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Research Article

Numerical Solution of 2-Point Boundary Value Problem by Subdivision Scheme

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Abstract. A numerical approximating collocation algorithm is formulated that is based on binary 6-point approximating subdivision scheme to generate the curves. It is examined that the scheme is generating more smooth continuous solutions of the problems. Numerical example is given to illustrate the algorithm with its graphically representation.

Keywords. Subdivision scheme; Boundary value problem; Convergence; Stability

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1. introduction

Subdivision scheme plays vital role in *Computer Aided Geometric Design* (CAGD). It produces smooth curves. Interpolating and approximating scheme are two classes of subdivision scheme. Approximating scheme produces more smooth curve as compare to interpolating scheme. Subdivision scheme plays significant role in engineering, medical, biological science, computer science and space science.

Initially the concept of subdivision scheme was introduced by French mathematician in 1956, whose name is de Rham [6]. He introduced corner cutting method and found limit curve of C^0 continuity. De-Boor [4] obtained the continues curves after generalization of Chaikin algorithms.

A binary m-point approximating subdivision scheme was introduced by Mustafa *et al.* [16], which gives limit curve of C^{2m-3} continuity, where $m \geq 2$.

A 4-point interpolating subdivision scheme was introduced by Levin [14], produces limit curve of C^1 continuity with finite mask. Hassan and Dodgson [11] examined a binary 3-point approximating scheme which produces limit curve of C^3 continuity ternary 3-point scheme that generates C^2 limiting curve, and ternary 3-point interpolating scheme that produces limit curve of C^1 continuity. A non-stationary 4-point binary interpolating subdivision scheme which regenerates limit curve of C^1 continuity was introduced by Jena *et al*. [12].

Zheng *et al*. [24] proposed a ternary 3-point interpolating subdivision scheme that develops C^1 limiting curve. A 3-point a-ary approximating scheme was introduced by Ghaffar *et al*. [9]. A binary 4-point non-stationary interpolating subdivision scheme that produces limit curve of C^1 continuity was develop by Beccari *et al.* [2]. Ghaffar and Mustafa [10] examined a binary even point approximating ternary scheme. Ghaffar *et al.* examined 4-point a-ary approximating subdivision scheme that generate a family of subdivision schemes.

The solution of 2-point b.v.p (boundary value problem) by interpolating subdivision scheme was introduced by Mustafa and Ejaz [18]. The solution of fourth order boundary value problem by interpolating and approximating presented by Ejaz *et al*. [8]. Different techniques and properties of the curve subdivision scheme was proposed in [1,3,5,7,13,15,17,23]. Ejaz *et al.* [19] presented an iterative collocation algorithm for nonlinear fourth order b.v.p illustrating the convergence of the scheme.

2. Approximating Schemes for Curve Design

We define the collocation algorithm by using 6-point approximating subdivision scheme to calculate the estimated solution of two-point second order boundary values problems. The collocation algorithm with the adjustment of the septic polynomial gives the approximation power $O(h^2)$ at the end points. Our reformulated collocation algorithm treats the following kind of two-point point second order boundary value problem:

$$\begin{aligned} &w''(r) = a(r)w(r) + b(r), \quad 0 \leq r \leq 1 \\ &w(0) = w_q, \qquad\qquad w(1) = w_l, \end{aligned} \tag{2.1}$$

where $a(r) \geq 0$, $a(r)$ and $b(r)$ are continuous on $[0,1]$.

We have following 6-point approximating subdivision scheme:

$$\begin{aligned} g^{j+1}_{2i} &= \frac{7}{128}g^j_{i-2} + \frac{36}{128}g^j_{i-1} + \frac{42}{128}g^j_i + \frac{36}{128}g^j_{i+1} + \frac{7}{128}g^j_{i+2}, \\ g^{j+1}_{2i+1} &= \frac{1}{128}g^j_{i-2} + \frac{21}{128}g^j_{i-1} + \frac{42}{128}g^j_i + \frac{42}{128}g^j_{i+1} + \frac{21}{128}g^j_{i+2} + \frac{1}{128}g^j_{i+3}. \end{aligned} \tag{2.2}$$

