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Research Article

Strongly Solid Varieties in Many-Sorted Algebras

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Abstract. In this paper, we apply the general theory of conjugate pairs of additive closure operators to characterize the strongly solid variety which is extended from one-sorted algebras to many-sorted algebras. Moreover, we give the concept of V-normal form which is useful for testing the strongly solid variety in many-sorted algebra.

Keywords. Many-sorted algebra; i-sorted Σ-generalized hypersubstitution; i-sorted Σ-algebras; Σ-terms; Σ-identity

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1. Introduction

Universal algebra is a branch of mathematics which can be applied to theoretical computer science. It can be used to describe the abstract data type. For example, colors, as we know all colors can be created by mixing the primary colors together. If we let the mixing of two colors and the mixing ratio be the operations and the collection of all colors and the amount of each color added be the base sets, then we can explain this situation using many-sorted algebra. For the usual definition of algebra, when we speak about an algebra, we always imagine an algebra which has only one base set. It is very interesting to study an algebra which has more than one base set and all of operations can be defined on different base sets. The concept of many-sorted algebras was introduced in 1970 by G. Birkhoff and John D. Lipson [1]. A vector space $\mathscr{V}$ over field $\mathbb{F}$ is one of examples of many-sorted algebra.

One of the most important study in Universal algebra is to classify algebras into varieties and to classify varieties into hypervarieties. A variety $\mathscr{V}$ is a class of algebras satisfying some equations. That means there is a class of equations K such that every equation in K holds for all algebras in $\mathscr{V}$. In some cases, there are some other equations which hold for all algebras in such variety, although they do not belong to K. This can be happenned because when we substitute operations of any equation in K with terms, the result is still an equation which holds for all algebra in $\mathscr{V}$. In this case, $\mathscr{V}$ is called a hypervariety. So, we can classify varieties into hypervarieties.

In one-sorted algebra, there are many papers focus on hypersubstitution and hyperidentity. In 2008, K. Denecke and S. Lekkoksung introduced the concept of terms and hypersubstitutions in many-sorted algebras, they proved some properties of hypersubstitutions and characterized the solid varieties of many-sorted algebras (see in [2, 3]).

Let I be a nonempty set, $I^* := \bigcup\limits_{n\geq 1} I^n$, $\Sigma \subseteq I^* \times I$ and $\Sigma_n := \Sigma \cap I^{n+1}$. For $\gamma \in I^*$, let $\gamma(j)$ denote the j-th component of γ. For $i \in I$, let $\Sigma_m(i) := \{\gamma \in \Sigma_m \mid \gamma(m+1) = i\}$ and $\Sigma(i) := \bigcup\limits_{n=1}^{\infty} \Sigma_m(i)$. We set $\Lambda_n(i) := \{\alpha \in I^{n+1} \mid \alpha(n+1) = i\}$, $\Lambda(i) := \bigcup\limits_{n=1}^{\infty} \Lambda_n(i)$ and $\Lambda := \bigcup\limits_{i\in I} \Lambda(i)$.

Let $A := (A_i)_{i\in I}$ which is called an I-sorted set or an I-indexed family of sets, where A_i is a set of elements of sort i of A, for $i \in I$. A structure $\mathscr{A} := (A, ((f_\gamma^{\mathscr{A}})_k)_{k\in K_\gamma, \gamma\in\Sigma})$ is called an I-sorted Σ-algebra where $f_\gamma^A : A_{k_1} \times \ldots \times A_{k_n} \to A_i$ is a mapping which is called an I-sorted n-ary operation on A, where $\gamma := (k_1, \ldots, k_n, i) \in \Sigma_n(i)$ and K_γ be the set of indices with respect to γ. Denote $Alg(\Sigma)$ the set of all I-sorted Σ-algebras.

Example 1.1. A vector space $\mathscr{V}$ over field F:
The structure $\mathscr{V} := \left(\{V, F\}, \left\{+^A_{(1,1,1)}, \cdot^A_{(2,1,1)}\right\}\right)$ is an I-sorted Σ-algebra.

Definition 1.2. Let I be an indexed set and $n \in \mathbb{N}^+$. Let $X^{(n)} := (X_i^{(n)})_{i\in I}$ which is called an I-sorted set of n variables or an n-element I-sorted alphabet, where $X_i^{(n)} = \{x_{i1}, x_{i2}, \ldots, x_{in}\}$, $i \in I$ and let $X := (X_i)_{i\in I}$ which is called an I-sorted set of variables or an I-sorted alphabet, where $X_i = \{x_{i1}, x_{i2}, x_{i3}, \ldots\}$, $i \in I$. Let $((f_\gamma)_k)_{k\in K_\gamma, \gamma\in\Sigma}$ be a set of I-sorted operation symbols. Then for each $i \in I$, a set $W_n(i)$ which is called the set of all n-ary Σ-terms of sort i, is a set that inductively defined as follows:

(1) $W_0^n(i) := X_i^{(n)}$,

(2) $W_{l+1}^n(i) := W_l^n(i) \cup \{f_\gamma(t_{k_1}, \ldots, t_{k_n}) \mid \gamma = (k_1, \ldots, k_n, i) \in \Sigma,\ t_{k_j} \in W_l^n(k_j)\}$, $l \in \mathbb{N}$. Here we inductively assume that the set $W_l^n(i)$ are already defined for sort $i \in I$.

Then $W_n(i) := \bigcup\limits_{l=0}^{\infty} W_l^n(i)$. Let $W(i) := \bigcup\limits_{n\in\mathbb{N}} W_n(i)$ which is called an I-sorted set of all Σ-terms of sort i. The set $W_\Sigma(X) := (W(i))_{i\in I}$ is called an I-sorted set of all Σ-terms and its elements are called I-sorted Σ-terms.

The aim of this paper is to characterized the strongly solid varieties by using the theory of conjugate pairs of additive closure operators. We first introduce the concept of Σ-generalized hypersubstitutions in many-sorted algebras and we need a monoid structure on a set of Σ-generalized hypersubstitutions as a first step. Using the concept of generalized hypersubstitutions for one-sorted algebras which was introduced by Leeratanavalee and Denecke [4], we defined a Σ-generalized hypersubstitution as follow;

For each $i \in I$, a Σ-generalized hypersubstitution of sort i is an arbitrary mapping

$$\sigma_i : \{f_\gamma \mid \gamma \in \Sigma(i)\} \to W(i).$$

The set of all Σ-generalized hypersubstitutions of sort i is denoted by $\Sigma(i)$-Hyp_G. We call $\sigma := (\sigma_i)_{i\in I}$ an I-sorted Σ-generalized hypersubstitution and let Σ-$Hyp_G := (\Sigma(i)\text{-}Hyp_G)_{i\in I}$ be the set of all I-sorted Σ-generalized hypersubstitutions. Define the superposition operation

$$S_\beta : W(i) \times W(k_1) \times \ldots \times W(k_n) \to W(i),$$

for $\beta = (k_1,\ldots,k_n,i) \in \Lambda$, by the following steps:

(1) If $t = x_{ij} \in X_i$, then

  (i) $S_\beta(x_{ij},t_1,\ldots,t_n) = x_{ij}$ if $i \neq k_j$, $\forall\ j$,

  (ii) $S_\beta(x_{ij},t_1,\ldots,t_n) = t_j$ if $i = k_j$, $1 \le j \le n$,

  (iii) $S_\beta(x_{ij},t_1,\ldots,t_n) = x_{ij}$ if $j > n$.

