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Research Article

On the Oscillation Criteria and Computation of Third Order Oscillatory Differential Equations

J. Sunday

Department of Mathematics, Adamawa State University, Mubi, Nigeria
joshuasunday2000@yahoo.com, sunday578@adsu.edu.ng

Abstract. One of the sets of differential equations that find applications in real life is the oscillatory problems. In this paper, the oscillation criteria and computation of third order oscillatory differential equations are studied. The conditions for a third order differential equation to have oscillatory solutions on the interval $I = [t_0, \infty)$ shall be analyzed. Further, a highly efficient and reliable one-step computational method (with three partitions) is formulated for the approximation of third order differential equations. The paper also analyzed the basic properties of the method so formulated. The results obtained on the application of the method on some sampled modeled third order oscillatory problems show that the method is computationally reliable and the method performed better than the ones with which we compared our results.

Keywords. Analysis; Computation; Bounded solutions; Infinite zeros; Oscillation; Third-order

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1. Introduction

One of the most challenging equations being encountered nowadays are the oscillatory differential equations. This is because their solutions are composed of smooth varying and 'nearly periodic' functions, i.e. they are oscillations whose wave form and period varies slowly with time (relative to the period), and where the solution is sought over a very large number of cycles ([17]). For such problems, one cannot and does not want to follow the trajectories; instead one resort to finding their approximate solutions or the computation of their quasi-envelops.

Oscillatory problems have some of their Eigen values near the imaginary axis, and their solutions are oscillation processes with slowly varying amplitudes. The difficulty of solving such problems is explained by the necessity to ensure correct values of the amplitude and phase angle over many periods.

In this research, we shall be interested in the analysis and computation of third order oscillatory problems of the form,

$$y'''(t) = f(t, y(t), y'(t), y''(t)), \quad y(t_0) = y_0, \quad y'(t_0) = y_0', \quad y''(t_0) = y_0''', \quad t \in [t_0, t_n] \tag{1}$$

where t_0 is the initial value/point, y_0 is the solution at t_0, f is continuous within the interval of integration. It is assumed that equation (1) satisfies the existence and uniqueness theorem of differential equations. It is also assumed that the solutions to equations of the form (1) are bounded. It is important to state that a solution $y(t)$to equation (1) is said to be bounded if,

$$\sup_{t\in\Re} \|y(t)\| < \infty. \tag{2}$$

Equation (1) has a wide range of applications in engineering, thermodynamics and other real life problems. They are also applied in studying thin-film flows [7], chaotic systems [9], electromagnetic waves [13], among other phenomenon.

A solution of equation (1) will be called oscillatory if it has infinity of zeros in $(0,\infty)$and non-oscillatory if it has but a finite number of zeros in this interval, [10]. An equation is termed oscillatory if there exists at least one oscillatory solution and non-oscillatory if all its solutions are non-oscillatory. This latter definition is necessary since an equation (1) may be both oscillatory and non-oscillatory.

Some methods have been derived by authors to directly solve third order differential equations of the form (1), see the works of [2–4, 13–15, 18]. Direct method for solving (1) has been reported to be more efficient than the method of reduction to system of first order differential equations ([1, 2]).

Definition 1.1. A differential equation is said to be oscillatory if

(i) all the nontrivial solution of (1) have an infinite number of zeros (roots) on $x_0 \le x < \infty$ (see [11]), and

(ii) it has at least one oscillating solution ([5]).

Definition 1.2 ([12])**.** A computational method is said to be A-stable if the whole of the left-half plane $\{z : Re(z) \le 0\}$ is contained in the region $\{z : |Re(z)| \le 1\}$, where $R(z)$ is called the stability polynomial of the method.

Definition 1.3 ([10])**.** The equation (1) is said to be of Class I (C_I), if any of its solutions $y(t)$ for which $y(a) = y'(a) = 0$, $y''(a) > 0$ $(0 < a < \infty)$ satisfies $y(t) > 0$ in $(0, a)$.

Definition 1.4 ([10])**.** The equation (1) is said to be of Class II (C_{II}), if any of its solutions $y(t)$ for which $y(a) = y'(a) = 0$, $y''(a) > 0$ $(0 < a < \infty)$ satisfies $y(t) > 0$ in (a, ∞).