Remark. Let μ be the limit curve generated from the cardinal data $\{p_i = (i, \delta_0)^T\}$; that is, $\mu(r)$ is the fundamental solution of the subdivision scheme (2.2); then

$$\mu(i) = \begin{cases} 1, & i = 0 \\ 0, & i \neq 0. \end{cases} \tag{2.3}$$

Furthermore, $\mu(r)$ satisfies the following two scale equation:

$$\begin{aligned}\mu(r) &= \mu_n(r)\\ &= \mu(2r) + \sum_{j=-n}^{n} L_{n,|j|}\mu(2r-2j+1), \quad r \in \mathbb{R}.\end{aligned} \tag{2.4}$$

Lemma 2.1 ([20, 21]). *The support of the fundamental solution $\mu_n(r)$ to scheme (2.2) is finite. Explicitly, support $\mu_n(r) = (-2n-1, 2n+1)$.*

Lemma 2.2 ([22]). *Given a square matrix A of order n, let the normalized left and right (generalized) eigenvectors of A be denoted by $\{\eta_i, \kappa_i\}$. Then for any vector $f \in \mathbb{R}^n$, there exists following Fourier expansion:*

$$f = \sum_{i=1}^{n} (f^T \eta_i)\kappa_i. \tag{2.5}$$

3. Subdivision Matrix

If S is subdivision matrix of scheme (2.2), then

$$S = \begin{pmatrix}
\frac{7}{128} & \frac{9}{32} & \frac{21}{64} & \frac{9}{32} & \frac{7}{128} & 0 & 0 & 0 & 0\\
\frac{1}{128} & \frac{21}{128} & \frac{21}{64} & \frac{21}{64} & \frac{21}{128} & \frac{1}{128} & 0 & 0 & 0\\
0 & \frac{7}{128} & \frac{9}{32} & \frac{21}{64} & \frac{9}{32} & \frac{7}{128} & 0 & 0 & 0\\
0 & \frac{1}{128} & \frac{21}{128} & \frac{21}{64} & \frac{21}{64} & \frac{21}{128} & \frac{1}{128} & 0 & 0\\
0 & 0 & \frac{7}{128} & \frac{9}{32} & \frac{21}{64} & \frac{9}{32} & \frac{7}{128} & 0 & 0\\
0 & 0 & \frac{1}{128} & \frac{21}{128} & \frac{21}{64} & \frac{21}{64} & \frac{21}{128} & \frac{1}{128} & 0\\
0 & 0 & 0 & \frac{7}{128} & \frac{9}{32} & \frac{21}{64} & \frac{9}{32} & \frac{7}{128} & 0\\
0 & 0 & 0 & \frac{1}{128} & \frac{21}{128} & \frac{21}{64} & \frac{21}{64} & \frac{21}{128} & \frac{1}{128}\\
0 & 0 & 0 & 0 & \frac{7}{128} & \frac{9}{32} & \frac{21}{64} & \frac{9}{32} & \frac{7}{128}
\end{pmatrix}, \tag{3.1}$$

its some real eigenvalues are

$$\zeta = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{-1}{128}. \tag{3.2}$$

For eigenvectors κ and η, and the eigenvalues ζ, condition $\eta S^T = \eta\zeta$ and $S\kappa = \zeta\kappa$ holds and are called left where κ and η right eigenvectors of the given subdivision matrix S, respectively. Few of normalized left and right eigenvectors given as corresponding to the first three eigenvalues are

$$\left.\begin{aligned}
&\kappa_0 = (1,1,1,1,1,1,1,1,1)^T, \eta_0 = \frac{1}{15120}(1,121,1312,3727,4798,4798,3727,1312,121,1)^T,\\
&\kappa_1 = (-4,-3,-2,-1,0,1,2,3,4)^T, \eta_1 = \frac{1}{2160}(-1,-57,-302,-301,0,301,302,57,1)^T,\\
&\kappa_2 = (-44,-23,-8,1,4,1,-8,-23,-44)^T, \eta_2 = \frac{1}{-2160}(1,25,40,-41,-50,-41,40,25,1)^T.
\end{aligned}\right\} \tag{3.3}$$

Since $\kappa^T \eta_j = 1$ for $i = j$ and 0 otherwise then by applying Lemmas 2.1 and 2.2, we obtain the following result.