(2) If $t = f_\gamma(s_1,\ldots,s_m) \in W(i)$, for $\gamma = (i_1,\ldots,i_m,i) \in \Sigma$ and $s_q \in W(i_q)$, $1 \le q \le m$. Assume that $S_{\beta_q}(s_q,t_1,\ldots,t_n)$ with $\beta_q = (k_1,\ldots,k_n,i) \in \Lambda(i_q)$ are already defined, then for $t_j \in W(k_j)$, $1 \le j \le n$,

$$S_\beta(f_\gamma(s_1,\ldots,s_m),t_1,\ldots,t_n) := f_\gamma(S_{\beta_1}(s_1,t_1,\ldots,t_n),\ldots,S_{\beta_m}(s_m,t_1,\ldots,t_n)).$$

Every $\sigma_i \in \Sigma(i)$-Hyp_G can be extended to a mapping $\hat{\sigma}_i : W(i) \to W(i)$ which is definded by the following steps:

(1) $\hat{\sigma}_i[x_{ij}] := x_{ij}$, for $x_{ij} \in X_i$,

(2) $\hat{\sigma}_i[f_\gamma(t_1,\ldots,t_n)] := S_\gamma(\sigma_i(f_\gamma),\hat{\sigma}_{k_1}[t_1],\ldots,\hat{\sigma}_{k_n}[t_n])$ where $\gamma = (k_1,\ldots,k_n,i)$ and $t_j \in W(k_j)$, $1 \le j \le n$. Assume that $\hat{\sigma}_{k_j}[t_j]$ are already defined.

So we can define a binary operation $\circ^i_G$ on $\Sigma(i)$-Hyp_G by $(\sigma_1)_i \circ^i_G (\sigma_2)_i := (\hat{\sigma_1})_i \circ (\sigma_2)_i$, for $(\sigma_1)_i,(\sigma_2)_i \in \Sigma(i)$-$Hyp_G$ and $\circ$ is the usual composition of mappings. Let $(\sigma_{id})_i \in \Sigma(i)$-$Hyp_G$ which maps each operation symbol f_γ to the Σ-term $f_\gamma(x_{k_1 1},\ldots,x_{k_n n})$, for $\gamma = (k_1,\ldots,k_n,i) \in \Sigma(i)$, i.e., $(\sigma_{id})_i(f_\gamma) := f_\gamma(x_{k_1 1},\ldots,x_{k_n n})$.

In general, there are many examples which show that operation $\circ^i_G$ is not associative. That is, $(\Sigma(i)\text{-}Hyp_G,\circ^i_G,(\sigma_{id})_i)$ is not a monoid. So, we will focus on special properties which make this structure being a monoid.

For each $i \in I$ and fixed $n \in \mathbb{N}$ with $n \ge 2$, let $\Sigma^{|I|,n}(i) \subseteq \Sigma(i)$ be a set of operation structures with all of operation symbols of sort i have the same arity n and the same structure. That is $\{\gamma\} = \Sigma(i)$ and each $k \in K_\gamma$, $(f_\gamma)_k$ is n-ary.

We can show that $(\Sigma^{|I|,n}(i)\text{-}Hyp_G,\circ^i_G,(\sigma_{id})_i)$ forms a monoid by the following:

Proposition 1.3. *For $(\sigma_1)_i,(\sigma_2)_i,(\sigma_3)_i \in \Sigma^{|I|,n}(i)$-$Hyp_G$ and $\alpha=(k_1,\dots,k_n,i)\in\Lambda$. Then*

(1) $\hat{\sigma}_i[S_\alpha(t,t_1,\dots,t_n)] = S_\alpha(\hat{\sigma}_i[t],\hat{\sigma}_{k_1}[t_1],\dots,\hat{\sigma}_{k_n}[t_n])$,

(2) $((\sigma_1)_i \circ^i_G (\sigma_2)_i)\hat{} = (\hat{\sigma}_1)_i \circ (\hat{\sigma}_2)_i$,

(3) $((\sigma_1)_i \circ^i_G (\sigma_2)_i)\circ^i_G (\sigma_3)_i = (\sigma_1)_i \circ^i_G ((\sigma_2)_i \circ^i_G (\sigma_3)_i)$,

(4) $(\sigma_1)_i \circ^i_G (\sigma_{id})_i = (\sigma_{id})_i \circ^i_G ((\sigma_1)_i = (\sigma_1)_i$.

Theorem 1.4. $(\Sigma^{|I|,n}(i)$-$Hyp_G, \circ^i_G, (\sigma_{id})_i)$ *is a monoid.*

Next, we introduce the concept and give some properties of Σ-term operation in many-sorted algebras.

Definition 1.5. Let $n,q\in\mathbb{N}^+$ with $1\le q\le n$. The q-th n-ary projection operation with input $\omega=(k_1,\dots,k_n)$ on $\mathscr{A}$ is a mapping

$$e^{\omega,A}_q : A_{k_1}\times\dots\times A_{k_n}\to A,$$

which is defined by $e^{\omega,A}_q(a_1,\dots,a_n)=a_q$.

Let $\mathscr{A}\in Alg(\Sigma)$ be a Σ-algebra and $i\in I$. For fixed $n\in\mathbb{N}$, let $f_i : X^{(n)}_i\to A_i$ be an evaluation mapping. We can extend each f_i to a homomorphism $\bar{f}_i : W(i)\to A_i$ which is defined by

$$\bar{f}_i(t)=\begin{cases} f_i(x_{ij}), & \text{if } t=x_{ij}\in X_i \text{ and } 1\le j\le n,\\ x_{ij}, & \text{if } t=x_{ij}\in X_i \text{ and } j>n,\\ f^{\mathscr{A}}_\gamma(\bar{f}_{i_1}(t_1),\dots,\bar{f}_{i_m}(t_m)), & \text{if } t=f_\gamma(t_1,\dots,t_m)\in W(i),\ \gamma=(i_1,\dots,i_m,i).\end{cases}$$

We set $f:=(f_i)_{i\in I}$ and $A^{X^{(n)}} := \{f:=(f_i)_{i\in I} \mid f_i : X^{(n)}_i\to A_i, \text{for } i\in I\}$.

