liminf_{m\to\infty} m \sum_{j=m}^{\infty} r(j) > \max\left\{\frac{\lambda}{\gamma}\gamma(1-\gamma) : \gamma \in [0,1]\right\}, \tag{38}$$

all solutions of (III) *is oscillate.*

By Hille's theorem, we derived the oscillatory theory of differential equations, with $\lambda = 1$.

Theorem 4.5. *Let k_0 be an integer, then*

$$\liminf_{m\to\infty} \frac{\tau(m)}{m} = \gamma \in [,\infty), \tag{39}$$

$$\delta^2 u(k) + r(k)u(k-k_0) = 0, \quad k \ge k_0. \tag{40}$$

It also oscillates.

Remark 4.1 ([21])**.** The constant 1/4 in (39) is maximum, as per ordinary differential equations only, in that a non-accurate value can't substitute the accurate inequality. Additionally, the same is accurate of condition (38) as well. To find this, use the symbol c to represent the right side of equation (38), i.e.,

$$\liminf_{m\to\infty} m \sum_{j=m}^{m} r(j) \ge c. \tag{41}$$

Therefore, Theorem 4.4 states that the limit cannot be higher than c. As per result, it is equal to c, and equation (38) is incorrect.

5. Conclusion

In the present study, we have considered the theory of oscillations of second-order delay differential equations. We applied the second-order functional equation and difference equation to solve many oscillatory solutions and verified that the solutions are oscillatory or non-oscillatory. In this paper, we follow the comparison theorem for deriving the oscillations, and we also follow many contributions of Wintner [30] and Hille [12] to solve the oscillations with delay. The advantages of the criteria are that we achieved a significant improvement to fill the gap between the conditions and all conditions are in the literature.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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