Journal of Atomic, Molecular, Condensate & Nano Physics
Vol. 4, Nos. 1 – 3, pp. 49–67, 2017
ISSN 2349-2716 (online); 2349-6088 (print)
Published by RGN Publications

http://www.rgnpublications.com

Photon Neutrino Interaction in a Magnetized Medium

Research Article

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Abstract. Loop induced interaction in a magnetized matter can make charge neutral fermions interact with photons. The matter or electromagnetic field induces an effective vertex and modifies the dispersion relation. In standard model of electroweak interaction, it has already been noted that in a medium neutrinos acquire an effective charge (from the vector type vertex of weak interaction). On the other hand in magnetized plasma, the axial vector part also start contributing to the effective charge of a neutrino. This contribution corresponding to the axial vector part in the interaction Lagrangian is denoted as the axial polarisation tensor. In an earlier paper the explicit form of the axial polarisation tensor to all odd orders in external magnetic field **B** was reported. In this note we complete that investigation by computing the same, to all even orders in external magnetic field. We further show its gauge invarience properties. Lastly the zero external momentum limit of this axial polarisation tensor is reported.

Keywords. Finite temperature field theory, Weak interaction, Neutrino

PACS. 05.30.-d, 3.35.+d, 11.10.-z, 14.80.MZ, 11.10-z

Received: July 17, 2017 **Accepted:** November 22, 2017

1. Introduction

Neutrino mediated processes are importance in cosmology and astrophysics. It is worth mentioning at this stage that various interesting possibilities have been looked into in the context of cosmology e.g., large scale structure formation in the universe, to name one of the few. In this note we would rather consider the astrophysical part of it.

Because of the effective neutrino photon interaction in a medium, it is possible that the neutrinos might dump a fraction of their energy inside the star during stellar evolution. For instance in a type II supernovae collapse neutrinos produced deep inside the proto-neutron star surge out carrying an effective energy $\sim 10^{51}$ erg/s. It is conjectured that the neutrinos deposit some fraction of its energy during the explosion through different kinds of neutrino electromagnetic interactions, to name a few. It is important to note here, that these processes are of order G_F^2. Our objective here is to take into account the presence of a strong magnetic field and estimate the corrections coming there off.

It is usually conjectured, taking into account the conservation of surface magnetic field of a proto-neutron star, that during a supernova collapse the magnetic field strength can reach up to 10^{13} Gauss or more. This conjecture makes it worthwhile to investigate the role of magnetic field in effective neutrino photon vertex.

Neutrinos do not couple with photons in the tree level in the standard model of particle physics, and this coupling can only take place at a loop level, mediated by the fermions and gauge bosons. This coupling can give birth to off-shell photons only, since for on-shell particles, the processes like $\nu \to \nu\gamma^*$ and $\gamma^* \to \nu\bar{\nu}$ are restrained kinematically. Only in presence of a medium can all the particles be on shell as there the dispersion relation of the photon changes, giving the much required phase space for the reactions. Intuitively when a neutrino moves inside a thermal medium composed of electrons and positrons, they interact with these background particles. The background electrons and positrons themselves have interaction with the electromagnetic fields, and this fact gives rise to an effective coupling of the neutrinos to the photons. Under these circumstance's the neutrinos may acquire an "effective electric charge" through which they interact with the ambient plasma.

In this paper, we concentrate upon the effective neutrino photon interaction vertex coming from the axial vector part of the interaction. From there we estimate the effective charge of the neutrino inside a magnetised medium. We name the axial contribution in the effective neutrino photon Lagrangian as the axial polarisation tensor $\Pi^5_{\mu\nu}$, which we will clearly define in the next section. We discuss the physical situations where the axial polarisation tensor arises, then show how it affects the physical processes. The effective charge of the neutrino has been calculated previously by many authors. In this regard, we also comment upon the contribution of this quantity on the effective charge of the neutrinos inside a magnetised plasma as well as other physical observable.

The plan of the paper is as follows, Section 2 deals with the formalism. Section 3, general form factor analysis of "axial polarisation tensor" on the basis of symmetry arguments is provided. Section 4 we discuss our results and conclude by touching upon the physical relevance of our work. Relevant details are relegated to the appendix.

2. Formalism

In this work, we consider neutrino momenta that are small compared to the masses of the W and Z bosons. We can, therefore, neglect the momentum dependence in the W and Z propagators, which is justified if we are performing a calculation to the leading order in the Fermi constant, G_F. In this limit four-fermions interaction is given by the following effective Lagrangian:

$$\mathscr{L}_{\text{eff}} = -\frac{1}{\sqrt{2}} G_F \bar{\nu}\gamma^{\mu}(1-\gamma_5)\nu \bar{l}_{\nu}\gamma_{\mu}(g_{\text{V}} + g_{\text{A}}\gamma_5) l_{\nu}\,, \tag{2.1}$$

where ν and l_ν are the neutrino and the corresponding lepton field respectively. For electron neutrinos,

$$g_{\rm V} = 1 - (1 - 4\sin^2\theta_{\rm W})/2, \tag{2.2}$$

$$g_{\rm A} = -1 + 1/2; \tag{2.3}$$

where the first terms in $g_{\rm V}$ and $g_{\rm A}$ are the contributions from the W exchange diagram and the second one from the Z exchange diagram.

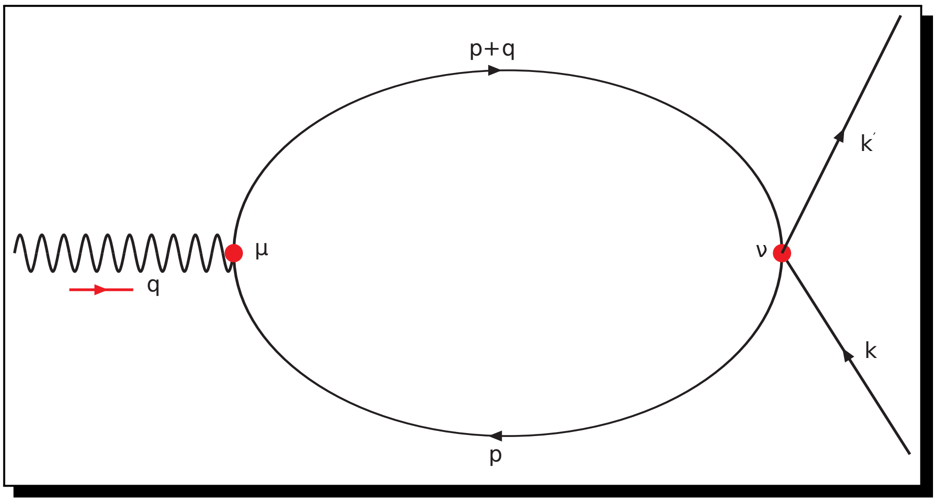

Figure 1. One-loop diagram for the effective electromagnetic vertex of the neutrino in the limit of infinitely heavy W and Z masses

With this interaction Lagrangian we can write down the matrix element for the Cherenkov amplitude as,

$$M = -\frac{G_F}{\sqrt{2}e} Z\epsilon^{\nu}\bar{\nu}\gamma^{\mu}(1-\gamma_5)\nu(g_V\Pi_{\mu\nu} + g_A\Pi^5_{\mu\nu}), \tag{2.4}$$

where ϵ^{ν} is the photon polarisation tensor, and Z is the wavefunction renormalisation factor inside a medium. The term $\Pi_{\mu\nu}$ is defined as

$$i\Pi_{\mu\nu} = (-ie)^2(-1)\int \frac{d^4p}{(2\pi)^4}{\rm Tr}[\gamma_\mu iS(p)\gamma_\nu iS(p')] \tag{2.5}$$

which looks exactly like the photon polarisation tensor, but does not have the same interpretation here. The momentum labels of the prapagators can be understood from Figure 1. Henceforth, we would call it the polarisation tensor. $\Pi^5_{\mu\nu}$ is defined as

$$i\Pi^5_{\mu\nu} = (-ie)^2(-1)\int \frac{d^4p}{(2\pi)^4}{\rm Tr}[\gamma_\mu\gamma_5 iS(p)\gamma_\nu iS(p')] \tag{2.6}$$

which we call the axial polarisation tensor. Both the polarisation tensor and the axial polarisation tensor are obtained by calculating the Feynman diagram given in Figure 1.