Let $t\in W(i)$, define $t^{\mathscr{A}} : A^{X^{(n)}}\to A_i$ by $t^{\mathscr{A}}(f)=\bar{f}_i(t)$, that is

$$t^{\mathscr{A}}(f)=\begin{cases} x^{\mathscr{A}}_{ij}(f)=f_i(x_{ij})=e^{\omega,\mathscr{A}}_j, & \text{if } t=x_{ij} \text{ with } 1\le j\le n,\\ x_{ij}, & \text{if } t=x_{ij} \text{ and } j>n,\\ f^{\mathscr{A}}_\gamma(t^{\mathscr{A}}_1(f),\dots,t^{\mathscr{A}}_m(f)) := f^{\mathscr{A}}_\gamma(t^{\mathscr{A}}_1,\dots,t^{\mathscr{A}}_m)(f), & \text{if } t=f_\gamma(t_1,\dots,t_m)\\ & \text{ with } \gamma=(i_1,\dots,i_m,i).\end{cases}$$

The operation $t^{\mathscr{A}}$ is called the Σ-term operation on $\mathscr{A}$ induced by the Σ-term t of sort i and denote $W^{\mathscr{A}}(i)$ the set of all Σ-term operations on $\mathscr{A}$ induced by the Σ-term of sort i.

Definition 1.6. The superposition operation

$$S^{\mathscr{A}}_\beta : W^{\mathscr{A}}(i)\times W^{\mathscr{A}}(k_1)\times\dots\times W^{\mathscr{A}}(k_n)\to W^{\mathscr{A}}(i),$$

for $\beta=(k_1,\dots,k_n,i)\in\Lambda$, is defined inductively by the following steps:

(1) If $t=x_{ij}\in X_i$, then

(i) $S_\beta(x^{\mathscr{A}}_{ij},t^{\mathscr{A}}_1,\dots,t^{\mathscr{A}}_n)=x^{\mathscr{A}}_{ij}$ if $i\ne k_j,\forall j$ and,

(ii) $S_\beta(x^{\mathscr{A}}_{ij},t^{\mathscr{A}}_1,\dots,t^{\mathscr{A}}_n)=t^{\mathscr{A}}_j$ if $i=k_j, 1\le j\le n$ and,

(iii) $S_\beta(x^{\mathscr{A}}_{ij},t^{\mathscr{A}}_1,\dots,t^{\mathscr{A}}_n)=x^{\mathscr{A}}_{ij}=x_{ij}$ if $j>n$.

(2) If $t = f_\gamma(s_1,\dots,s_m) \in W(i)$, for $\gamma = (i_1,\dots,i_m,i) \in \Sigma$ and $s_q \in W(i_q)$, $1 \le q \le m$. Assume that $S^{\mathscr{A}}_{\beta_q}(s^{\mathscr{A}}_q, t^{\mathscr{A}}_1,\dots,t^{\mathscr{A}}_n)$ with $\beta_q = (k_1,\dots,k_n,i) \in \Lambda(i_q)$ are already defined, then

$$S^{\mathscr{A}}_\beta(f_\gamma(s_1,\dots,s_m)^{\mathscr{A}}, t^{\mathscr{A}}_1,\dots,t^{\mathscr{A}}_n) := f^{\mathscr{A}}_\gamma(S^{\mathscr{A}}_{\beta_1}(s^{\mathscr{A}}_1, t^{\mathscr{A}}_1,\dots,t^{\mathscr{A}}_n),\dots,S^{\mathscr{A}}_{\beta_m}(s^{\mathscr{A}}_m, t^{\mathscr{A}}_1,\dots,t^{\mathscr{A}}_n)),$$

for $t_j \in W(k_j)$, $1 \le j \le n$.

Lemma 1.7. *Let $\mathscr{A} \in Alg(\Sigma)$ and $\alpha = (k_1,\dots,k_m,i) \in \Lambda$,*

$$S^{\mathscr{A}}_\alpha(t^{\mathscr{A}}, t^{\mathscr{A}}_1,\dots,t^{\mathscr{A}}_m) = (S_\alpha(t,t_1,\dots,t_m))^{\mathscr{A}}.$$

Proof. We prove by induction on the complexity of Σ-term t of sort i.
If $t = x_{ij} \in X(i)$,

Case 1: $i \neq k_j$,

$$\begin{aligned} S^{\mathscr{A}}_\alpha(t^{\mathscr{A}}, t^{\mathscr{A}}_1,\dots,t^{\mathscr{A}}_m) = S^{\mathscr{A}}_\alpha(x^{\mathscr{A}}_{ij}, t^{\mathscr{A}}_1,\dots,t^{\mathscr{A}}_m) = x^{\mathscr{A}}_{ij} &= (S_\alpha(x_{ij},t_1,\dots,t_m))^{\mathscr{A}} \\ &= (S_\alpha(t,t_1,\dots,t_m))^{\mathscr{A}}. \end{aligned}$$

Case 2: $i = k_j$, $1 \le j \le m$,

$$\begin{aligned} S^{\mathscr{A}}_\alpha(t^{\mathscr{A}}, t^{\mathscr{A}}_1,\dots,t^{\mathscr{A}}_m) = S^{\mathscr{A}}_\alpha(x^{\mathscr{A}}_{ij}, t^{\mathscr{A}}_1,\dots,t^{\mathscr{A}}_m) = t^{\mathscr{A}}_j &= (S_\alpha(x_{ij},t_1,\dots,t_m))^{\mathscr{A}} \\ &= (S_\alpha(t,t_1,\dots,t_m))^{\mathscr{A}}. \end{aligned}$$

Case 3: $j > m$,

$$\begin{aligned} S^{\mathscr{A}}_\alpha(t^{\mathscr{A}}, t^{\mathscr{A}}_1,\dots,t^{\mathscr{A}}_m) = S^{\mathscr{A}}_\alpha(x^{\mathscr{A}}_{ij}, t^{\mathscr{A}}_1,\dots,t^{\mathscr{A}}_m) = x^{\mathscr{A}}_{ij} &= (S_\alpha(x_{ij},t_1,\dots,t_m))^{\mathscr{A}} \\ &= (S_\alpha(t,t_1,\dots,t_m))^{\mathscr{A}}. \end{aligned}$$

If $t = f_\gamma(s_1,\dots,s_n) \in W(i)$ with $\gamma = (i_1,\dots,i_n,i) \in \Sigma^{|I|,n}(i)$. Assume that

$$S^{\mathscr{A}}_{\alpha_q}(s^{\mathscr{A}}_q, t^{\mathscr{A}}_1,\dots,t^{\mathscr{A}}_m) = (S_{\alpha_q}(s_q,t_1,\dots,t_m))^{\mathscr{A}},$$

for all $\alpha_q = (k_1,\dots,k_m,i_q)$. Then,

$$\begin{aligned} S^{\mathscr{A}}_\alpha(t^{\mathscr{A}}, t^{\mathscr{A}}_1,\dots,t^{\mathscr{A}}_m) &= S^{\mathscr{A}}_\alpha((f_\gamma(s_1,\dots,s_n))^{\mathscr{A}}, t^{\mathscr{A}}_1,\dots,t^{\mathscr{A}}_m) \\ &= f^{\mathscr{A}}_\gamma(S^{\mathscr{A}}_{\alpha_1}(s^{\mathscr{A}}_1, t^{\mathscr{A}}_1,\dots,t^{\mathscr{A}}_m),\dots,S^{\mathscr{A}}_{\alpha_n}(s^{\mathscr{A}}_n, t^{\mathscr{A}}_1,\dots,t^{\mathscr{A}}_m)) \\ &= f^{\mathscr{A}}_\gamma((S_{\alpha_1}(s_1,t_1,\dots,t_m))^{\mathscr{A}},\dots,(S_{\alpha_n}(s_n,t_1,\dots,t_m))^{\mathscr{A}}) \\ &= (f_\gamma(S_{\alpha_1}(s_1,t_1,\dots,t_m),\dots,S_{\alpha_n}(s_n,t_1,\dots,t_m)))^{\mathscr{A}} \\ &= (S_\alpha(f_\gamma(s_1,\dots,s_n),t_1,\dots,t_m))^{\mathscr{A}} \\ &= (S_\alpha(t,t_1,\dots,t_m))^{\mathscr{A}}. \end{aligned}$$