Lemma 3.1. *The fundamental solution (Cardinal basis) $\mu(r)$ is 2 times continuously differentiable and supported on $[-4,4]$ and its derivatives at integers are given by*

$$\begin{aligned} \mu'(i) &= 2\ \mathrm{sign}(i) e_{|i|}^T \eta_1, \\ \mu''(i) &= 2^2 e_{|i|}^T \eta_2, \qquad -4 \le i \le 4, \end{aligned} \tag{3.4}$$

where

$$\left.\begin{aligned} &e_0 = (0,0,0,0,1,0,0,0,0)^T, \qquad e_1 = (0,0,0,1,0,0,0,0,0)^T, \\ &e_2 = (0,0,1,0,0,0,0,0,0)^T, \qquad e_3 = (0,1,0,0,0,0,0,0,0)^T, \\ &e_4 = (1,0,0,0,0,0,0,0,0)^T, \end{aligned}\right\} \tag{3.5}$$

$$\left.\begin{aligned} &\mu'(0) = 0, \quad \mu'(\pm 1) = \mp\frac{301}{1080}, \quad \mu'(\pm 2) = \mp\frac{151}{540}, \quad \mu'(\pm 3) = \mp\frac{19}{360}, \quad \mu'(\pm 4) = \mp\frac{1}{1080}, \\ &\mu''(0) = \tfrac{5}{54}, \quad \mu''(\pm 1) = \frac{41}{540}, \quad \mu''(\pm 2) = \frac{-2}{27}, \quad \mu''(\pm 3) = \frac{-5}{54}, \quad \mu''(\pm 4) = \frac{-1}{540}. \end{aligned}\right\} \tag{3.6}$$

By using (3.6) *we obtain the the numerical values of first and second derivative.*

4. Numerical Approximating Collocation Algorithm

Now first we define a numerical approximating collocation algorithm for linear second order two-point boundary value problems. Then we arrange the boundary conditions to obtain unique solution.

4.1 The Collocation Algorithm

Let N be a positive integer ($N \ge 4$), $h = \frac{1}{N}$, and $r_i = \frac{i}{N} = ih$, $i = 0,1,2,3,\dots,N$, and set $a_i = a(r_i)$, $b_i = b(r_i)$. Let

$$V(r) = \sum_{i=-4}^{N+4} V_i \mu\left(\frac{r-r_i}{h}\right), \quad 0 \le r \le 1 \tag{4.1}$$

be the assume approximate solution to (2.1), where unknown that is $\{V_i\}$ are to be determined by (2.1). The collocation algorithm, along with the boundary conditions to be explained, is settle by

$$V''(r_j) = a(r_j)V(r_j) + b(r_j), \qquad j = 0,1,2,3,\dots,N, \tag{4.2}$$

$$\begin{aligned} V'(r_j) &= \frac{1}{h}\sum_{i=-4}^{N+4} v_i \mu'\left(\frac{r_j - r_i}{h}\right), \\ V''(r_j) &= \frac{1}{h^2}\sum_{i=-4}^{N+4} v_i \mu''\left(\frac{r_j - r_i}{h}\right). \end{aligned} \tag{4.3}$$

Applying (4.1) and (4.3) in (4.2), we obtain following $N+1$ system of equations:

$$\begin{aligned} &\frac{1}{h^2}\sum_{i=-4}^{N+4} v_i \mu''\left(\frac{r_j - r_i}{h}\right) - a_j \sum_{i=-4}^{N+4} v_i \mu\left(\frac{r_j - r_i}{h}\right) = b_j, \\ \Rightarrow \ &\sum_{i=-4}^{N+4} v_i \mu''\left(\frac{r_j - r_i}{h}\right) - h^2 a_j \sum_{i=-4}^{N+4} v_i \mu\left(\frac{r_j - r_i}{h}\right) = h^2 b_j, \quad j = 0,1,2,\dots,N. \end{aligned} \tag{4.4}$$

Now we explain the above system of equations in the following theorems.

Theorem 4.1. *For $j=0$ (4.4), one gets*

$$v_{-4}\mu''_4+v_{-3}\mu''_3+v_{-2}\mu''_2+v_{-1}\mu''_1+v_0g_0+v_1\mu''_{-1}+v_2\mu'''_{-2}+v_3\mu''_{-3}+v_4\mu''_{-4}=h^2b_0, \tag{4.5}$$

where $\mu''_j=\mu''(j)$ and $g_0=\mu'''_0-a_0h^2$.