The off-shell electromagnetic vertex function Γ_ν is defined in such a way that, for on-shell neutrinos, the $\nu\nu\gamma$ amplitude is given by:

$$\mathscr{M} = -i\bar{u}(q')\Gamma_\nu u(q)A^\nu(k), \tag{2.7}$$

where k is the photon momentum. Here, $u(q)$ is the neutrino spinor and A^ν stands for the electromagnetic vector potential. In general, Γ_ν would depend on k and the characteristics of the medium. With our effective Lagrangian Γ_ν is given by

$$\Gamma_\nu = -\frac{1}{\sqrt{2}e}G_F\gamma^\mu(1-\gamma_5)(g_{\rm V}\Pi_{\mu\nu} + g_{\rm A}\Pi^5_{\mu\nu}). \tag{2.8}$$

The effective charge of the neutrinos is defined in terms of the vertex function by the following relation [14]:

$$e_{\rm eff} = \frac{1}{2q_0}\bar{u}(q)\Gamma_0(k_0 = 0, \mathbf{k} \to 0)u(q). \tag{2.9}$$

For massless Weyl spinors this definition can be rendered into the form:

$$e_{\rm eff} = \frac{1}{2q_0}{\rm Tr}[\Gamma_0(k_0 = 0, \mathbf{k} \to 0)(1 + \lambda\gamma^5)\not{q}] \tag{2.10}$$

where $\lambda = \pm 1$ is the helicity of the spinors.

While discussing about $\Pi^5_{\mu\nu}$ it should be remembered that for the electromagnetic vertex, we have the current conservation relation,

$$k^\nu \Pi^5_{\mu\nu} = 0 \tag{2.11}$$

which is the gauge invariance condition.

In order to calculate the Cherenkov amplitude or the effective charge of the neutrinos inside a medium, we have to calculate $\Pi^5_{\mu\nu}$. The formalism so discussed is a general one and we extend the calculations previously done based upon this formalism to the case where we have a constant background magnetic field in addition to a thermal medium. In doing so we give the full expression of $\Pi^5_{\mu\nu}$ in a magnetised medium and explicitly show its gauge invariance. We also comment on the effective charge contribution from the axial polarisation part.

Discussing about the effective charge of the neutrinos in a medium, the way we have done, it should be mentioned that although it is interesting to find it theoretically, it is not the "charge" with which the neutrinos couple with a magnetic field. From the definition of the electromagnetic vertex as given in eq. (2.7) and the definition of charge in eq. (2.9) it is clear that we are interested to find the coupling of the photon field with $\bar{u}\Gamma^0 u$ and not $\bar{u}\Gamma^i u$. The magnetic interaction will come from the term $\bar{u}(q)\Gamma^i u(q)A_i$, but we will see at the end of the calculation that no Γ^i exists in the limit $k_0 \to 0$, $\vec{k} \to 0$, whereby we cant say of any possible interaction of the neutrinos with the external static uniform magnetic field.

3. General Analysis

We start this section with a discussion on the possible tensor structure and form factor analysis of $\Pi^5_{\mu\nu}(k)$, based on the symmetry of the interaction. To begin with we note that, $\Pi^5_{\mu\nu}(k)$ in vacuum should vanish. This follows from the following arguments. In vacuum the available vectors and tensors at hand are the following,

$$k_\mu,\ g_{\mu\nu} \text{ and } \epsilon_{\mu\nu\lambda\sigma}. \tag{3.1}$$

The two point axial-vector correlation function $\Pi^5_{\mu\nu}$ can be expanded in a basis, constructed out of tensors $g_{\mu\nu}$, $\epsilon_{\mu\nu\lambda\sigma}$, and vector k_λ. Given the parity structure of the theory it is impossible to construct a tensor of rank two using $g_{\mu\nu}$ and k_μ. So the only available tensor (with the right parity structure) we have at hand is $\epsilon_{\mu\nu\lambda\sigma}$. The other vector needed to make it a tensor of rank two is k_λ. As we contract $\epsilon_{\mu\nu\lambda\sigma}$ with k_λ, k_σ, since $\epsilon_{\mu\nu\lambda\sigma}$ is completely antisymmetric tensor of rank four, the corresponding term vanishes.

On the other hand, in a medium, we have an additional vector u^μ, i.e., the velocity of the centre of mass of the medium. Therefore, the polarisation tensor can be expanded in terms of form factors along with the new tensors constructed out of u^μ and the ones we already had in absence of a medium as,

$$\Pi_{\mu\nu}(k) = \Pi_T T_{\mu\nu} + \Pi_L L_{\mu\nu}, \tag{3.2}$$

where

$$T_{\mu\nu} = \widetilde{g}_{\mu\nu} - L_{\mu\nu}, \tag{3.3}$$

$$L_{\mu\nu} = \frac{\widetilde{u}_\mu \widetilde{u}_\nu}{\widetilde{u}^2} \tag{3.4}$$

with

$$\widetilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}, \tag{3.5}$$

$$\widetilde{u}_\mu = \widetilde{g}_{\mu\rho} u^\rho. \tag{3.6}$$

In the rest frame of the medium the four velocity is given by $u^\mu = (1,0,0,0)$. It is easy to see that the longitudinal projector $L_{\mu\nu}$ is not zero in the limit $k_0 = 0, \vec{k} \to 0$ and Π_L is also not zero in the above mentioned limit. This fact is responsible for giving nonzero contribution to the effective charge of neutrino.

As has already been mentioned, that in a medium, we have another extra four vector u^μ and hence it is possible to construct the axial polarisation tensor of rank two, out of $\varepsilon_{\mu\nu\alpha\beta}$, u_μ, k_μ, i.e, $\varepsilon_{\mu\nu\alpha\beta} u^\alpha k^\beta$, that would verify the Ward identity for the two point function. An explicit calculation of $\Pi^5_{\mu\nu}(k)$ verifies the tensor structure of it as predicted here. It is worth noting that this contributes to the Cherenkov amplitude, but not to the effective electric charge of the neutrinos since for charge calculation we have to put the index $\nu = 0$. In the rest frame only u^0 exists, that forces the totally antisymmetric tensor to vanish.

In a constant background magnetic field in addition to the ones mentioned in eq. (3.1) one has the freedom of having other extra vectors and tensors (to first order in field strength), such as

$$F_{\mu\nu},\ \widetilde{F}_{\mu\nu} \tag{3.7}$$

along with

$$e^{(1)}_{\mu} = k^{\lambda} F_{\lambda\mu}, \quad e^{(2)}_{\mu} = k^{\lambda} \widetilde{F}_{\lambda\mu}. \tag{3.8}$$

Explicit evaluation of the axial polarisation tensor, in a constant background magnetic field is (however the metric used by the authors in references mentioned is different from that of us) [5,12],

$$\Pi^{5}_{\mu\nu}(k) = \frac{e^3}{(4\pi)^2 m^2}[-C_{\parallel} k_{\nu_{\parallel}} (\widetilde{F}k)_{\mu} + C_{\perp}\{k_{\nu_{\perp}}(k\widetilde{F})_{\mu} + k_{\mu_{\perp}}(k\widetilde{F})_{\nu} - k^2_{\perp}\widetilde{F}_{\mu\nu}\}], \tag{3.9}$$

where $\widetilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, $F_{12} = -F_{21} = \mathscr{B}$ and $(k\widetilde{F})_{\nu} = \widetilde{F}^{\mu\nu}k_{\mu}$. According to the notation used in eq. (3.9), $k_{\parallel} = (k_0, 0, 0, k_3)$ and $k_{\perp} = (0, k_x, k_y, 0)$. Lastly, $C_{\parallel}$ and $C_{\perp}$ are functions of $\mathscr{B}, k^2_{\parallel}, k^2_{\perp}$. It is easy to note that in consonance with the general parity structure of the theory the basis tensors for this case are $\widetilde{F}_{\mu\nu}$, $e^{(2)}_{\mu} k_{\nu_{\perp}}$ and $e^{(2)}_{\nu} k_{\mu_{\perp}}$.