Lemma 1.1. *Let* $y(t) \in C^3([t_0,\infty)) > 0$ *be bounded for* $t \in [t_0,\infty)$. *Then,* $y(t) > 0$, $y'(t) < 0$, $y''(t) < 0$ *and* $y'''(t) > 0$ *cannot hold for all* $t \geq t_0$.
Note that $C^3([t_0,\infty))$ *are the set of continuous functions on the interval* $I \in [t_0,\infty)$.

2. Analysis of Oscillation Criteria for Third Order Oscillatory Differential Equations

Equation (1) shall be rewritten as

$$y'''(t) + p(t)y''(t) + q(t)y'(t) + r(t)y(t) = 0, \quad y'_n(t_0) = y^i_n, \quad i = 0,1,2 \tag{3}$$

where $p(t)$, $q(t)$ and $r(t)$ are continuous functions on the interval $I \in (a,\infty)$.

Theorem 2.1. *Let* $p(t) \geq 0$, $q(t) \geq 0$, $r(t) < 0$ *for* $t \in (a,\infty)$ *and*

$$\int_{t_0}^{\infty} r(s)ds = -\infty, \quad t_0 > a. \tag{4}$$

Then every bounded solution $y(t)$ *of* (1) *on* $t \in [t_0,\infty)$ *is oscillatory on the interval* $I = [t_0,\infty)$.

Proof. Assume without loss of generality that $y(t) > 0$ be bounded on $[t_0,\infty)$, $t_0 > a$.
Firstly, we consider a situation where $y'(t) > 0$, $t \geq T \geq t_0$
Integrating equation (3) between t_0 and t, we obtain

$$\begin{aligned}&\int_{t_0}^{t} \left(y'''(t) + p(t)y''(t) + q(t)y'(t) + r(t)y(t)\right)dt \\ &\quad = y''(t) + p(t)y'(t) + q(t)y(t) + \int_{t_0}^{t} r(s)ds = k\end{aligned} \tag{5}$$

where the constant k is given by

$$k = y''(t_0) + p(t_0)y'(t_0) + q(t_0)y(t_0).$$

Using equation (4), the conditions that $p(t) \geq 0$, $q(t) \geq 0$, $r(t) < 0$ and also the fact that $y(t)$ is bounded implies that $y''(t) \to \infty$ as $t \to \infty$. Thus, $y(t)$cannot be bounded on $t \in [t_0,\infty)$. This is a contradiction.
Secondly, we consider a situation where $y'(t) \leq 0$ for $t \geq T \geq t_0$.

Equation (3) can be written as

$$y'''(t) = -p(t)y''(t) - q(t)y'(t) - r(t)y(t) > 0 \tag{6}$$

for $t \geq T$ using the conditions in the statement of the theorem. But, by Lemma 1.1, this is not possible.
Thirdly, we consider the situation where $y'(t)$ has infinitely many null points at which it changes its signs. Let $y(t) > k > 0$, then for equation (5), we obtain $y''(t) > 0$ for $t \geq T \geq t_0$ and this implies that $y'(t) > 0$ increases for $t \geq T$. This is a contradiction with the case $y'(t)$ being oscillatory. Thus,

$$\liminf_{t\to\infty} y(t) = 0. \tag{7}$$

□

Theorem 2.2 ([10]). *If in the equation*

$$y'''(t)+q(t)y'(t)+r(t)y(t)=0 \tag{8}$$

$2r(t)-q'(t)>0$ $\left(2r(t)-q'(t)<0\right)$, *except at isolated points at which* $2r(t)-q'(t)$ *may vanish, then equation* (8) *is of* C_I *((8) is of* C_{II}*).*