Proof. We have

$$\begin{aligned}&\Big\{v_{-4}\mu''\left(\frac{r_j-r_{-4}}{h}\right)+r_{-3}\mu''\left(\frac{r_j-r_{-3}}{h}\right)+r_{-2}\mu''\left(\frac{r_j-r_{-2}}{h}\right)+\dots\\&+r_{N+2}\mu''\left(\frac{r_j-r_{N+2}}{h}\right)+v_{N+3}\mu'''\left(\frac{x_j-r_{N+3}}{h}\right)+v_{N+4}\mu'''\left(\frac{r_j-r_{N+4}}{h}\right)\Big\}\\&-a_0h^2\Big\{v_{-4}\mu\left(\frac{r_j-r_{-4}}{h}\right)+v_{-3}\mu\left(\frac{r_j-r_{-3}}{h}\right)+v_{-2}\mu'''\left(\frac{r_j-r_{-2}}{h}\right)+\dots\\&+v_{N+2}\mu\left(\frac{r_j-r_{N+2}}{h}\right)+v_{N+3}\mu\left(\frac{r_j-r_{N+3}}{h}\right)+v_{N+4}\mu\left(\frac{r_j-r_{N+4}}{h}\right)\Big\}=h^2b_0.\end{aligned} \tag{4.6}$$

By putting $j=0$ in (4.4), we obtain

$$\begin{aligned}&\Big\{v_{-4}\mu''\left(\frac{r_0-r_{-4}}{h}\right)+r_{-3}\mu''\left(\frac{r_0-x_{-3}}{h}\right)+r_{-2}\mu''\left(\frac{r_0-r_{-2}}{h}\right)+\dots\\&+r_{N+2}\mu''\left(\frac{r_0-r_{N+2}}{h}\right)+v_{N+3}\mu'''\left(\frac{r_0-r_{N+3}}{h}\right)+v_{N+4}\mu'''\left(\frac{r_0-r_{N+4}}{h}\right)\Big\}\\&-a_0h^2\Big\{v_{-4}\mu\left(\frac{x_0-r_{-4}}{h}\right)+v_{-3}\mu\left(\frac{r_0-r_{-3}}{h}\right)+v_{-2}\mu'''\left(\frac{r_0-r_{-2}}{h}\right)+\dots\\&+v_{N+2}\mu\left(\frac{r_0-r_{N+2}}{h}\right)+v_{N+3}\mu\left(\frac{r_0-r_{N+3}}{h}\right)+v_{N+4}\mu\left(\frac{r_0-r_{N+4}}{h}\right)\Big\}=h^2b_0.\end{aligned} \tag{4.7}$$

For $x_j=jh$, $j=0,1,2,3,\dots,N$, this implies

$$\begin{aligned}&v_{-4}\mu''(4)+v_{-3}\mu''(3)+v_{-2}\mu''(2)+\dots+v_{N+3}\mu''(-N-3)+v_{N+4}\mu''(-N-4)\\&+a_0h^2\big[v_{-4}\mu(4)+v_{-3}\mu(3)+v_{-2}\mu(2)+\dots+v_{N+2}\mu(-N-2)+v_{N+3}\mu(-N-3)\\&+v_{N+4}\mu(-N-4)\big]=h^2b_0.\end{aligned} \tag{4.8}$$

Since $(-5,5)$ is the support of $\mu(r)$, $\mu'(r)$ and $\mu''(r)$ are zero away from the interval $(-5,5)$; also by (2.4) and (3.6), we obtain

$$\begin{aligned}&v_{-4}\mu''(4)+v_{-3}\mu''(3)+v_{-2}\mu''(2)+v_{-1}\mu''(1)+v_0\mu''(0)+v_1\mu''(-1)\\&+v_2\mu''(-2)+v_3\mu''(-3)+v_4\mu''(-4)-a_0h^2v_0\mu(0)=h^2b_0.\end{aligned} \tag{4.9}$$