From the above expression we can see that the axial polarisation tensor in a background magnetic field does not survive when the momentum of the external photon vanishes, and as a result there cannot be any effective electric charge of the neutrinos in a constant background magnetic field. Actually this formal statement could have been spoil by the presence of possible infrared divergence in the loop; i.e., to say in $C_{\parallel}$ and $C_{\perp}$. Since the particle inside the loop is massive so there is no scope of having infrared divergence, hence it does not contribute to neutrino effective charge.

In the presence of an external magnetic field (to even and odd order in field strength) plus medium, we have other vectors available. The ones important for our purpose, are the following,

$$\begin{aligned} e^{(3)}_{\mu} &= F_{\mu\alpha} F^{\alpha\beta} k_{\beta}, \\ e^{(4)}_{\mu} &= \epsilon_{\mu\nu\lambda\sigma} F^{\lambda\sigma} u^{\nu}. \end{aligned} \tag{3.10}$$

It is worth noting that, in a background magnetic field pointed in the z direction, the possible vector with highest power of external magnetic field that can occur in the axial polarisation tensor is $e^{(3)}_{\mu}$. Other constructions with higher powers of field strength tensor would simply be a linear combination of the same and/or $F^{\mu\nu}k_{\nu}$. The structure of axial polarisation tensor for odd powers in magnetic field, had already been analysed in [3], so we would directly comment on the same with even powers of external magnetic field $\mathscr{B}$. Since the underlying theory is CP conserving, therefore, the possibilities of decomposing $\Pi^{5}_{\mu\nu}$ in terms of the basis vectors are,

$$\Pi^{5(O(\mathscr{B}^2))}_{\mu\nu} = F_1 \epsilon_{\mu\nu\lambda\sigma} u^{\lambda} k^{\sigma} + F_2 \epsilon_{\mu\nu\alpha\beta} k^{\alpha} e^{(3)\beta}, \tag{3.11}$$

where F_1 and F_2, i.e., the form factors, are functions of Lorentz scalars, constructed out of all the vectors or tensors we have at our disposal as well other parameters like temperature and chemical potential. The first term on the right hand side of eq. (3.11) have already been discussed in [14], with the exception that the function F_1 now is a even function of external magnetic field $\mathscr{B}$; on the other hand the appearance of the second term is new. One can also observe that, in keeping with CP invariance of the theory (i.e., background along with the interaction), both the functions F_1 and F_2 should be odd functions of chemical potential. However, this would become clear from eq. (B.4) of Section 2.

3.1 Effective Charge at Even Order in the External Field and Coupling With Magnetic Fields

From the part of $\Pi_{\mu\nu}$ which is even in the external fields we see from eq. (B.4) that

$$\mathrm{R}^{(e)}_{\mu 0} = 4i\eta_-(p_0)\Big[\varepsilon_{\mu 0\alpha\beta}p^\alpha k^\beta(1+\tan(e\mathscr{B}s)\tan(e\mathscr{B}s')) + \varepsilon_{\mu 0\alpha\beta_\perp}k^\alpha k^{\beta_\perp}\tan(e\mathscr{B}s)\tan(e\mathscr{B}s') \times \frac{\tan(e\mathscr{B}s)-\tan(e\mathscr{B}s')}{\tan(e\mathscr{B}s)+\tan(e\mathscr{B}s')}\Big] \tag{3.12}$$

which shows that $\Pi^5_{\mu\nu}(k)$ to even orders in the external field will vanish when $k_0 \to 0$, $\vec{k} \to 0$. This implies that there will be no contribution to the effective neutrino charge from the sector which is even in the powers of $\mathscr{B}$.

Can the neutrinos which are propagating in a magnetised plasma couple with the classical magnetic field? The situation is a little bit subtle here, as the vertex of the neutrinos with the dynamical photons do get changed here due to the presence of the magnetic field, but this change cannot induce any electromagnetic form factor responsible for coupling of the neutrinos with any magnetic field. In order to find the effective charge of the neutrinos which couples them with time independent magnetic field one should look for (as given in eq. (2.7)), the Γ^i's, where $i = 1,2$. A magnetic field in the z-direction, is given by a gauge where A_1, A_2 are both nonzero, or one of them is nonzero. So to calculate the charge which is essential for the neutrino current to couple with a magnetic field, one has to put the index $\nu = 1,2$ in eq. (C.1) and take the limit $k_0 \to 0$, $\vec{k} \to 0$ and see which component of $\Pi^5_{\mu 1}(k)$ exists in the prementioned momentum limit. For the odd $\mathscr{B}$ part we see from eq. (A.25) that

$$\mathrm{R}^{(o)}_{\mu 1} = 4i\eta_+(p_0)\left[g_{\mu\alpha_\parallel}k_1\left\{p^{\widetilde{\alpha_\parallel}}(\tan(e\mathscr{B}s)-\tan(e\mathscr{B}s')) - k^{\widetilde{\alpha_\parallel}}\frac{\sec^2(e\mathscr{B}s)\tan^2(e\mathscr{B}s')}{\tan(e\mathscr{B}s)+\tan(e\mathscr{B}s')}\right\} + g_{\mu 1}(p\cdot\widetilde{k})(\tan(e\mathscr{B}s)-\tan(e\mathscr{B}s'))\right] \tag{3.13}$$

which goes to zero as the photon momentum tends to zero. By the same argument it follows that for $\nu = 2$ there is vanishing contribution. Thus it shows that there is no effective magnetic coupling from $\mathscr{B}$ odd part.

For the $\mathscr{B}$ even part, as is seen from eq. (A.24), that only $\mathrm{R}^{(e)}_{12}$ survives, and is given by

$$\mathrm{R}^{(e)}_{12} = 4i\eta_-(p_0)[\varepsilon_{1203}(p\cdot\widetilde{k})(1+\tan(e\mathscr{B}s)\tan(e\mathscr{B}s'))], \tag{3.14}$$

which also perishes in the limit when the external momentum goes to zero. So from this we can say that $\Pi^5_{\mu\nu}$ has no contribution for any charge of the neutrinos which can couple them with the magnetic field.

4. Conclusion

In our analysis, we have calculated the contributions to $\Pi^{5(o)}_{\mu\nu}(k)$ to odd and even orders in the external constant magnetic field. The main reason for doing so is the fact that, $\Pi^{5(o)}_{\mu\nu}(k)$ and $\Pi^{5(e)}_{\mu\nu}(k)$, the axial polarisation tensors to odd and even powers in $e\mathscr{B}$, have different dependence on the background matter. Pieces proportional to even powers in $\mathscr{B}$ are proportional to $\eta_-(p_0)$, an odd function of the chemical potential. On the other hand pieces proportional to odd powers in $\mathscr{B}$ depend on $\eta_+(p_0)$, and are even in μ and as a result it survives in the limit $\mu \to 0$. As has already been noted, this is a direct consequence on the charge and parity symmetries of the underlying theory.

In a background magnetic field the field dependence of the form factors, which are usually scalars, can be of the following form:

$$k^\mu F_{\mu\nu} F^{\nu\lambda} k^\lambda \quad \text{and} \quad F_{\mu\nu} F^{\mu\nu}. \tag{4.1}$$

These forms dose not exhaust all the possibilities, but whatever they are they must contain an even number of F's and k's and hence they will be even functions of $\mathscr{B}$.

Of all possible tensorial structures for the axial polarisation tensor in a magnetised plasma, there exists one which satisfies the current conservation condition in the ν vertex, that is given by,

$$\phi_{\mu\nu} = \epsilon_{\mu\alpha\lambda\sigma} F^{\lambda\sigma} u^\alpha \left[u_\nu - \frac{(k \cdot u) k_\nu}{k^2} \right]. \tag{4.2}$$

Its worth noting that the first term in the square bracket in eq. (4.2), which is odd in the external field, survives in the zero external momentum limit in the rest frame of the medium. The tensorial structures which are explicitly even in powers of $\mathscr{B}$ do have and k's also, and so they vanish in the limit when the external momentum goes to zero. We have earlier noted that the form factors which exist in the rest frame of the medium and in the zero momentum limit are even in powers of the external field. This tells us directly that the axial polarisation tensor must be odd in the external field in the zero external momentum limit, a result which we have verified in this work. However, it should be noted that the other form factors contribute in other situations like neutrino Cherenkov radiation, which is not discussed in this work.