Proof. Suppose $y(t)$ is the solution of equation (8) such that $y(b)=y'(b)=0$ and assume that equation (8) is not of Class I, that is, let $t=a$ $(a<b)$ be a zero of $y(t)$. Multiplying equation (8) by $y(t)$ and integrating from a to b, we obtain

$$\left[y(t)y''(t)-\frac{1}{2}y'^2(t)+\frac{1}{2}q(t)y^2(t)\right]_a^b-\int_a^b\left(q'(t)-2r(t)\right)y^2(t)dt=0,$$

$$-\left[y'(a)\right]^2=\int_a^b\left(2r(t)-q'(t)\right)y^2(t)dt. \tag{9}$$

This contradiction completes the prove. □

3. Formulation of the Computational Method

A one-step computational method of the form,

$$A^{(0)}Y_m^{(i)}=\sum_{i=0}^{1}\frac{(jh)^{(i)}}{i!}e_iy_n^{(i)}+h^{(3-i)}[d_if(y_n)+b_iF(Y_m)] \tag{10}$$

for the computation of third order oscillatory problems (1) shall be formulated. In formulating the method, we employ a power series approximate solution of the form,

$$y(t)=\sum_{j=0}^{r+s-1}a_jt^j \tag{11}$$

where r and s are the numbers of collocation and interpolation points, respectively.

Equation (11) is differentiated three times and substituted into equation (10), that is,

$$f(t,y,y',y'')=\sum_{j=2}^{r+s-1}j(j-1)(j-2)a_jt^{j-3}. \tag{12}$$

A grid of one-steplength is considered in this paper with a constant step size h given by $h=t_{n+i}-t_n$, $i=0,1$ and off-step points at $t_{n+\frac{1}{4}}$, $t_{n+\frac{1}{2}}$ and $t_{n+\frac{3}{4}}$.

Interpolating (11) at point t_{n+s}, $s=\frac{1}{4}\left(\frac{1}{4}\right)\frac{3}{4}$ and collocating (12) at points t_{n+r}, $r=0\left(\frac{1}{4}\right)1$, give a system of nonlinear equation of the form,

$$TA=U \tag{13}$$

where

$$A=[a_0\quad a_1\quad a_2\quad a_3\quad a_4\quad a_5\quad a_6\quad a_7]^T$$

$$U=\left[y_{n+\frac{1}{4}}\quad y_{n+\frac{1}{2}}\quad y_{n+\frac{3}{4}}\quad f_n\quad f_{n+\frac{1}{4}}\quad f_{n+\frac{1}{2}}\quad f_{n+\frac{3}{4}}\quad f_{n+1}\right]^T$$

$$T=\begin{bmatrix} 1 & t_{n+\frac{1}{4}} & t^2_{n+\frac{1}{4}} & t^3_{n+\frac{1}{4}} & t^4_{n+\frac{1}{4}} & t^5_{n+\frac{1}{4}} & t^6_{n+\frac{1}{4}} & t^7_{n+\frac{1}{4}} \\ 1 & t_{n+\frac{1}{2}} & t^2_{n+\frac{1}{2}} & t^3_{n+\frac{1}{2}} & t^4_{n+\frac{1}{2}} & t^5_{n+\frac{1}{2}} & t^6_{n+\frac{1}{2}} & t^7_{n+\frac{1}{2}} \\ 1 & t_{n+\frac{3}{4}} & t^2_{n+\frac{3}{4}} & t^3_{n+\frac{3}{4}} & t^4_{n+\frac{3}{4}} & t^5_{n+\frac{3}{4}} & t^6_{n+\frac{3}{4}} & t^7_{n+\frac{3}{4}} \\ 0 & 0 & 0 & 6 & 24t_n & 60t_n^2 & 120t_n^3 & 210t_n^4 \\ 0 & 0 & 0 & 6 & 24t_n & 60t^2_{n+\frac{1}{4}} & 120t^3_{n+\frac{1}{4}} & 210t^4_{n+\frac{1}{4}} \\ 0 & 0 & 0 & 6 & 24t_n & 60t^2_{n+\frac{1}{2}} & 120t^3_{n+\frac{1}{2}} & 210t^4_{n+\frac{1}{2}} \\ 0 & 0 & 0 & 6 & 24t_n & 60t^2_{n+\frac{3}{4}} & 120t^3_{n+\frac{3}{4}} & 210t^4_{n+\frac{3}{4}} \\ 0 & 0 & 0 & 6 & 24t_n & 60t^2_{n+1} & 120t^3_{n+1} & 210t^4_{n+1} \end{bmatrix}$$