So $S^{\mathscr{A}}_\alpha(t^{\mathscr{A}}, t^{\mathscr{A}}_1,\dots,t^{\mathscr{A}}_m) = (S_\alpha(t,t_1,\dots,t_m))^{\mathscr{A}}$. □

Let $\mathscr{A} := (A, ((f^{\mathscr{A}}_\gamma)_k)_{k \in K_\gamma, \gamma \in \Sigma})$ be a Σ-algebra and $\sigma \in \Sigma^{|I|,n}$-$Hyp_G$. The Σ-algebra derived from $\mathscr{A}$ by σ is a Σ-algebra which consists of A together with family of operations $((\sigma_i(f_\gamma)_k)^{\mathscr{A}})_{k \in K_\gamma, \gamma \in \Sigma(i)}$, i.e.,

$$\sigma(\mathscr{A}) := (A, ((\sigma_i(f_\gamma)_k)^{\mathscr{A}})_{k \in K_\gamma, \gamma \in \Sigma(i)}).$$

Lemma 1.8. *Let $\mathscr{A} \in Alg(\Sigma)$ and $\sigma \in \Sigma^{|I|,n}$-$Hyp_G$. For $t \in W(i)$, we have $t^{\sigma(\mathscr{A})} = (\hat{\sigma}_i[t])^{\mathscr{A}}$.*

Proof. For $f \in A^{X^{(n)}}$. We prove by induction on the complexity of $t \in W(i)$.
If $t = x_{ij} \in X_i$,

$$t^{\sigma(\mathscr{A})}(f) = x_{ij}^{\sigma(\mathscr{A})}(f) = x_{ij}^{\mathscr{A}}(f) = (\hat{\sigma}_i[x_{ij}])^{\mathscr{A}}(f) = (\hat{\sigma}_i[t])^{\mathscr{A}}(f).$$

If $t = f_\gamma(s_1,\dots,s_n) \in W(i)$, for $\gamma = (i_1,\dots,i_n,i) \in \Sigma^{|I|,n}$ and $s_q \in W(i_q), 1 \le q \le n$, and assume that $s_q^{\sigma(\mathscr{A})} = (\hat{\sigma}_{i_q}[s_q])^{\mathscr{A}}$. Then

$$\begin{aligned} t^{\sigma(\mathscr{A})}(f) = (f_\gamma(s_1,\dots,s_n))^{\sigma(\mathscr{A})}(f) &= f_\gamma^{\sigma(\mathscr{A})}(s_1^{\sigma(\mathscr{A})},\dots,s_n^{\sigma(\mathscr{A})})(f) \\ &= \sigma_i(f_\gamma)^{\mathscr{A}}((\hat{\sigma}_{i_1}[s_1])^{\mathscr{A}},\dots,(\hat{\sigma}_{i_n}[s_n])^{\mathscr{A}})(f) \\ &= (\sigma_i(f_\gamma)(\hat{\sigma}_{i_1}[s_1],\dots,\hat{\sigma}_{i_n}[s_n]))^{\mathscr{A}}(f) \\ &= (S_\gamma(\sigma_i(f_\gamma),\hat{\sigma}_{i_1}[s_1],\dots,\hat{\sigma}_{i_n}[s_n]))^{\mathscr{A}}(f) \\ &= (\hat{\sigma}_i[f_\gamma(s_1,\dots,s_n)])^{\mathscr{A}}(f) = (\hat{\sigma}_i[t])^{\mathscr{A}}. \end{aligned}$$ □

Lemma 1.9. *Let $\mathscr{A} \in Alg(\Sigma)$ and $\sigma_1,\sigma_2 \in \Sigma^{|I|,n}$-$Hyp_G$. For $i \in I$ and $\gamma \in \Sigma^{|I|,n}(i)$, then*

$$((\sigma_1)_i(f_\gamma))^{\sigma_2(\mathscr{A})} = (((\sigma_2)_i \circ_G^i (\sigma_1)_i)(f_\gamma))^{\mathscr{A}}.$$

Proof. By the previous lemma,

$$((\sigma_1)_i(f_\gamma))^{\sigma_2(\mathscr{A})} = ((\hat{\sigma}_2)_i[(\sigma_1)_i(f_\gamma)])^{\mathscr{A}} = (((\sigma_2)_i \circ_G^i (\sigma_1)_i)(f_\gamma))^{\mathscr{A}}.$$ □

For $\sigma_1,\sigma_2 \in \Sigma^{|I|,n}$-$Hyp_G$, we define $\sigma_1 \diamond \sigma_2 := (((\sigma_1)_i \circ_G^i (\sigma_2)_i))_{i\in I}$.

Lemma 1.10. *Let $\mathscr{A} \in Alg(\Sigma)$ and $\sigma_1,\sigma_2 \in \Sigma^{|I|,n}$-$Hyp_G$. Then*

$$\sigma_1(\sigma_2(\mathscr{A})) = (\sigma_2 \diamond \sigma_1)(\mathscr{A}).$$

Proof. From the previous lemma,

$$\begin{aligned} \sigma_1(\sigma_2(\mathscr{A})) &= (\mathscr{A}, (((\sigma_1)_i(f_\gamma^{\mathscr{A}})_k)^{\sigma_2(\mathscr{A})})_{k\in K_\gamma,\gamma\in\Sigma(i),i\in I}) \\ &= (\mathscr{A}, ((((\sigma_2)_i \circ_G^i (\sigma_1)_i)(f_\gamma^{\mathscr{A}})_k)^{\mathscr{A}})_{k\in K_\gamma,\gamma\in\Sigma(i),i\in I}) \\ &= (\sigma_2 \diamond \sigma_1)(\mathscr{A}). \end{aligned}$$ □

Theorem 1.11 ([2]). *Let $\mathscr{A} \in Alg(\Sigma)$, then we have $\sigma_{id}(\mathscr{A}) = \mathscr{A}$.*

2. Strong Hyper Identities in Many-sorted Algebra

Let $\mathscr{A} = (A,(f_\gamma^{\mathscr{A}})_{\gamma\in\Sigma})$ be a Σ-algebra and $i \in I$. The Σ-equation $s_i \approx_i t_i$ is said to be a Σ-identity of sort $i \in \mathscr{A}$ if $s^{\mathscr{A}} = t^{\mathscr{A}}$, i.e., if the induced Σ-term operations are equal. In this case we write $\mathscr{A} \models_i s_i \approx_i t_i$.