If $\mu''_j=\mu''(j)$, then

$$v_{-4}\mu''_4+v_{-3}\mu''_3+v_{-2}\mu''_2+v_{-1}\mu''_1+v_0(\mu''_0-a_0h^2)+v_1\mu''_{-1}+v_2\mu''_{-2}+v_3\mu''_{-3}+v_4\mu''_{-4}=h^2b_0. \tag{4.10}$$

For $g_0=\mu''_0-a_0h^2$, we acquire (4.5). This complete the proof. □

Theorem 4.2. *For $j=1,2,3,\dots,N$ the system of equation* (4.4) *is equal to*

$$\begin{aligned}&v_{-4}\mu''_{j+4}+v_{-3}\mu''_{j+3}+\dots+v_0\mu''_j+v_1\left(\mu''_{j-1}-a_j\mu_{j-1}\right)+v_2\left(\mu''_{j-2}-a_jh^2\mu_{j-2}\right)+\dots\\&+v_{N-1}\left(\mu''_{j-N+1}-a_jh^2\mu_{j-N+1}\right)+v_N\left(\mu''_{j-N}-a_jh^2\mu_{j-N}\right)+z_{N+1}\mu''_{j-N-1}+\dots\\&+v_{N+4}\mu''_{j-N-4}=h^2b_j.\end{aligned} \tag{4.11}$$

Proof. By expending (4.4), we obtain

$$\Big\{v_{-4}\mu''\left(\frac{r_j-r_{-4}}{h}\right)+v_{-3}\mu''\left(\frac{r_j-r_{-3}}{h}\right)+v_{-2}\mu''\left(\frac{r_j-r_{-2}}{h}\right)+\dots$$

$$+v_{N+2}\mu''\left(\frac{r_j-r_{N+2}}{h}\right)+v_{N+3}\mu''\left(\frac{r_j-x_{N+3}}{h}\right)+v_{N+4}\mu''\left(\frac{r_j-r_{N+4}}{h}\right)\bigg\}$$
$$-a_0h^2\left\{v_{-4}\mu\left(\frac{r_j-r_{-4}}{h}\right)+v_{-3}\mu\left(\frac{r_j-r_{-3}}{h}\right)+v_{-2}\mu''\left(\frac{r_j-r_{-2}}{h}\right)+\dots\right.$$
$$\left.+v_{N+2}\mu\left(\frac{r_j-r_{N+2}}{h}\right)+v_{N+3}\mu\left(\frac{r_j-r_{N+3}}{h}\right)+v_{N+4}\mu\left(\frac{r_j-r_{N+4}}{h}\right)\right\}=h^2b_j. \tag{4.12}$$

For $r_j=jh$, $j=1,2,3,\dots,N$, we obtain

$$v_{-4}\mu'''(j+4)+v_{-3}\mu'''(j+3)+\dots+v_{N+3}\mu'''(j-N-3)+v_{N+4}\mu'''(j-N-4)$$
$$-a_jh^2\left\{v_{-4}\mu(j+4)+v_{-3}\mu(j+3)+\dots+v_{N+3}\mu(j-N-3)+v_{N+4}\mu(j-N-4)\right\}=h^2b_j. \tag{4.13}$$

If $\mu'''_j=\mu'''(j)$ for $j=1,2,3,\dots,N$,

$$v_{-4}\left(\mu'''_{j+4}-a_jh^2\mu_{j+4}\right)+v_{-3}\left(\mu'''_{j+3}-a_jh^2\mu_{j+3}\right)+\dots$$
$$+v_{N+3}\left(\mu'''_{j-N-3}-a_jh^2\mu_{j-N-3}\right)+v_{N+4}\left(\mu'''_{j-N-4}-a_jh^2\mu_{j-N-4}\right)=h^2b_j. \tag{4.14}$$

Since $\mu'(r)$ and $\mu''(r)$ are zero away from the interval $(-5,5)$ then from (2.4) and (3.6), we obtain (4.11). From (4.5) and (4.11), we obtain following undetermined $(N+1)$ system of equations with $(N+9)$ unknown $\{v_i\}$.