In this work we have elucidated upon the physical significance of the axial polarisation tensor in various neutrino mediated processes in a magnetised medium, and explicitly written down its form in a gauge invariant way. It has been shown that the part of $\Pi^5_{\mu\nu}$ even in $\mathscr{B}$ does not contribute to the effective electric charge. However, it does contribute to physical processes, e.g., neutrino Cherenkov radiation or neutrino decay in a medium. It is worth noting that in the low density high temperature limit, the magnitude of $e^{\nu_a}_{\rm eff}$ can become greater than the effective charge of the neutrino in ordinary medium provided $e\mathscr{B}$ is large enough. On the other hand in the high density limit $e^{\nu_a}_{\rm eff}$ can dominate over the effective charge of the neutrino as found in an unmagnetized medium, provided the temperature is low enough. However, in standard astrophysical objects, e.g., core of type II Supernova temperature is of the order of 30-60 MeV with Fermi momentum around 300 MeV, for red giants the same are 10 keV and 400 keV, for young white dwarves temperature is around 0.1-1 keV and Fermi momentum 500 keV. In these systems one can have relatively large induced neutrino charge, provided the field strength is large enough. The effective neutrino photon coupling in a magnetised medium can also shed some light in understanding the observed gamma ray bursts or gamma ray repeaters observed in nature.

Acknowledgment

We would like to thank Prof. P. B. Pal for sharing his view.

Appendices

A. One Loop Calculation of the Axial Polarisation Tensor

Since we investigate the case with a background magnetic field, without any loss of generality it can be taken to be in the z-direction. We denote the magnitude of this field by $\mathscr{B}$. Ignoring first the presence of the medium, the electron propagator in such a field can be written down following Schwinger's approach [**?**, 6, 17]:

$$iS_B^V(p) = \int_0^\infty ds e^{\Phi(p,s)} G(p,s), \tag{A.1}$$

where Φ and G are as given below

$$\Phi(p,s) \equiv is\left(p_\parallel^2 - \frac{\tan(e\mathscr{B}s)}{e\mathscr{B}s} p_\perp^2 - m^2\right) - \epsilon|s|, \tag{A.2}$$

$$\begin{aligned} G(p,s) &\equiv \frac{e^{ie\mathscr{B}s\sigma_z}}{\cos(e\mathscr{B}s)}\left(\not{p}_\parallel + \frac{e^{-ie\mathscr{B}s\sigma_z}}{\cos(e\mathscr{B}s)}\not{p}_\perp + m\right) \\ &= [(1 + i\sigma_z \tan(e\mathscr{B}s))(\not{p}_\parallel + m) + \sec^2(e\mathscr{B}s)\not{p}_\perp], \end{aligned} \tag{A.3}$$

where

$$\sigma_z = i\gamma_1\gamma_2 = -\gamma_0\gamma_3\gamma_5, \tag{A.4}$$

and we have used,

$$e^{ie\mathscr{B}s\sigma_z} = \cos(e\mathscr{B}s) + i\sigma_z \sin(e\mathscr{B}s). \tag{A.5}$$

To make the expressions transparent we specify our convention in the following way,

$$\begin{aligned} \not{p}_\parallel &= \gamma_0 p^0 + \gamma_3 p^3, \\ \not{p}_\perp &= \gamma_1 p^1 + \gamma_2 p^2, \\ p_\parallel^2 &= p_0^2 - p_3^2, \\ p_\perp^2 &= p_1^2 + p_2^2. \end{aligned}$$

Of course in the range of integration indicated in eq. (A.1) s is never negative and hence $|s|$ equals s. In the presence of a background medium, the above propagator is now modified to [8]:

$$iS(p) = iS_B^V(p) + S_B^\eta(p), \tag{A.6}$$

where

$$S_B^\eta(p) \equiv -\eta_F(p)[iS_B^V(p) - i\overline{S}_B^V(p)], \tag{A.7}$$

and

$$\overline{S}_B^V(p) \equiv \gamma_0 S_B^{V\dagger}(p)\gamma_0, \tag{A.8}$$

for a fermion propagator, such that

$$S_B^\eta(p) = -\eta_F(p)\int_{-\infty}^{\infty} ds\, e^{\Phi(p,s)} G(p,s), \tag{A.9}$$

where $\eta_F(p)$ contains the distribution function for the fermions and the anti-fermions:

$$\eta_F(p) = \Theta(p\cdot u) f_F(p,\mu,\beta) + \Theta(-p\cdot u) f_F(-p,-\mu,\beta), \tag{A.10}$$

f_F denotes the Fermi-Dirac distribution function:

$$f_F(p,\mu,\beta) = \frac{1}{e^{\beta(p\cdot u-\mu)}+1}, \tag{A.11}$$

and Θ is the step function given by:

$$\Theta(x) = \begin{cases} 1, & \text{for } x > 0, \\ 0, & \text{for } x < 0, \end{cases}$$

where the four velocity of the medium is u, in the rest frame it looks like $u^\mu = (1,0,0,0)$.

A.1 The Expression for $\Pi^5_{\mu\nu}$ in Thermal Medium and in the Presence of a Background Uniform Magnetic Field

The axial polarisation tensor $\Pi^5_{\mu\nu}$ is expressed as

$$i\Pi^5_{\mu\nu} = (-ie)^2(-1)\int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}[\gamma_\mu\gamma_5 iS(p)\gamma_\nu iS(p')]. \tag{A.12}$$

Leaving out the vacuum contribution (the contribution devoid of any thermal effects) and the contributions with two thermal factors, we are left with

$$i\Pi^5_{\mu\nu}(k) = (-ie)^2(-1)\int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}[\gamma_\mu\gamma_5 S_B^\eta(p)\gamma_\nu iS_B^V(p') + \gamma_\mu\gamma_5 iS_B^V(p)\gamma_\nu S_B^\eta(p')]. \tag{A.13}$$

The vacuum part has already been done in [5] and the thermal part is related with pure absorption effects in the medium, which we are leaving out for the time being.

Using the form of the fermion propagator in a magnetic field in presence of a thermal medium, as given by expressions (A.1) and (A.9), we get

$$\begin{aligned} i\Pi^5_{\mu\nu}(k) &= -(-ie)^2(-1)\int \frac{d^4p}{(2\pi)^4}\int_{-\infty}^{\infty} ds\, e^{\Phi(p,s)}\int_0^\infty ds' e^{\Phi(p',s')}[\mathrm{Tr}[\gamma_\mu\gamma_5 G(p,s)\gamma_\nu G(p',s')]\eta_F(p) \\ &\quad + \mathrm{Tr}[\gamma_\mu\gamma_5 G(-p',s')\gamma_\nu G(-p,s)]\eta_F(-p)] \\ &= -(-ie)^2(-1)\int \frac{d^4p}{(2\pi)^4}\int_{-\infty}^{\infty} ds\, e^{\Phi(p,s)}\int_0^\infty ds'\, e^{\Phi(p',s')}\mathrm{R}_{\mu\nu}(p,p',s,s'), \end{aligned} \tag{A.14}$$

where $\mathrm{R}_{\mu\nu}(p,p',s,s')$ contains the trace part.