Let $K \subseteq Alg(\Sigma)$ and $L(i) \subseteq W(i)^2$. We write $K \models_i s_i \approx_i t_i$ if $s_i \approx_i t_i$ is satisfied as Σ-identity in $\mathscr{A} \in K$ and $\mathscr{A} \models_i L(i)$ if $s_i \approx_i t_i$ is satisfied as Σ-identity in every $\mathscr{A}$, for all $(s_i,t_i) \in L(i)$.

The relation $\models_i \subseteq Alg(\Sigma) \times W(i)^2$ induces a Galois connection $(\Sigma(i)\text{-}Id, \Sigma(i)\text{-}Mod)$ between $K_0 \subseteq Alg(\Sigma)$ and $L(i) \subseteq W(i)^2$ which is defined as follows:

$$\Sigma(i)\text{-}Id\,K_0 = \{(s_i,t_i) \in W(i)^2 \mid \mathscr{A} \models_i s_i \approx_i t_i, \forall \mathscr{A} \in K_0\},$$

$$\Sigma(i)\text{-}Mod\,L(i) = \{\mathscr{A} \in Alg(\Sigma) \mid \mathscr{A} \models_i s_i \approx_i t_i, \forall (s_i,t_i) \in L(i)\}.$$

Definition 2.1. Let $\mathscr{V} \subseteq Alg(\Sigma)$ and $L(i) \subseteq W(i)^2$. $\mathscr{V}$ is said to be a Σ-variety of sort i if $\mathscr{V} = \Sigma(i)\text{-}Mod\,\Sigma(i)\text{-}Id\,\mathscr{V}$ and $L(i)$ is said to be a Σ-equational theory of sort i if $L(i) = \Sigma(i)\text{-}Id\,\Sigma(i)\text{-}Mod\,L(i)$.

Definition 2.2. Let $\mathscr{A} \in Alg(\Sigma)$. A Σ-identity $s_i \approx_i t_i$ is said to be a Σ-strong hyperidentity of sort i in $\mathscr{A}$ if $\mathscr{A} \models_i \hat{\sigma}[s_i] \approx_i \hat{\sigma}[t_i]$, for all $\sigma \in \Sigma^{|I|,n}(i)\text{-}Hyp_G$. In this case we write $\mathscr{A} \underset{\Sigma\text{-}hyp_G}{\models_i} s_i \approx_i t_i$.

We define operator $\chi^{\Sigma\text{-}A}$ and $\chi^{\Sigma\text{-}E(i)}$ on $Alg(\Sigma)$ and $W(i)^2$, respectively, by

$$\chi^{\Sigma\text{-}A}[K_0] = \bigcup_{\mathscr{A} \in K_0} \chi^{\Sigma\text{-}A}[\mathscr{A}],$$

$$\chi^{\Sigma\text{-}E(i)}[L(i)] = \bigcup_{(s_i,t_i) \in L(i)} \chi^{\Sigma\text{-}E(i)}[s_i \approx_i t_i]$$

where $\chi^{\Sigma\text{-}A}[\mathscr{A}] := \{\sigma(\mathscr{A}) \mid \sigma \in \Sigma^{|I|,n}\text{-}Hyp_G\}$ and $\chi^{\Sigma\text{-}E(i)}[s_i \approx_i t_i] := \{\hat{\sigma}_i[s_i] \approx_i \hat{\sigma}_i[t_i] \mid \sigma_i \in \Sigma^{|I|,n}(i)\text{-}Hyp_G\}$.

Straight from definition of $\chi^{\Sigma\text{-}A}$ and $\chi^{\Sigma\text{-}E(i)}$,we can show that $(\chi^{\Sigma\text{-}E(i)}, \chi^{\Sigma\text{-}A})$ is a conjugate pair of additive closure operators with respect to the relation $\models_i$ by the following propositions:

Proposition 2.3. *Let $K_0, K_1, K_2 \subseteq Alg(\Sigma)$ and $L(i), L_1(i), L_2(i) \subseteq W(i)^2$. Then*

(1) $K_0 \subseteq \chi^{\Sigma\text{-}A}[K_0]$,

(2) *If $K_1 \subseteq K_2$, then* $\chi^{\Sigma\text{-}A}[K_1] \subseteq \chi^{\Sigma\text{-}A}[K_2]$,

(3) $\chi^{\Sigma\text{-}A}[K_0] = \chi^{\Sigma\text{-}A}[\chi^{\Sigma\text{-}A}[K_0]]$,

(4) $L(i) \subseteq \chi^{\Sigma\text{-}E(i)}[L(i)]$,

(5) *If $L_1(i) \subseteq L_2(i)$, then* $\chi^{\Sigma\text{-}E(i)}[L_1(i)] \subseteq \chi^{\Sigma\text{-}E(i)}[L_2(i)]$,

(6) $\chi^{\Sigma\text{-}E(i)}[L(i)] = \chi^{\Sigma\text{-}E(i)}[\chi^{\Sigma\text{-}E(i)}[L(i)]]$.

This shows that $\chi^{\Sigma\text{-}A}$ and $\chi^{\Sigma\text{-}E(i)}$ are closure operators on $Alg(\Sigma)$ and $W(i)^2$, respectively. By definition, both operators are additive. The next proposition shows that $(\chi^{\Sigma\text{-}E(i)}, \chi^{\Sigma\text{-}A})$ is a conjugate pair.

Proposition 2.4. *Let $\mathscr{A} \in Alg(\Sigma), (s_i,t_i) \in W(i)^2$ and $\sigma \in \Sigma^{|I|,n}\text{-}Hyp_G$. We have*

$$\sigma(\mathscr{A}) \models_i s_i \approx_i t_i \iff \mathscr{A} \models_i \hat{\sigma}_i[s_i] \approx_i \hat{\sigma}_i[t_i].$$

Using the property of closure operator, the set of all fixed points $\{K_0 \subseteq Alg(\Sigma) \mid \chi^{\Sigma\text{-}A}[K_0] = K_0\}$ and $\{L(i) \subseteq W(i)^2 \mid \chi^{\Sigma\text{-}E(i)}[L(i)] = L(i)\}$ form complete sublattice of $P(Alg(\Sigma))$ and $P(W(i)^2)$, respectively.

The relation $\underset{\Sigma\text{-}hyp_G}{\models_i}$ induces a Galois connection $(H\Sigma(i)\text{-}Id, H\Sigma(i)\text{-}Mod)$ between $Alg(\Sigma)$ and $W(i)^2$ which is defined by

$$H\Sigma(i)\text{-}Id\,K_0 = \{(s_i,t_i)\in W(i)^2 \mid \mathscr{A} \underset{\Sigma\text{-}hyp_G}{\models_i} s_i \approx_i t_i, \forall \mathscr{A}\in K\},$$

$$H\Sigma(i)\text{-}Mod\,L(i) = \{\mathscr{A}\in Alg(\Sigma) \mid \mathscr{A} \underset{\Sigma\text{-}hyp_G}{\models_i} s_i \approx_i t_i, \forall (s_i,t_i)\in L(i)\},$$

where $K_0 \subseteq Alg(\Sigma)$ and $L(i)\subseteq W(i)^2$.