Solving (13) for a_j, $j=0(1)7$ which are constants to be determined and putting back into (11) gives a one-step continuous computational method of the form

$$y(t)=\alpha_{\frac{1}{4}}(t)y_{n+\frac{1}{4}}+\alpha_{\frac{1}{2}}(t)y_{n+\frac{1}{2}}+\alpha_{\frac{3}{4}}(t)y_{n+\frac{3}{4}}+h^3\left[\sum_{j=0}^{1}\beta_j(t)f_{n+j}+\beta_s(t)f_{n+s}\right],\quad s=\frac{1}{4}\left(\frac{1}{4}\right)\frac{3}{4},\tag{14}$$

where $\alpha_s(t)$, $\beta_j(t)$ and $\beta_s(t)$are expressed as functions of x with

$$x=\frac{t-t_n}{h}\tag{15}$$

to obtain the continuous form as follows:

$$\left.\begin{array}{l}
\alpha_{\frac{1}{4}}(t)=8x^2-10x+3,\quad \alpha_{\frac{1}{2}}(t)=-16x^2+16x-3,\quad \alpha_{\frac{3}{4}}(t)=8x^2-6x+1\\
\beta_0(t)=\frac{1}{322560}(16384x^7-71680x^6+125440x^5-112000x^4+53760x^3-13216x^2+1354x-21)\\
\beta_{\frac{1}{4}}(t)=-\frac{1}{80640}(16384x^7-64512x^6+93184x^5-53760x^4+13356x^2-5240x+609)\\
\beta_{\frac{1}{2}}(t)=\frac{1}{53760}(16384x^7-57344x^6+68096x^5-26880x^4-1792x^2+2418x-441)\\
\beta_{\frac{3}{4}}(t)=-\frac{1}{80640}(16384x^7-50176x^6+50176x^5-17920x^4+980x^2-32x-21)\\
\beta_1(t)=\frac{1}{322560}(16384x^7-43008x^6+39424x^5-13440x^4+672x^2+10x-21)
\end{array}\right\}$$

Solving (14) for the independent solution gives a continuous computational method of the form

$$y(t)=\sum_{i=0}^{1}\frac{(jh)^i}{i!}y_n^{(i)}+h^3\left[\sum_{j=0}^{1}\sigma_j(t)f_{n+j}+\sigma_s(t)f_{n+s}\right],\quad s=\frac{1}{4}\left(\frac{1}{4}\right)\frac{3}{4},\tag{16}$$

where

$$\left.\begin{array}{l}
\sigma_0(t)=\frac{1}{2520}(128x^7-560x^6+980x^5-875x^4+420x^3)\\
\sigma_{\frac{1}{4}}(t)=-\frac{2}{315}(32x^7-126x^6+182x^5-105x^4)\\
\sigma_{\frac{1}{2}}(t)=\frac{1}{210}(64x^7-224x^6+266x^5-105x^4)\\
\sigma_{\frac{3}{4}}(t)=-\frac{2}{315}(32x^7-98x^6+98x^5-35x^4)\\
\sigma_1(t)=\frac{1}{2520}(128x^7-336x^6+308x^5-105x^4)
\end{array}\right\}$$

and t is as defined in equation (15).

Evaluating (16) at $t=\frac{1}{4}\left(\frac{1}{4}\right)1$, gives a discrete one-step computational method of the form (1), where

$$Y_m^{(i)}=\begin{bmatrix} y^{(i)}_{n+\frac{1}{4}} & y^{(i)}_{n+\frac{1}{2}} & y^{(i)}_{n+\frac{3}{4}} & y^{(i)}_{n+1}\end{bmatrix}^T,\quad F(Y_m)=\begin{bmatrix} f_{n+\frac{1}{4}} & f_{n+\frac{1}{2}} & f_{n+\frac{3}{4}} & f_{n+1}\end{bmatrix}^T,$$

$$y_n^{(i)} = \begin{bmatrix} y_{n-\frac{1}{4}}^{(i)} & y_{n-\frac{1}{2}}^{(i)} & y_{n-\frac{3}{4}}^{(i)} & y_n^{(i)} \end{bmatrix}^T$$

and $A^{(0)}$ is a 4×4 identity matrix.