A.2 $R_{\mu\nu}$ To Even and Odd Orders in Magnetic Field

We calculate $R_{\mu\nu}(p,p',s,s')$ to even and odd orders in the external magnetic field and call them $R^{(e)}_{\mu\nu}$ and $R^{(o)}_{\mu\nu}$. The reason for doing this is that the two contributions have different properties as far as their dependence on medium is concerned, a topic which will be discussed in the concluding section. Calculating the traces we obtain,

$$\begin{aligned} R^{(e)}_{\mu\nu} = 4i\eta_-(p)[&\varepsilon_{\mu\nu\alpha_\parallel\beta_\parallel}p^{\alpha_\parallel}p'^{\beta_\parallel}(1+\tan(e\mathscr{B}s)\tan(e\mathscr{B}s'))+\varepsilon_{\mu\nu\alpha_\parallel\beta_\perp}p^{\alpha_\parallel}p'^{\beta_\perp}\sec^2(e\mathscr{B}s') \\ &+\varepsilon_{\mu\nu\alpha_\perp\beta_\parallel}p^{\alpha_\perp}p'^{\beta_\parallel}\sec^2(e\mathscr{B}s)+\varepsilon_{\mu\nu\alpha_\perp\beta_\perp}p^{\alpha_\perp}p'^{\beta_\perp}\sec^2(e\mathscr{B}s)\sec^2(e\mathscr{B}s')] \end{aligned} \tag{A.15}$$

and

$$\begin{aligned} R^{(o)}_{\mu\nu} = 4i\eta_+(p)[&m^2\varepsilon_{\mu\nu12}(\tan(e\mathscr{B}s)+\tan(e\mathscr{B}s'))+\{(g_{\mu\alpha_\parallel}p^{\widetilde{\alpha_\parallel}}p'_{\nu_\parallel}-g_{\mu\nu}p'_{\alpha_\parallel}p^{\widetilde{\alpha_\parallel}}+g_{\nu\alpha_\parallel}p^{\widetilde{\alpha_\parallel}}p'_{\mu_\parallel}) \\ &+(g_{\mu\alpha_\parallel}p^{\widetilde{\alpha_\parallel}}p'_{\nu_\perp}+g_{\nu\alpha_\parallel}p^{\widetilde{\alpha_\parallel}}p'_{\mu_\perp})\sec^2(e\mathscr{B}s')\}\tan(e\mathscr{B}s) \\ &+\{(g_{\mu\alpha_\parallel}p'^{\widetilde{\alpha_\parallel}}p_{\nu_\parallel}-g_{\mu\nu}p_{\alpha_\parallel}p'^{\widetilde{\alpha_\parallel}}+g_{\nu\alpha_\parallel}p'^{\widetilde{\alpha_\parallel}}p_{\mu_\parallel}) \\ &+(g_{\mu\alpha_\parallel}p'^{\widetilde{\alpha_\parallel}}p_{\nu_\perp}+g_{\nu\alpha_\parallel}p'^{\widetilde{\alpha_\parallel}}p_{\mu_\perp})\sec^2(e\mathscr{B}s)\}\tan(e\mathscr{B}s')], \end{aligned} \tag{A.16}$$

where

$$\eta_+(p)=\eta_F(p)+\eta_F(-p), \tag{A.17}$$

$$\eta_-(p)=\eta_F(p)-\eta_F(-p) \tag{A.18}$$

which contain the information about the distribution functions. Also, it should be noted that, in our convention

$$a_\mu b^{\widetilde{\mu}_\parallel}=a_0b^3+a_3b^0.$$

If we concentrate on the rest frame of the medium, then $p\cdot u=p_0$. Thus, the distribution function does not depend on the spatial components of p. In this case we can write the expressions of $R^{(e)}_{\mu\nu}$ and $R^{(o)}_{\mu\nu}$ using the relations derived earlier [10] inside the integral sign, as

$$p^{\beta_\perp}\overset{\circ}{=}-\frac{\tan(e\mathscr{B}s')}{\tan(e\mathscr{B}s)+\tan(e\mathscr{B}s')}k^{\beta_\perp}, \tag{A.19}$$

$$p'^{\beta_\perp}\overset{\circ}{=}\frac{\tan(e\mathscr{B}s)}{\tan(e\mathscr{B}s)+\tan(e\mathscr{B}s')}k^{\beta_\perp}, \tag{A.20}$$

$$p^2_\perp\overset{\circ}{=}\frac{1}{\tan(e\mathscr{B}s)+\tan(e\mathscr{B}s')}\left[-ie\mathscr{B}+\frac{\tan(e\mathscr{B}s')^2}{\tan(e\mathscr{B}s)+\tan(e\mathscr{B}s')}k^2_\perp\right], \tag{A.21}$$

$$p'^2_\perp\overset{\circ}{=}\frac{1}{\tan(e\mathscr{B}s)+\tan(e\mathscr{B}s')}\left[-ie\mathscr{B}+\frac{\tan(e\mathscr{B}s')^2}{\tan(e\mathscr{B}s)+\tan(e\mathscr{B}s)}k^2_\perp\right], \tag{A.22}$$

$$m^2\overset{\circ}{=}\left(i\frac{d}{ds}+(p^2_\parallel-\sec^2(e\mathscr{B}s)p^2_\perp)\right) \tag{A.23}$$

and get

$$\begin{aligned} R^{(e)}_{\mu\nu}\overset{\circ}{=}4i\eta_-(p_0)[&\varepsilon_{\mu\nu\alpha_\parallel\beta_\parallel}p^{\alpha_\parallel}p'^{\beta_\parallel}(1+\tan(e\mathscr{B}s)\tan(e\mathscr{B}s'))+\varepsilon_{\mu\nu\alpha_\parallel\beta_\perp}p^{\alpha_\parallel}p'^{\beta_\perp}\sec^2(e\mathscr{B}s') \\ &+\varepsilon_{\mu\nu\alpha_\perp\beta_\parallel}p^{\alpha_\perp}p'^{\beta_\parallel}\sec^2(e\mathscr{B}s)] \end{aligned} \tag{A.24}$$

and

$$\mathrm{R}^{(o)}_{\mu\nu} \overset{\circ}{=} 4i\eta_+(p_0)\Big[-\varepsilon_{\mu\nu 12}\left\{\frac{\sec^2(e\mathscr{B}s)\tan^2(e\mathscr{B}s')}{\tan(e\mathscr{B}s)+\tan(e\mathscr{B}s')}k_\perp^2+(k\cdot p)_\parallel(\tan(e\mathscr{B}s)+\tan(e\mathscr{B}s'))\right\}$$
$$+2\varepsilon_{\mu 12\alpha_\parallel}(p'_{\nu_\parallel}p^{\alpha_\parallel}\tan(e\mathscr{B}s)+p_{\nu_\parallel}p'^{\alpha_\parallel}\tan(e\mathscr{B}s'))$$
$$+g_{\mu\alpha_\parallel}k_{\nu_\perp}\left\{p^{\tilde{\alpha}_\parallel}(\tan(e\mathscr{B}s)-\tan(e\mathscr{B}s'))-k^{\tilde{\alpha}_\parallel}\frac{\sec^2(e\mathscr{B}s)\tan^2(e\mathscr{B}s')}{\tan(e\mathscr{B}s)+\tan(e\mathscr{B}s')}\right\}$$
$$+\{g_{\mu\nu}(p\cdot\tilde{k})_\parallel+g_{\nu\alpha_\parallel}p^{\tilde{\alpha}_\parallel}k_{\mu_\perp}\}(\tan(e\mathscr{B}s)-\tan(e\mathscr{B}s'))+g_{\nu\alpha_\parallel}k^{\tilde{\alpha}_\parallel}p_{\mu_\perp}\sec^2(e\mathscr{B}s)\tan(e\mathscr{B}s')\Big]. \tag{A.25}$$

The $\overset{\circ}{=}$ symbol signifies that the above relations are not proper equations, the equality holds only inside the momentum integrals in eq. (A.14).

B. Gauge Invarience

B.1 Gauge Invarience for $\Pi^5_{\mu\nu}$ to Even Orders in the External Field

The axial polarisation tensor even in the external field is given by

$$\Pi^{5(e)}_{\mu\nu} = -(-ie)^2(-1)\int\frac{d^4p}{(2\pi)^4}\int_{-\infty}^{\infty}ds\,e^{\Phi(p,s)}\int_0^{\infty}ds'\,e^{\Phi(p',s')}\mathrm{R}^{(e)}_{\mu\nu}(p,p',s,s'). \tag{B.1}$$

Using eq. (A.24) in the rest frame of the medium, we have

$$\mathrm{R}^{(e)}_{\mu\nu} \overset{\circ}{=} 4i\eta_-(p_0)[\varepsilon_{\mu\nu\alpha_\parallel\beta_\parallel}p^{\alpha_\parallel}p'^{\beta_\parallel}(1+\tan(e\mathscr{B}s)\tan(e\mathscr{B}s'))+\varepsilon_{\mu\nu\alpha_\parallel\beta_\perp}p^{\alpha_\parallel}p'^{\beta_\perp}\sec^2(e\mathscr{B}s')$$
$$+\varepsilon_{\mu\nu\alpha_\perp\beta_\parallel}p^{\alpha_\perp}p'^{\beta_\parallel}\sec^2(e\mathscr{B}s)]. \tag{B.2}$$