The two closure operators $H\Sigma(i)\text{-}Mod\,H\Sigma(i)\text{-}Id$ and $H\Sigma(i)\text{-}Id\,H\Sigma(i)\text{-}Mod$ are obtained by above Galois connection and their fixed points, $\{K_0\subseteq Alg(\Sigma) \mid H\Sigma(i)\text{-}Mod\,H\Sigma(i)\text{-}Id\,K_0 = K_0\}$ and $\{L(i)\subseteq W(i)^2 \mid H\Sigma(i)\text{-}Id\,H\Sigma(i)\text{-}Mod\,L(i) = L(i)\}$, form complete lattice.

Definition 2.5. Let $\mathscr{V}\subseteq Alg(\Sigma)$ and M be a submonoid of $\Sigma^{|I|,n}(i)\text{-}Hyp_G$. $\mathscr{V}$ is called M-strongly solid variety of sort i if for Σ-identity $s_i\approx_i t_i$ of sort i in $\mathscr{V}$ and for $\sigma\in M$, the Σ-equation $\hat{\sigma}[s_i]\approx_i \hat{\sigma}[t_i]$holds in $\mathscr{A}$, for all $\mathscr{A}\in\mathscr{V}$. If $M=\Sigma^{|I|,n}(i)\text{-}Hyp_G$, $\mathscr{V}$ is said to be a strongly solid variety of sort i, i.e., $\mathscr{V} \underset{\Sigma\text{-}hyp_G}{\models_i} \Sigma^{|I|,n}(i)\text{-}Id\,\mathscr{V}$. $\mathscr{V}$ is called a Σ-strongly solid variety if $\mathscr{V} \underset{\Sigma\text{-}hyp_G}{\models_i} \Sigma^{|I|,n}(i)\text{-}Id\,\mathscr{V}$, for all $i\in I$.

Example 2.6. Let $I=\{1\}$, $\Sigma^{|I|,2}=\{(1,1,1)\}$ and $i=1$. Let $\mathscr{V}$ be a Σ-variety of sort i with $\Sigma(i)\text{-}Id\,\mathscr{V}=\{f_{(1,1,1)}(f_{(1,1,1)}(x_{11},x_{12}),x_{13})\approx_i f_{(1,1,1)}(x_{11},f_{(1,1,1)}(x_{12},x_{13}))\}$. Then $\mathscr{V}$ is a strongly solid variety of sort i.

Proof. Let $\sigma\in\Sigma^{|I|,2}(i)\text{-}Hyp_G$ with $\sigma(f_{(1,1,1)})=t_1\in W(i)$. We show that

$$\hat{\sigma}[f_{(1,1,1)}(f_{(1,1,1)}(x_{11},x_{12}),x_{13})]\approx_i \hat{\sigma}[f_{(1,1,1)}(x_{11},f_{(1,1,1)}(x_{12},x_{13}))]\in\Sigma(i)\text{-}Id\,\mathscr{V}$$

. Firstly,

$$\begin{aligned}
\hat{\sigma}[f_{(1,1,1)}(f_{(1,1,1)}(x_{11},x_{12}),x_{13})] &= S_{(1,1,1)}(\sigma(f_{(1,1,1)}),\hat{\sigma}[f_{(1,1,1)}(x_{11},x_{12})],\hat{\sigma}[x_{13}])\\
&= S_{(1,1,1)}(\sigma(f_{(1,1,1)}),S_{(1,1,1)}(\sigma(f_{(1,1,1)}),x_{11},x_{12}),x_{13})\\
&= S_{(1,1,1)}(t_1,S_{(1,1,1)}(t_1,x_{11},x_{12}),x_{13}),
\end{aligned}$$

$$\begin{aligned}
\hat{\sigma}[f_{(1,1,1)}(x_{11},f_{(1,1,1)}(x_{12},x_{13}))] &= S_{(1,1,1)}(\sigma(f_{(1,1,1)}),\hat{\sigma}[x_{11}],\hat{\sigma}[f_{(1,1,1)}(x_{12},x_{13})])\\
&= S_{(1,1,1)}(t_1,x_{11},S_{(1,1,1)}(t_1,x_{12},x_{13})).
\end{aligned}$$

If $t_1=x_{11}$, we get

$$\hat{\sigma}[f_{(1,1,1)}(f_{(1,1,1)}(x_{11},x_{12}),x_{13})] = S_{(1,1,1)}(x_{11},S_{(1,1,1)}(x_{11},x_{11},x_{12}),x_{13}) = x_{11}$$

and

$$\hat{\sigma}[f_{(1,1,1)}(x_{11},f_{(1,1,1)}(x_{12},x_{13}))] = S_{(1,1,1)}(x_{11},x_{11},S_{(1,1,1)}(x_{11},x_{12},x_{13})) = x_{11}.$$

If $t_1=x_{12}$, we have

$$\hat{\sigma}[f_{(1,1,1)}(f_{(1,1,1)}(x_{11},x_{12}),x_{13})]=x_{13}$$

and

$$\hat{\sigma}[f_{(1,1,1)}(x_{11}, f_{(1,1,1)}(x_{12}, x_{13}))] = x_{13}.$$

If $t_1 = x_{1k}$ and $k \geq 3$,

$$\hat{\sigma}[f_{(1,1,1)}(f_{(1,1,1)}(x_{11}, x_{12}), x_{13})] = x_{1k}$$

and

$$\hat{\sigma}[f_{(1,1,1)}(x_{11}, f_{(1,1,1)}(x_{12}, x_{13}))] = x_{1k}.$$

If $t_1 = f_{(1,1,1)}(h_1, h_2)$ with $h_1, h_2 \in W(i)$. Assume that

$$S_{(1,1,1)}(h_j, \hat{\sigma}[f_{(1,1,1)}(x_{11}, x_{12})], x_{13}) \approx_i S_{(1,1,1)}(h_j, x_{11}, \hat{\sigma}[f_{(1,1,1)}(x_{12}, x_{13})]) \in \Sigma(i)\text{-}Id\mathscr{V}, j = 1, 2.$$

By induction on the complexity of Σ-term t_1, we get

$$\begin{aligned}
\hat{\sigma}[f_{(1,1,1)}(f_{(1,1,1)}(x_{11}, x_{12}), x_{13})] &= S_{(1,1,1)}(f_{(1,1,1)}(h_1, h_2), S_{(1,1,1)}(f_{(1,1,1)}(h_1, h_2), x_{11}, x_{12}), x_{13}) \\
&= f_{(1,1,1)}(S_{(1,1,1)}(h_1, S_{(1,1,1)}(f_{(1,1,1)}(h_1, h_2), x_{11}, x_{12}), x_{13}), \\
&\quad S_{(1,1,1)}(h_2, S_{(1,1,1)}(f_{(1,1,1)}(h_1, h_2), x_{11}, x_{12}), x_{13})) \\
&= f_{(1,1,1)}(S_{(1,1,1)}(h_1, x_{11}, S_{(1,1,1)}(f_{(1,1,1)}(h_1, h_2), x_{12}, x_{13})), \\
&\quad S_{(1,1,1)}(h_2, x_{11}, S_{(1,1,1)}(f_{(1,1,1)}(h_1, h_2), x_{12}, x_{13}))) \\
&= S_{(1,1,1)}(f_{(1,1,1)}(h_1, h_2), x_{11}, S_{(1,1,1)}(f_{(1,1,1)}(h_1, h_2), x_{12}, x_{13})) \\
&= S_{(1,1,1)}(t_1, x_{11}, S_{(1,1,1)}(t_1, x_{12}, x_{13})) \\
&= \hat{\sigma}[f_{(1,1,1)}(x_{11}, f_{(1,1,1)}(x_{12}, x_{13}))].
\end{aligned}$$

So $\mathscr{V}$ is a strongly solid variety of sort i. □

Now, we can apply the general theory of conjugate pairs of additive closure operators.