When $i=0$

$$e_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{32} \\ 0 & 0 & 0 & \frac{1}{8} \\ 0 & 0 & 0 & \frac{9}{32} \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix},$$

$$d_0 = \begin{bmatrix} 0 & 0 & 0 & \frac{113}{71680} \\ 0 & 0 & 0 & \frac{331}{40320} \\ 0 & 0 & 0 & \frac{1431}{71680} \\ 0 & 0 & 0 & \frac{31}{840} \end{bmatrix}, \quad b_0 = \begin{bmatrix} \frac{107}{64512} & \frac{-103}{107520} & \frac{43}{107520} & \frac{-47}{645120} \\ \frac{83}{5040} & \frac{-1}{168} & \frac{13}{5040} & \frac{-19}{40320} \\ \frac{1863}{35840} & \frac{-243}{35840} & \frac{45}{7168} & \frac{-81}{71680} \\ \frac{34}{315} & \frac{1}{210} & \frac{2}{105} & \frac{-1}{504} \end{bmatrix}.$$

When $i=1$

$$e_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad d_1 = \begin{bmatrix} 0 & 0 & 0 & \frac{367}{23040} \\ 0 & 0 & 0 & \frac{53}{1440} \\ 0 & 0 & 0 & \frac{147}{2560} \\ 0 & 0 & 0 & \frac{7}{90} \end{bmatrix},$$

$$b_1 = \begin{bmatrix} \frac{3}{128} & \frac{-47}{3840} & \frac{29}{5760} & \frac{-7}{7680} \\ \frac{1}{10} & \frac{-1}{48} & \frac{1}{90} & \frac{-1}{480} \\ \frac{117}{640} & \frac{27}{1280} & \frac{3}{128} & \frac{-9}{2560} \\ \frac{4}{15} & \frac{1}{15} & \frac{4}{45} & 0 \end{bmatrix}.$$

When $i=2$

$$e_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad d_2 = \begin{bmatrix} 0 & 0 & 0 & \frac{251}{2880} \\ 0 & 0 & 0 & \frac{29}{360} \\ 0 & 0 & 0 & \frac{27}{320} \\ 0 & 0 & 0 & \frac{7}{90} \end{bmatrix}, \quad b_2 = \begin{bmatrix} \frac{323}{1440} & \frac{-11}{120} & \frac{53}{1440} & \frac{-19}{2880} \\ \frac{31}{90} & \frac{1}{15} & \frac{1}{90} & \frac{-1}{360} \\ \frac{51}{160} & \frac{9}{40} & \frac{21}{160} & \frac{-3}{320} \\ \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} \end{bmatrix}.$$

4. Analysis of Basic Properties of the Computational Method

4.1 Order of the Computational Method

Let the linear operator $\ell\{y(t):h\}$ be defined on equation (10) when $i=0$ such that

$$\ell\{y(t):h\} = A^{(0)}Y_m^{(i)} - \sum_{i=0}^{1}\frac{(jh)^{(i)}}{i!}e_i y_n^{(i)} + h^{(3-i)}[d_i f(y_n) + b_i F(Y_m)] \tag{17}$$

From equation (17), expanding Y_m *and* $F(Y_m)$ in Taylor's series and comparing the coefficients of h gives

$$\ell\{y(t):h\} = C_0 y(t) + C_1 y'(t) + \ldots + C_p h^p y^p(t) + C_{p+1} h^{p+1} y^{p+1}(t) + C_{p+2} h^{p+2} y^{p+2}(t) + \ldots$$

Definition 4.1 ([12])**.** The linear operator ℓ and the associated computational method (10) are said to be of order p if $C_0 = C_1 = \ldots = C_p = C_{p+1} = C_{p+2} = 0$, $C_{p+3} \neq 0$. C_{p+3} is called the error constant and implies that the truncation error is given by $T_{n+k} = C_{p+3}h^{p+3}y^{p+3}(t) + O(h^{p+4})$