Noting that it is possible to write,

$$q^\alpha p_\alpha = q^{\alpha_\parallel}p_{\alpha_\parallel}+q^{\alpha_\perp}p_{\alpha_\perp}.$$

Eq. (B.2) can be written as,

$$\mathrm{R}^{(e)}_{\mu\nu} \overset{\circ}{=} 4i\eta_-(p_0)[(\varepsilon_{\mu\nu\alpha\beta}p^\alpha p'^\beta-\varepsilon_{\mu\nu\alpha\beta_\perp}p^\alpha p'^{\beta_\perp}-\varepsilon_{\mu\nu\alpha_\perp\beta}p^{\alpha_\perp}p'^\beta)(1+\tan(e\mathscr{B}s)\tan(e\mathscr{B}s'))$$
$$+\varepsilon_{\mu\nu\alpha\beta_\perp}p^\alpha p'^{\beta_\perp}\sec^2(e\mathscr{B}s')+\varepsilon_{\mu\nu\alpha_\perp\beta}p^{\alpha_\perp}p'^\beta\sec^2(e\mathscr{B}s)], \tag{B.3}$$

where throughout we have omitted terms such as $\varepsilon_{\mu\nu\alpha_\perp\beta_\perp}p^{\alpha_\perp}p'^{\beta_\perp}$, since by the application of eq. (A.19), we have

$$\varepsilon_{\mu\nu\alpha_\perp\beta_\perp}p^{\alpha_\perp}p'^{\beta_\perp} = \varepsilon_{\mu\nu\alpha_\perp\beta_\perp}p^{\alpha_\perp}p^{\beta_\perp}+\varepsilon_{\mu\nu\alpha_\perp\beta_\perp}p^{\alpha_\perp}k^{\beta_\perp}$$
$$\overset{\circ}{=} -\frac{\tan(e\mathscr{B}s')}{\tan(e\mathscr{B}s')+\tan(e\mathscr{B}s')}\varepsilon_{\mu\nu\alpha_\perp\beta_\perp}k^{\alpha_\perp}k^{\beta_\perp}$$

which is zero.

After rearranging the terms appearing in eq. (B.3), and by the application of eqs. (A.19) and (A.20), we arrive at the expression

$$\mathrm{R}^{(e)}_{\mu\nu} \overset{\circ}{=} 4i\eta_-(p_0)\Big[\varepsilon_{\mu\nu\alpha\beta}p^\alpha k^\beta(1+\tan(e\mathscr{B}s)\tan(e\mathscr{B}s'))$$
$$+\varepsilon_{\mu\nu\alpha\beta_\perp}k^\alpha k^{\beta_\perp}\tan(e\mathscr{B}s)\tan(e\mathscr{B}s')\frac{\tan(e\mathscr{B}s)-\tan(e\mathscr{B}s')}{\tan(e\mathscr{B}s)+\tan(e\mathscr{B}s')}\Big]. \tag{B.4}$$

Because of the presence of terms like $\varepsilon_{\mu\nu\alpha\beta}k^\beta$ and $\varepsilon_{\mu\nu\alpha\beta_\perp}k^\alpha$ if we contract $\mathrm{R}^{(e)}_{\mu\nu}$ by k^ν, it vanishes.

B.2 Gauge Invarience for $\Pi^5_{\mu\nu}$ To Odd Orders in the External Field

The axial polarisation tensor odd in the external field is given by

$$\Pi^{5(o)}_{\mu\nu} = -(-ie)^2(-1)\int\frac{d^4p}{(2\pi)^4}\int_{-\infty}^{\infty}ds\,e^{\Phi(p,s)}\int_0^\infty ds'\,e^{\Phi(p',s')}\mathrm{R}^{(o)}_{\mu\nu}(p,p',s,s'), \tag{B.5}$$

where $\mathrm{R}^{(o)}_{\mu\nu}(p,p',s,s')$ is given by eq. (A.25). The general gauge invarience condition in this case

$$k^\nu\Pi^{5(o)}_{\mu\nu} = 0 \tag{B.6}$$

can always be written down in terms of the following two equations,

$$k^\nu\Pi^{5(o)}_{\mu_\parallel\nu} = 0, \tag{B.7}$$
$$k^\nu\Pi^{5(o)}_{\mu_\perp\nu} = 0, \tag{B.8}$$

where $\Pi^{5(o)}_{\mu_\parallel\nu}$ is that part of $\Pi^{5(o)}_{\mu\nu}$ where the index μ can take the values 0 and 3 only. Similarly, $\Pi^{5(o)}_{\mu_\perp\nu}$ stands for the part of $\Pi^{5(o)}_{\mu\nu}$, where μ can take the values 1 and 2 only. $\Pi^{5(o)}_{\mu_\parallel\nu}$ contains $\mathrm{R}^{(o)}_{\mu_\parallel\nu}(p,p',s,s')$ which from eq. (A.25) is as follows,

$$\mathrm{R}^{(o)}_{\mu_\parallel\nu} \overset{\circ}{=} 4i\eta_+(p_0)\Big[-\varepsilon_{\mu_\parallel\nu12}\left\{\frac{\sec^2(e\mathscr{B}s)\tan^2(e\mathscr{B}s')}{\tan(e\mathscr{B}s)+\tan(e\mathscr{B}s')}k_\perp^2+(k\cdot p)_\parallel(\tan(e\mathscr{B}s)+\tan(e\mathscr{B}s'))\right\}$$
$$+\ 2\varepsilon_{\mu_\parallel 12\alpha_\parallel}(p'_{\nu_\parallel}p^{\alpha_\parallel}\tan(e\mathscr{B}s)+p_{\nu_\parallel}p'^{\alpha_\parallel}\tan(e\mathscr{B}s'))$$
$$+g_{\mu_\parallel\alpha_\parallel}k_{\nu_\perp}\left\{p^{\widetilde{\alpha}_\parallel}(\tan(e\mathscr{B}s)-\tan(e\mathscr{B}s'))-k^{\widetilde{\alpha}_\parallel}\frac{\sec^2(e\mathscr{B}s)\tan^2(e\mathscr{B}s')}{\tan(e\mathscr{B}s)+\tan(e\mathscr{B}s')}\right\}$$
$$+g_{\mu_\parallel\nu}(p\cdot\widetilde{k})_\parallel(\tan(e\mathscr{B}s)-\tan(e\mathscr{B}s'))] \tag{B.9}$$

and $\Pi^{5(o)}_{\mu_\perp\nu}$ contains $\mathrm{R}^{(o)}_{\mu_\perp\nu}(p,p',s,s')$ which is

$$\mathrm{R}^{(o)}_{\mu_\perp\nu} \overset{\circ}{=} 4i\eta_+(p_0)[\{g_{\mu_\perp\nu}(p\cdot\widetilde{k})_\parallel+g_{\nu\alpha_\parallel}p^{\widetilde{\alpha}_\parallel}k_{\mu_\perp}\}(\tan(e\mathscr{B}s)-\tan(e\mathscr{B}s'))$$
$$+g_{\nu\alpha_\parallel}k^{\widetilde{\alpha}_\parallel}p_{\mu_\perp}\sec^2(e\mathscr{B}s)\tan(e\mathscr{B}s')]. \tag{B.10}$$

Eqs. (B.7), (B.8) implies one should have the following relations satisfied,

$$k^\nu\int\frac{d^4p}{(2\pi)^4}\int_{-\infty}^{\infty}ds\,e^{\Phi(p,s)}\int_0^\infty ds'\,e^{\Phi(p',s')}\mathrm{R}^{(o)}_{\mu_\perp\nu} = 0 \tag{B.11}$$

and

$$k^{\nu}\int\frac{d^4p}{(2\pi)^4}\int_{-\infty}^{\infty}ds\,e^{\Phi(p,s)}\int_0^{\infty}ds'\,e^{\Phi(p',s')}\mathrm{R}^{(o)}_{\mu_{\|}\nu}=0. \tag{B.12}$$