Theorem 2.7. *Let $\mathscr{V}$ be a Σ-variety of sort i, the following conditions are equivalent:*

(1) $\mathscr{V} = H\Sigma(i)\text{-}Mod\,H\Sigma(i)\text{-}Id\mathscr{V}$,

(2) $\chi^{\Sigma\text{-}A}[\mathscr{V}] = \mathscr{V}$,

(3) $\Sigma(i)\text{-}Id\mathscr{V} = H\Sigma(i)\text{-}Id\mathscr{V}$,

(4) $\chi^{\Sigma\text{-}E(i)}[\Sigma(i)\text{-}Id\mathscr{V}] = \Sigma(i)\text{-}Id\mathscr{V}$

and, let $L(i)$ be a Σ-equational theory of sort i, the following are equivalent:

(1) $L(i) = H\Sigma(i)\text{-}Id\,H\Sigma(i)\text{-}Mod\,L(i)$,

(2) $\chi^{\Sigma\text{-}E(i)}[L(i)] = L(i)$,

(3) $\Sigma(i)\text{-}Mod\,L(i) = H\Sigma(i)\text{-}Mod\,L(i)$,

(4) $\chi^{\Sigma\text{-}A}[\Sigma(i)\text{-}Mod\,L(i)] = \Sigma(i)\text{-}Mod\,L(i)$.

3. $\mathscr{V}$-Normal Form Σ-Generalized Hyper substitutions

In this section, we give the concept of V-normal form Σ-generalized hypersubstitution which is useful for testing a strongly solid Σ-variety.

Definition 3.1. Let $\mathscr{V}$ be a Σ-variety of sort i. $\sigma_i \in \Sigma^{|I|,n}(i)\text{-}Hyp_G$ is called a $\mathscr{V}$-proper Σ-generalized hypersubstitution of sort i if for every Σ-identuty $s_i \approx_i t_i$ in $\mathscr{V}$, the Σ-identity $\hat{\sigma}_i[s_i] \approx_i \hat{\sigma}_i[t_i]$ holds in $\mathscr{V}$.

Denote $P^i_G(\mathscr{V})$ a set of $\mathscr{V}$-proper Σ-generalized hypersubstitutions of sort i. We see that $P^i_G(\mathscr{V}) \subseteq \Sigma^{|I|,n}(i)\text{-}Hyp_G$ and $P^i_G(\mathscr{V}) \neq \emptyset$, since $(\sigma_{id})_i \in P^i_G(\mathscr{V})$.

Let $\mathscr{V}$ be a Σ-variety of sort i.

Lemma 3.2. *$(P^i_G(\mathscr{V}), \circ^i_G, (\sigma_{id})_i)$ is a submonoid of $(\Sigma^{|I|,n}(i)\text{-}Hyp_G, \circ^i_G, (\sigma_{id})_i)$.*

Proof. Let $(\sigma_1)_i, (\sigma_2)_i \in P^i_G(\mathscr{V})$ and $(s_i, t_i) \in W(i)^2$ with $\mathscr{V} \models_i s_i \approx_i t_i$. We have

$$\begin{aligned}((\sigma_1)_i \circ^i_G (\sigma_2)_i)\hat{}[s_i] &= ((\hat{\sigma}_1)_i \circ (\hat{\sigma}_2)_i)[s_i]\\ &= (\hat{\sigma}_1)_i[(\hat{\sigma}_2)_i[s_i]]\\ &= (\hat{\sigma}_1)_i[(\hat{\sigma}_2)_i[t_i]]\\ &= ((\sigma_1)_i \circ^i_G (\sigma_2)_i)\hat{}[t_i].\end{aligned}$$

So $P^i_G(\mathscr{V})$ is closed under $\circ^i_G$ and we see that $\sigma_{id} \in P^i_G(\mathscr{V})$. We can conclude that $(P^i_G(\mathscr{V}), \circ^i_G, (\sigma_{id})_i)$ is a submonoid of $(\Sigma^{|I|,n}(i)\text{-}Hyp_G, \circ^i_G, (\sigma_{id})_i)$. □

Now, we define relation $\sim_{VG(i)}$ on $\Sigma^{|I|,n}(i)\text{-}Hyp_G$ by for $\sigma_1, \sigma_2 \in \Sigma^{|I|,n}(i)\text{-}Hyp_G$,

$$\sigma_1 \sim_{VG(i)} \sigma_2 \iff \mathscr{V} \models_i \sigma_1(f_\gamma) \approx_i \sigma_2(f_\gamma),\ \forall\ \gamma \in \Sigma(i).$$

It's easy to prove that the relation $\sim_{VG(i)}$ is an equivalence relation on $\Sigma^{|I|,n}(i)\text{-}Hyp_G$, but it may not be a congruence relation.

Theorem 3.3. *For $\sigma_1, \sigma_2 \in \Sigma^{|I|,n}(i)\text{-}Hyp_G$. The following are equivalent:*

(1) $\sigma_1 \sim_{VG(i)} \sigma_2$.

(2) $\mathscr{V} \models_i \hat{\sigma}_1[t] \approx_i \hat{\sigma}_2[t]$.