$$\begin{bmatrix}
\sum\limits_{j=0}^{\infty} \frac{\left(\frac{1}{4}h\right)^j}{j!} - y_n - \frac{1}{4}hy'_n - \frac{1}{32}h^2y''_n - \frac{113}{71680}h^3y'''_n - \sum\limits_{j=0}^{\infty} \frac{h^{j+3}}{j!}y_n^{j+3} \begin{bmatrix} \frac{107}{64512}\left(\frac{1}{4}\right)^j - \frac{103}{107520}\left(\frac{1}{2}\right)^j \\ + \frac{43}{107520}\left(\frac{3}{4}\right)^j - \frac{47}{645120}(1)^j \end{bmatrix} \\
\sum\limits_{j=}^{\infty} \frac{\left(\frac{1}{2}h\right)^j}{j!} - y_n - \frac{1}{2}hy'_n - \frac{1}{8}h^2y''_n - \frac{331}{40320}h^3y'''_n - \sum\limits_{j=0}^{\infty} \frac{h^{j+3}}{j!}y_n^{j+3} \begin{bmatrix} \frac{83}{5040}\left(\frac{1}{4}\right)^j - \frac{1}{168}\left(\frac{1}{2}\right)^j \\ + \frac{13}{5040}\left(\frac{3}{4}\right)^j - \frac{19}{40320}(1)^j \end{bmatrix} \\
\sum\limits_{j=0}^{\infty} \frac{\left(\frac{3}{4}h\right)^j}{j!} - y_n - \frac{3}{4}hy'_n - \frac{9}{32}h^2y''_n - \frac{1431}{71680}h^3y'''_n - \sum\limits_{j=0}^{\infty} \frac{h^{j+3}}{j!}y_n^{j+3} \begin{bmatrix} \frac{1863}{35840}\left(\frac{1}{4}\right)^j - \frac{243}{35840}\left(\frac{1}{2}\right)^j \\ + \frac{45}{7168}\left(\frac{3}{4}\right)^j - \frac{81}{71680}(1)^j \end{bmatrix} \\
\sum\limits_{j=0}^{\infty} \frac{(h)^j}{j!} - y_n - hy'_n - \frac{1}{2}h^2y''_n - \frac{31}{840}h^3y'''_n - \sum\limits_{j=0}^{\infty} \frac{h^{j+3}}{j!}y_n^{j+3} \begin{bmatrix} \frac{34}{315}\left(\frac{1}{4}\right)^j + \frac{1}{210}\left(\frac{1}{2}\right)^j \\ + \frac{2}{105}\left(\frac{3}{4}\right)^j - \frac{1}{504}(1)^j \end{bmatrix}
\end{bmatrix} = 0.$$

Comparing the coefficients of h, the order p of the computational method and error constants are given respectively by $p = [5 \quad 5 \quad 5 \quad 5]^T$ and
$[5.260 \times 10^{-8} \quad 3.3908 \times 10^{-7} \quad 8.2765 \times 10^{-7} \quad 1.5501 \times 10^{-6}]^T$.

4.2 Consistency of the Computational Method

A computational method is said to be consistent if its order $p \geq 1$. Our method is thus consistent since it is of uniform order 5. Note that consistency controls the magnitude of the local truncation error committed at each stage of the computation [8].

4.3 Zero-stability of the Computational Method

Definition 4.2 ([8])**.** The computational method is said to be zero-stable, if the roots z_s, $s = 1, 2, \ldots, k$ of the first characteristic polynomial $\rho(z)$ defined by $\rho(z) = \det(zA^{(0)} - E)$ satisfies $|z_s| \leq 1$ and every root satisfying $|z_s| = 1$ have multiplicity not exceeding the order of the differential equation. The first characteristic polynomial is given by,

$$\rho(z) = \left| z \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} z & 0 & 0 & -1 \\ 0 & z & 0 & -1 \\ 0 & 0 & z & -1 \\ 0 & 0 & 0 & z-1 \end{bmatrix} \right| = z^3(z-1)$$

Thus, solving for $z$in

$$z^3(z-1) = 0 \tag{18}$$

gives $z = 0, 0, 0, 1$. Hence, the computational method (10) is said to be zero-stable.

4.4 Convergence of the Computational Method

Theorem 4.1 ([19])**.** *The necessary and sufficient conditions that a continuous LMM be convergent are that it be consistent and zero-stable.*

Thus, the computational method formulated is convergent.

4.5 Stability Region of the Computational Method

Definition 4.3 (Yan, 2011)**.** Region of absolute stability is a region in the complex z plane, where $z = \lambda h$ for which the method is absolutely stable. It is defined as those values of z such that the numerical solutions of $y''' = -\lambda y$ satisfy $y_j \to 0$ as $j \to \infty$ for any initial condition.