Out of the two above equations, eq. (B.11) can be verified easily since

$$k^{\nu}\mathrm{R}_{\mu_{\perp}\nu}=0. \tag{B.13}$$

Now we look at eq. (B.12). We explicitly consider the case $\mu_{\|}=3$ (the $\mu_{\|}=0$ case lead to similar result). For $\mu_{\|}=3$,

$$k^{\nu}\mathrm{R}^{(o)}_{3\nu}\stackrel{\circ}{=}-p_0[(p'^2_{\|}-p^2_{\|})(\tan(e\mathscr{B}s)+\tan(e\mathscr{B}s'))-k^2_{\perp}(\tan(e\mathscr{B}s)-\tan(e\mathscr{B}s'))](4i\eta_+(p_0)). \tag{B.14}$$

Apart from the small convergence factors,

$$\frac{i}{e\mathscr{B}}(\Phi(p,s)+\Phi(p',s'))=(p'^2_{\|}+p^2_{\|}-2m^2)\xi-(p'^2_{\|}-p^2_{\|})\zeta-p'^2_{\perp}\tan(\xi-\zeta)-p^2_{\perp}\tan(\xi+\zeta), \tag{B.15}$$

where we have defined the parameters

$$\begin{aligned}\xi&=\frac{1}{2}e\mathscr{B}(s+s'),\\ \zeta&=\frac{1}{2}e\mathscr{B}(s-s').\end{aligned} \tag{B.16}$$

From the last two equations, we can write

$$ie\mathscr{B}\frac{d}{d\zeta}e^{\Phi(p,s)+\Phi(p',s')}=e^{\Phi(p,s)+\Phi(p',s')}(p'^2_{\|}-p^2_{\|}-p'^2_{\perp}\sec^2(\xi-\zeta)+p^2_{\perp}\sec^2(\xi+\zeta)), \tag{B.17}$$

which implies

$$p'^2_{\|}-p^2_{\|}=ie\mathscr{B}\frac{d}{d\xi}+[p'^2_{\perp}\sec^2(e\mathscr{B}s')-p^2_{\perp}\sec^2(e\mathscr{B}s)]. \tag{B.18}$$

The equation above is valid in the sense that both sides of it actually acts upon $e^{\widetilde{\Phi}(p,s,p',s')}$, where

$$\widetilde{\Phi}(p,p',s,s')=\Phi(p,s)+\Phi(p',s'). \tag{B.19}$$

From eqs. (B.14) and (B.18), we have

$$\begin{aligned}k^{\nu}\mathrm{R}_{3\nu}e^{\widetilde{\Phi}}\stackrel{\circ}{=}-4i\eta_+(p_0)p_0\Big[&(p'^2_{\perp}\sec^2(e\mathscr{B}s)-p^2_{\perp}\sec^2(e\mathscr{B}s))(\tan(e\mathscr{B}s)+\tan(e\mathscr{B}s'))\\ &-k^2_{\perp}(\tan(e\mathscr{B}s)-\tan(e\mathscr{B}s'))+ie\mathscr{B}p_0(\tan(e\mathscr{B}s)+\tan(e\mathscr{B}s'))\frac{d}{d\xi}\Big]e^{\widetilde{\Phi}}.\end{aligned} \tag{B.20}$$

Now using the expressions for $p^2_{\perp}$ and $p'^2_{\perp}$ from Eqs. (A.21) and (A.22), we can write

$$k^{\nu}\mathrm{R}_{3\nu}e^{\widetilde{\Phi}}\stackrel{\circ}{=}4e\mathscr{B}\eta_+(p_0)p_0\left[(\sec^2(e\mathscr{B}s)-\sec^2(e\mathscr{B}s'))+(\tan(e\mathscr{B}s)+\tan(e\mathscr{B}s'))\frac{d}{d\xi}\right]e^{\widetilde{\Phi}}. \tag{B.21}$$

The above equation can also be written as

$$k^\nu \mathrm{R}_{3\nu} e^{\widetilde{\Phi}} \overset{\circ}{=} 4e\mathscr{B}\eta_+(p_0)p_0 \frac{d}{d\xi}[e^{\widetilde{\Phi}}(\tan(e\mathscr{B}s)+\tan(e\mathscr{B}s'))]. \tag{B.22}$$

Transforming to ξ,ζ variables and using the above equation we can write the parametric integrations (integrations over s and s') on the left-hand side of eq. (B.12) as

$$\int_{-\infty}^{\infty} ds \int_0^{\infty} ds' k^\nu \mathrm{R}_{3\nu} e^{\widetilde{\Phi}} = \frac{8\eta_+(p_0)p_0}{e\mathscr{B}} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \Theta(\xi-\zeta)\frac{d}{d\xi}\mathscr{F}(\xi,\zeta), \tag{B.23}$$

where

$$\mathscr{F}(\xi,\zeta) = e^{\widetilde{\Phi}}(\tan(e\mathscr{B}s)+\tan(e\mathscr{B}s')).$$

The integration over ξ and ζ variables in eq. (B.23) can be represented as,

$$\begin{aligned}\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \Theta(\xi-\zeta)\frac{d}{d\xi}\mathscr{F}(\xi,\zeta) &= \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \left[\frac{d}{d\xi}\{\Theta(\xi-\zeta)\mathscr{F}(\xi,\zeta)\} - \delta(\xi-\zeta)\mathscr{F}(\xi,\zeta)\right] \\ &= -\int_{-\infty}^{\infty} d\xi \mathscr{F}(\xi,\xi),\end{aligned} \tag{B.24}$$

here the second step follows from the first one as the first integrand containing the Θ function vanishes at both limits of the integration. The remaining integral is now only a function of ξ and is even in p_0. But in eq. (B.23), we have $\eta_+(p_0)p_0$ sitting, which makes the the integrand odd under p_0 integration in the left-hand side of eq. (B.12), as $\eta_+(p_0)$ is an even function in p_0. So the p_0 integral as it occurs in the left-hand side of eq. (B.12) vanishes as expected, yielding the required result shown in eq. (B.7).

C. Effective Charge

Now we concentrate on the neutrino effective charge. From the onset it is to be made clear that we are only calculating the axial contribution to the effective charge[1]. We can now write the full expression of the axial polarisation tensor as

$$i\Pi^5_{\mu\nu}(k) = -(-ie)^2(-1)\int \frac{d^4p}{(2\pi)^4}\int_{-\infty}^{\infty} ds\, e^{\Phi(p,s)} \int_0^{\infty} ds'\, e^{\Phi(p',s')}[\mathrm{R}^{(o)}_{\mu\nu} + \mathrm{R}^{(e)}_{\mu\nu}], \tag{C.1}$$

where $\mathrm{R}^{(o)}_{\mu\nu}$ and $\mathrm{R}^{(e)}_{\mu\nu}$ are given by eqs. (A.25) and (A.24) in the rest frame of the medium.

C.1 Effective Charge to Odd Orders in External Field

In the limit when the external momentum tends to zero only two terms survive from $\Pi^5_{\mu\nu}(k)$. Denoting $\Pi^5_{\mu\nu}(k_0=0,\vec{k}\to 0) = \Pi^5_{\mu\nu}$, we obtain

$$\begin{aligned}\Pi^5_{\mu 0} = \lim_{k_0=0\vec{k}\to 0} 4e^2 \int \frac{d^4p}{(2\pi)^4}\int_{-\infty}^{\infty} ds\, e^{\Phi(p,s)} \int_0^{\infty} ds' e^{\Phi(p',s')}(\tan(e\mathscr{B}s)+\tan(e\mathscr{B}s')) \\ \times \eta_+(p_0)[2p_0^2 - (k\cdot p)_\parallel]\varepsilon_{\mu 012}.\end{aligned} \tag{C.2}$$

[1]In a forthcoming publication we will comment on the vector contribution to the effective charge of the neutrino [9].