Proof. First part, we prove by induction on the complexity of Σ-term $t \in W(i)$.
If $t = x_{ij} \in X_i$. Since $\hat{\sigma}_1[x_{ij}] = x_{ij} = \hat{\sigma}_2[x_{ij}]$, $\mathscr{V} \models_i \hat{\sigma}_1[x_{ij}] \approx_i \hat{\sigma}_2[x_{ij}]$.
If $t = f_\gamma(t_1, \ldots, t_n) \in W(i)$ with $\gamma = (i_1, \ldots, i_n, i) \in \Sigma^{|I|,n}(i)$.
Assume that $\mathscr{V} \models_{i_j} (\hat{\sigma}_1)_{i_j}[t_j] \approx_j (\hat{\sigma}_2)_{i_j}[t_j], \forall j$.

$$\begin{aligned}\hat{\sigma}_1[f_\gamma(t_1, \ldots, t_n)] &= S_\gamma(\sigma_1(f_\gamma), (\hat{\sigma}_1)_{i_1}[t_1], \ldots, (\hat{\sigma}_1)_{i_n}[t_n])\\ &= S_\gamma(\sigma_1(f_\gamma), (\hat{\sigma}_2)_{i_1}[t_1], \ldots, (\hat{\sigma}_2)_{i_n}[t_n])\\ &\approx S_\gamma(\sigma_2(f_\gamma), (\hat{\sigma}_2)_{i_1}[t_1], \ldots, (\hat{\sigma}_2)_{i_n}[t_n])\\ &= \hat{\sigma}_2[f_\gamma(t_1, \ldots, t_n)].\end{aligned}$$

Therefore $\mathscr{V} \models_i \hat{\sigma}_1[t] \approx_i \hat{\sigma}_2[t]$. Conversely, let $\gamma = (i_1, \ldots, i_n, i) \in \Sigma^{|I|,n}(i)$. Put $t = f_\gamma(t_1, \ldots, t_n)$. By assumption, $\mathscr{V} \models_i \hat{\sigma}_1[t] \approx_i \hat{\sigma}_2[t]$. That is for every $\mathscr{A} \in \mathscr{V}$,

$$\begin{aligned}&\hat{\sigma}_1[f_\gamma(t_1, \ldots, t_n)]^{\mathscr{A}} = \hat{\sigma}_2[f_\gamma(t_1, \ldots, t_n)]^{\mathscr{A}}\\ \Rightarrow\quad &S_\gamma(\sigma_1(f_\gamma), \hat{\sigma}_{i_1}[t_1], \ldots, \hat{\sigma}_{i_n}[t_n])^{\mathscr{A}} = S_\gamma(\sigma_2(f_\gamma), \hat{\sigma}_{i_1}[t_1], \ldots, \hat{\sigma}_{i_n}[t_n])^{\mathscr{A}}\end{aligned}$$

$$\Rightarrow \quad S_\gamma^{\mathscr{A}}(\sigma_1(f_\gamma)^{\mathscr{A}}, \hat{\sigma}_{i_1}[t_1]^{\mathscr{A}}, \ldots, \hat{\sigma}_{i_n}[t_n]^{\mathscr{A}}) = S_\gamma^{\mathscr{A}}(\sigma_2(f_\gamma)^{\mathscr{A}}, \hat{\sigma}_{i_1}[t_1]^{\mathscr{A}}, \ldots, \hat{\sigma}_{i_n}[t_n]^{\mathscr{A}})$$
$$\Rightarrow \quad \sigma_1(f_\gamma)^{\mathscr{A}} = \sigma_2(f_\gamma)^{\mathscr{A}}$$
$$\Rightarrow \quad \mathscr{A} \models_i \sigma_1(f_\gamma) \approx_i \sigma_2(f_\gamma).$$

So $\mathscr{V} \models_i \hat{\sigma}_1[t] \approx_i \hat{\sigma}_2[t]$. □

Lemma 3.4. *Let $\mathscr{V}$ be a Σ-variety of sort i. Then*

(1) *For $\sigma_1, \sigma_2 \in \Sigma^{|I|,n}(i)$-$Hyp_G$ with $\sigma_1 \sim_{VG(i)} \sigma_2$ and for $s_i, t_i \in W(i)$,*

$$\mathscr{V} \models_i \hat{\sigma}_1[s_i] \approx_i \hat{\sigma}_1[t_i] \iff \mathscr{V} \models_i \hat{\sigma}_2[s_i] \approx_i \hat{\sigma}_2[t_i].$$

(2) *For $\sigma_1, \sigma_2 \in \Sigma^{|I|,n}(i)$-$Hyp_G$ with $\sigma_1 \sim_{VG(i)} \sigma_2$,*

$$\sigma_1 \in P_G^i(\mathscr{V}) \iff \sigma_2 \in P_G^i(\mathscr{V}).$$

Proof. In the first part, since $\sigma_1 \sim_{VG(i)} \sigma_2$ and by the previous theorem, $\hat{\sigma}_1[s_i] \approx_i \hat{\sigma}_2[s_i]$ and $\hat{\sigma}_1[t_i] \approx_i \hat{\sigma}_2[t_i]$ hold in $\mathscr{V}$. So we have $\mathscr{V} \models_i \hat{\sigma}_2[s_i] \approx_i \hat{\sigma}_2[t_i]$ (or $\sigma_2 \in P_G^i(\mathscr{V})$). Conversely, we can prove it in a similar way in the first part. □

Definition 3.5. Let $\mathscr{V}$ be a Σ-variety of sort i and M be a submonoid of $\Sigma^{|I|,n}(i) - Hyp_G$. Let $\phi : M/{\sim_{VG(i)|_M}} \to M$ be a choice function which chooses one Σ-generalized hypersubstitution, which is called a $\mathscr{V}$-normal form Σ-generalized hypersubstitution, from each equivalence class of the relation $\sim_{VG(i)|_M}$ and denote the set of $\mathscr{V}$-normal form Σ-generalized hypersubstitutions by $N_\phi^M(\mathscr{V})$.

$\mathscr{V}$ is called $N_\phi^M(\mathscr{V})$-strongly solid if for Σ-identity $s_i \approx_i t_i$ of sort i in $\mathscr{V}$ and for $\sigma \in N_\phi^M(\mathscr{V})$, the Σ-identity $\hat{\sigma}[s_i] \approx_i \hat{\sigma}[t_i]$holds in $\mathscr{A}$, for all $\mathscr{A} \in \mathscr{V}$.

We see that $N_\phi^M(\mathscr{V}) \subseteq M$ and it is not always a submonoid of $\Sigma^{|I|,n}(i)$-Hyp_G because the product of any two elements in $N_\phi^M(\mathscr{V})$ need not be in $N_\phi^M(\mathscr{V})$.

Theorem 3.6. *Let $\mathscr{V}$ be a Σ-variety of sort i and M be a submonoid of $\Sigma^{|I|,n}(i)$-Hyp_G. For any choice function ϕ,*

$$\mathscr{V} \text{ is } M\text{-strongly solid} \iff \mathscr{V} \text{ is } N_\phi^M(\mathscr{V})\text{-strongly solid.}$$

Proof. Assume that $\mathscr{V}$ is M-strongly solid. Since $N_\phi^M(\mathscr{V}) \subseteq M$, $\mathscr{V}$ is $N_\phi^M(\mathscr{V})$-strongly solid. Conversely, suppose that $\mathscr{V}$ is $N_\phi^M(\mathscr{V})$-strongly solid. For Σ-identity $s_i \approx_i t_i$ in $\mathscr{V}$, then $\mathscr{V} \models_i \hat{\sigma}[s_i] \approx_i \hat{\sigma}[t_i], \forall \sigma \in N_\phi^M(\mathscr{V})$. Let $\sigma' \in M$. Then there exists $\sigma \in N_\phi^M(\mathscr{V})$ such that $\sigma \sim_{VG(i)|_M} \sigma'$. By the previous lemma, $\mathscr{V} \models_i \hat{\sigma}'[s_i] \approx_i \hat{\sigma}'[t_i]$. So $\mathscr{V}$ is M-strongly solid. □

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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