$$AV=D, \tag{4.15}$$

where $(N+1)\times(N+9)$, $(N+9)$ and $(N+1)$ are orders of matrices A, V, and D respectively, are given by

$$A=\begin{pmatrix} \mu''_4 & \mu''_3 & \mu''_2 & \mu''_1 & q_0 & \mu''_{-1} & \mu''_{-2} & \mu''_{-3} & \mu''_{-4} & \dots & 0 & 0 & 0\\ 0 & \mu''_4 & \mu''_3 & \mu''_2 & \mu''_1 & q_1 & \mu''_{-1} & \mu''_{-2} & \mu''_{-3} & \dots & 0 & 0 & 0\\ 0 & 0 & \mu''_4 & \mu''_3 & \mu''_2 & \mu''_1 & q_2 & \mu''_{-1} & \mu''_{-2} & \dots & 0 & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \mu''_{-2} & \mu''_{-3} & \mu''_{-4} \end{pmatrix} \tag{4.16}$$

$V=(v_{-4},v_{-3},v_{-2},\dots,v_{N+4})^T$ and $D=(b_0h^2,b_1h^2,b_3h^2,\dots,b_Nh^2)^T$, where $\mu''_j=\mu''(j)$ and $g_j=\mu''_0-a_jh^2$.

4.2 Adjustment of Boundary Conditions

$(N+1)\times(N+9)$ is the order of the coefficient matrix (4.16). We required eight more conditions to obtain unique solution of the system. If we suppose $v_0=t_q$ then two conditions can be obtained by applying following boundary conditions i.e:

$$v_0=w_q,\quad v_N=w_l. \tag{4.17}$$

Still we require six more conditions to obtain stable solution. As the polynomial regenerates polynomials of degree seven then the boundary conditions for solution (4.15) will be of order eight. The convenience is employed by discussing the values at the left and right end points with the use of septic polynomial $g(x)$ that is v_{-3},v_{-2},v_{-1} and v_{N+1},v_{N+2},v_{N+3} at (x_i,v_i), we have

$$v_{-i}=g(-r_i),\quad i=1,2,3, \tag{4.18}$$

where

$$g(r_i) = \sum_{j=1}^{6} \binom{6}{j} (-1)^{j+1} Z(x_{i-j}). \tag{4.19}$$

Since from (4.1) $V(x_i) = r_i$ for $i = 1,2,3$ then, by substituting r_i by $-r_i$, we have

$$g(-r_i) = \sum_{j=1}^{6} \binom{6}{j} (-1)^{j+1} v_{-i+j}. \tag{4.20}$$

Hence at the left end following boundary conditions can be used:

$$\sum_{j=0}^{6} \binom{6}{j} (-1)^{j} v_{-i+j} = 0, \quad i = 3,2,1. \tag{4.21}$$

For the right end similarly we can explain $v_i = g(-r_i)$, $i = N+1, N+2, N+3$, and

$$g(r_i) = \sum_{j=1}^{6} \binom{6}{j} (-1)^{j+1} v_{i-j}. \tag{4.22}$$

So at the right end we have following boundary conditions:

$$\sum_{j=0}^{6} \binom{6}{j} (-1)^{j} v_{i-j} = 0, \quad i = N+1, N+2, N+3. \tag{4.23}$$

Lastly, we obtain $(N+9)$ unknown $\{v_i\}$ with $(N+1)$ new system of linear equation, in which two equation obtained from (4.17), six equation from boundary conditions (4.21) and (4.23) and $N+1$ equations are achieved from (4.5) and (4.11):

$$BV = R, \tag{4.24}$$

where

$$B = (B_0^T, A^T, B_1^T)^T,$$

A is obtained from (4.16), and B_0 and B_1 are obtained from (4.17), (4.21), and (4.23)

$$B_0 = \begin{pmatrix} 0 & 1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \tag{4.25}$$

in matrix B_0 first three rows come from (4.21) and last row comes from (4.17) at $v_0 = w_q$. Consider

$$B_1 = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 \end{pmatrix}, \tag{4.26}$$

in B_1 first row comes from (4.17) at $v_N = w_l$ remaining three rows come from (4.23) and the matrices V and R are given as

$$V = \left(v_{-4}, v_{-3}, \dots, v_{N+3}, v_{N+4}\right)^T,$$
$$R = \left(0,0,0,W_q,D^T,W_l,0,0,0\right)^T. \tag{4.27}$$