The other terms turn out to be zero in this limit. The above equation shows that, except the exponential functions, the integrand is free of the perpendicular components of momenta. This implies we can integrate out the perpendicular component of the loop momentum. Upon performing the Gaussian integration over the perpendicular components and taking the limit $k_\perp \to 0$, we obtain

$$\Pi^5_{\mu 0} = \lim_{k_0=0,\vec{k}\to 0} \frac{(-4ie^3B)}{4\pi} \int \frac{d^2p_\parallel}{(2\pi)^2} \int_{-\infty}^{\infty} ds\, e^{is(p_\parallel^2-m^2)-\varepsilon|s|} \int_0^{\infty} ds' e^{is'(p_\parallel'^2-m^2)-\varepsilon|s'|}$$
$$\times \eta_+(p_0)\left[2p_0^2-(k\cdot p)_\parallel\right]\varepsilon_{\mu 012}. \tag{C.3}$$

It is worth noting that the s integral gives

$$\int_{\infty}^{\infty} ds\, e^{is(p_\parallel^2-m^2)-\varepsilon|s|} = 2\pi\delta(p_\parallel^2-m^2) \tag{C.4}$$

and the s' integral gives

$$\int_0^{\infty} ds'\, e^{is'(p_\parallel'^2-m^2)-\varepsilon|s'|} = \frac{i}{(p_\parallel'^2-m^2)+i\varepsilon}. \tag{C.5}$$

Using the above results in eq. (C.3) and using the delta function constraint, we arrive at

$$\Pi^5_{\mu 0} = \lim_{k_0=0,\vec{k}\to 0} 2(e^3B)\int \frac{d^2p_\parallel}{(2\pi)^2}\delta(p_\parallel^2-m^2)\eta_+(p_0)\left[\frac{2p_0^2}{(k_\parallel^2+2(p\cdot k)_\parallel)}-\frac{1}{2}\right]\varepsilon_{\mu 012}. \tag{C.6}$$

In deriving eq. (C.6), pieces proportional to $k_\parallel^2$ in the numerator were neglected. Now if one makes the substitution, $p'_\parallel \to (p_\parallel + k_\parallel/2)$ and sets $k_0 = 0$ one arrives at,

$$\Pi^5_{\mu 0} = -\lim_{k_0=0,\vec{k}\to 0} 2(e^3\mathscr{B})\int \frac{dp_3}{(2\pi)^2}\left(n_+(E'_p)+n_-(E'_p)\right)\left[\frac{E'_p}{p_3k_3}+\frac{1}{2E'_p}\right]\varepsilon_{\mu 012}, \tag{C.7}$$

where $n_\pm(E'_p)$ are the functions $f_F(E'_p,-\mu,\beta)$, and $f_F(E'_p,\mu,\beta)$, as given in eq. (A.11), which are nothing but the Fermi-Dirac distribution functions of the particles and the antiparticles in the medium. The new term E'_p is defined as follows,

$$E'^2_p = [(p_3-k_3/2)]^2+m^2,$$

and it can be expanded for small external momenta in the following way

$$E'^2_p \simeq p_3^2+m^2-p_3k_3 = E_p^2-p_3k_3$$

where $E_p^2 = p_3^2+m^2$. Noting, that

$$E'_p = E_p - \frac{p_3k_3}{2E_p}+O(k_3^2), \tag{C.8}$$

one can use this expansion in eq. (C.7), to arrive at:

$$\Pi^5_{\mu 0} = -\lim_{k_0=0,\vec{k}\to 0} 2(e^3\mathscr{B})\int \frac{dp_3}{(2\pi)^2}(n_+(E'_p)+n_-(E'_p))\left[\frac{E_p}{p_3k_3}\right]\varepsilon_{\mu 012}. \tag{C.9}$$

The expression for $\eta_+(E'_p) = n_+(E'_p) + n_-(E'_p)$ when expanded in powers of the external momentum k_3 is given by

$$\eta_+(E'_p) = \left(1 + \frac{1}{2}\frac{\beta p_3 k_3}{E_p}\right)\eta_+(E_p) \tag{C.10}$$

up to first order terms in the external momentum k_3.

C.1.1 Effective Charge for $\mu \ll m$

In the limit, when $\mu \ll m$ one can use the following expansion,

$$\begin{aligned}\eta_+(E'_p) &= [n_+(E'_p) + n_-(E'_P)] \\ &= 2\sum_{n=0}^{\infty}(-1)^n \cosh([n+1]\beta\mu)e^{-(n+1)\beta E_p}\left(1 + \frac{\beta p_3 k_3}{2E_p} + O(k_3^2) + \dots\right).\end{aligned} \tag{C.11}$$

Now using eq. (C.11) in eq. (C.9), we get

$$\Pi^5_{\mu 0} = -\varepsilon_{\mu 012}\lim_{k_0=0,\vec{k}\to 0}(4e^3\mathscr{B})\sum_{n=0}^{\infty}(-1)^n\cosh([n+1]\beta\mu)\int\frac{dp_3}{(2\pi)^2}e^{-(n+1)\beta E_p}\left[\frac{E_p}{(p_3k_3)} + \frac{\beta}{2}\right]. \tag{C.12}$$

The first term vanishes by symmetry of the integral, but the second term is finite, thus we get

$$\Pi^5_{\mu 0} = -\beta\varepsilon_{\mu 012}\lim_{k_0=0\vec{k}\to 0}\frac{(e^3\mathscr{B})}{2\pi^2}\sum_{n=0}^{\infty}(-1)^n\cosh([n+1]\beta\mu)\int dp_3 e^{-(n+1)\beta E_p}. \tag{C.13}$$

To perform the momentum integration, use of the following integral transform turns out to be extremely convenient

$$e^{-\alpha\sqrt{s}} = \frac{\alpha}{2\sqrt{\pi}}\int_0^{\infty} du e^{-us-\frac{\alpha^2}{4u}}u^{-3/2}. \tag{C.14}$$

Identifying $\sqrt{s}$ with E_p and $[(n+1)\beta]$ as α (since the square root opens up), one can easily perform the Gaussian p_3 integration without any difficulty. The result is:

$$\begin{aligned}\Pi^5_{\mu 0} &= -\beta\varepsilon_{\mu 012}\frac{(e^3\mathscr{B})}{2\pi^2}\sum_{n=0}^{\infty}(-1)^n\cosh([n+1]\beta\mu) \\ &\quad\times(\beta(n+1)/2)\int du e^{-m^2u-\frac{((n+1)\beta/2)^2}{u}}u^{-2}.\end{aligned} \tag{C.15}$$

Performing the integration the axial part of the effective charge of neutrino in the limit of $m > \mu$ turns out to be,

$$e^{\nu_a}_{\rm eff} = -\sqrt{2}g_A m\beta G_F\frac{e^2\mathscr{B}}{\pi^2}(1-\lambda)\cos(\theta)\sum_{n=0}^{\infty}(-1)^n\cosh((n+1)\beta\mu)K_{-1}(m\beta(n+1)), \tag{C.16}$$

where θ is the angle between the neutrino three momentum and the background magnetic field. The superscript ν_a on $e^{\nu_a}_{\rm eff}$ denotes that we are calculating the axial contribution of the effective charge. $K_{-1}(m\beta(n+1))$ is the modified Bessel function (of the second kind) of order one (for this function $K_{-1}(x) = K_1(x)$) which sharply falls off as we move away from the origin in the positive direction. Although as temperature tends to zero eq. (C.16) seems to blow up because of the presence of $m\beta$, but $K_{-1}(m\beta(n+1))$ would damp its growth as $e^{-m\beta}$, hence the result remains finite.

C.1.2 Effective Charge for $\mu \gg m$

Here we would try to estimate neutrino effective charge when $\mu \gg m$ and $\beta \neq \infty$. We would like to emphasize that the last condition should be strictly followed, i.e., temperature $T \neq 0$. Using eqs. (C.9) and (C.10), we would obtain

$$\Pi^5_{30} = \frac{e^3 \mathscr{B}}{2\pi} \beta \int \frac{dp}{2\pi} \eta_+(E_p). \tag{C.17}$$

Neglecting m in the expression in E_p, we would obtain

$$\Pi^5_{30} = \frac{e^3 \mathscr{B}}{2\pi^2} \ln[(1+e^{\beta\mu})(1+e^{-\beta\mu})]. \tag{C.18}$$

The same can also be written as

$$\Pi^5_{30} = \frac{e^3 \mathscr{B}}{\pi^2} \ln\left(2\cosh\left(\frac{\beta\mu}{2}\right)\right). \tag{C.19}$$

The expression for the effective charge then turns out to be

$$e^{\nu_a}_{\text{eff}} = -\sqrt{2} g_A G_F \frac{e^2 \mathscr{B}}{\pi^2} \ln\left(2\cosh\left(\frac{\beta\mu}{2}\right)\right)(1-\lambda)\cos(\theta), \tag{C.20}$$

where λ is the helicity of the neutrino spinors.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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