5. Numerical Example

In this section second order boundary value problem is solved by using approximating subdivision scheme and absolute error is also calculate. Result is tabulated for the sake of evaluation. Graphical representation is also presented. We have,

$$\begin{aligned} &w''(r) = w(r^2), \quad 0 \le w \le 1, \\ &w(0) = 0, \qquad w(1) = 1. \end{aligned} \tag{5.1}$$

The analytic solution of given problem is

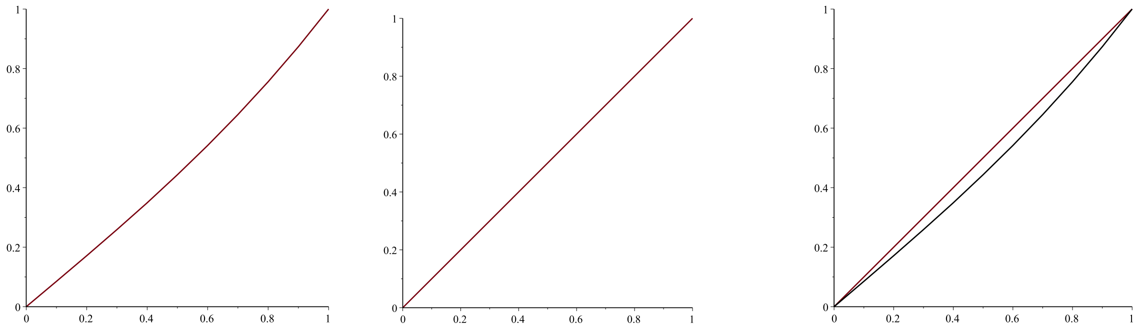
$$w(r) = \frac{e}{e^2-1}(e^w) - \frac{e}{e^2-1}(e^{-w}). \tag{5.2}$$

Now we present the numerical solution of the above problem by approximating subdivision scheme. We solve the above problem at step size 10. We obtained the following solution of the given problem: $V_j = \sum_{i=-4}^{N+4} v_i \phi(j-i)$, we obtain values of $\{v_{-4}, v_{-3}, v_{-2}, \dots, v_{13}, v_{14}\}$ by applying (4.24) are

$$\begin{aligned} &-0.399999999844310, -0.299999999891543, -0.199999999923264, -0.0999999999603605, \\ &0, 0.0999999999603885, 0.199999999923213, 0.299999999890819, 0.399999999865585, \\ &0.499999999850024, 0.599999999846413, 0.699999999857199, 0.799999999884802, \\ &0.899999999931609, 1, 1.100000000092381, 1.20000000021123, 1.300000000035910, \\ &1.40000000052849. \end{aligned} \tag{5.3}$$

Table 1. Solutions and errors estimation of Example

w_i	Analytic sol W	Approximating sol V by (4.24)	Absolute err by (4.24)
0	0	0	0
0.1	0.08523370	0.0999999999603885	0.014766299
0.2	0.17132045	0.199999999923213	0.028679549
0.3	0.2591218382	0.299999999890819	0.040878161
0.4	0.349516002	0.399999999865585	0.050483997
0.5	0.443409442	0.499999999850024	0.056590557
0.6	0.5417400745	0.599999999846413	0.058259925
0.7	0.6454926237	0.699999999857199	0.054507376
0.8	0.75570548	0.799999999884802	0.026518309
0.9	0.8734816908	0.899999999931609	0.026518391
1	1	1	0



(a) Analytic solution (b) Approximate solution (c) Analytic and approximate solution

Figure 1. Comparison between analytic and approximate solution

Boundary adjustments gives the approximate solution of (4.24) presented in Table 1 with the absolute errors calculated. These solutions are graphically represented in Figure 1. Figure 1(a) represents the analytic solution and the approximate solution is given by Figure 1(b). The comparison between the analytic and approximate solution is represented by Figure 1(c).

6. Conclusion and Future Work

This research is devoted to the solution of numerical problem of second order differential equation using 6-point approximating subdivision scheme. Adjustment of the boundary conditions are obtained by using septic polynomial at the end points which will give a number of equations that will be further more solved for the values of unknown by applying numerical method. The adjustment of boundary conditions has an influence on the approximation of solution for the considered problem. The proper adjustment of the boundary conditions improves the order of accuracy. The improvement of approximation order is expected in future for betterment of solution.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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