To determine the regions of absolute stability of the block integrators, a method that requires neither the computation of roots of a polynomial nor solving of simultaneous inequalities was adopted. This method according to [12] is called the Boundary Locus Method. Applying this method we obtain the stability polynomial below

$$\begin{aligned}\overline{h}(w) = h^{12}\left(\frac{1}{4404019200}w^4 - \frac{41}{44040192000}w^4\right) - h^9\left(\frac{233}{77414400}w^3 - \frac{47}{1238630400}w^4\right)\\ + h^6\left(\frac{23}{6881280}w^4 - \frac{66841}{41287680}w^3\right) - \frac{13}{60}h^3w^3 + w^4 - \frac{5}{2}w^3 \end{aligned} \tag{19}$$

On the application of the stability polynomial in equation (19), we obtain the region of absolute stability in the figure below.

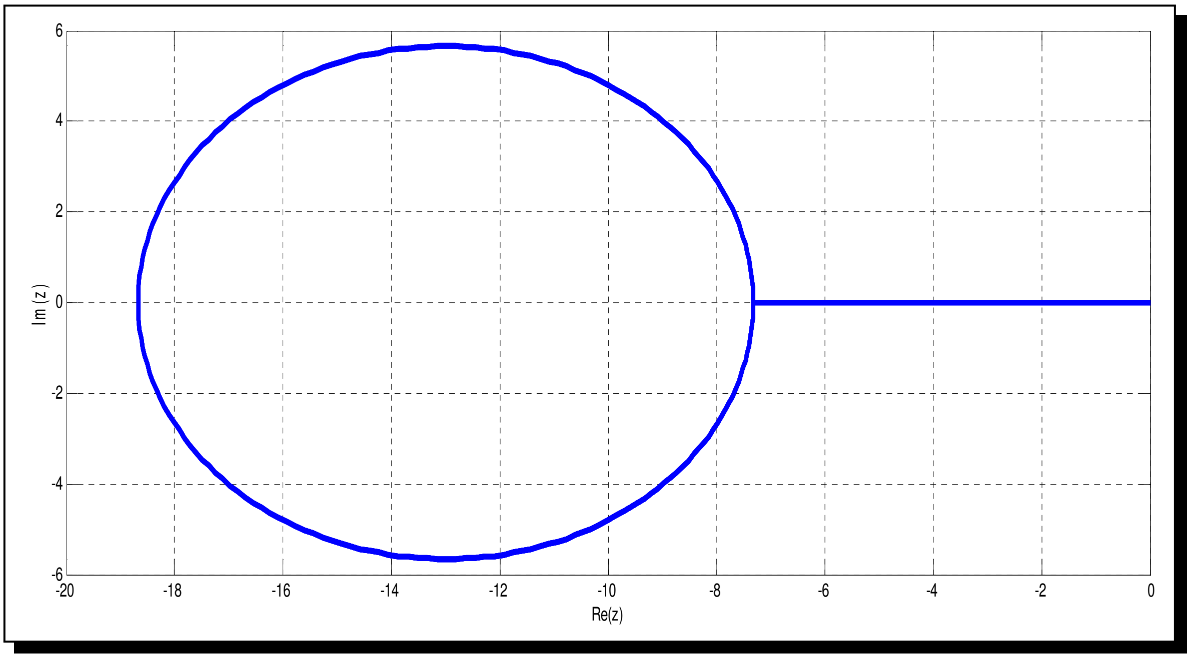

Figure 1. Region of absolute stability of the computational method

The stability region obtained in Figure 1 is A-stable, since it contains the whole of the left-half plane of the figure. Note that the unstable region is the exterior of the curve (when the curve is on the negative plane) while the stability region is the interior of the curve.

5. Implementation of the Computational Method

It is important to state that the one-step computational method formulated can be used to implement higher differential equations of the form (1) without the need to reduce it to an equivalent system of first order. For the computational method formulated which is of uniform order $p = 5$, we use Taylor series expansion to calculate y_{n+1} and its first, second and third derivatives up to order $p = 5$.

$$y_{n+j} \equiv y(t_n + jh) \cong y(t_n) + jhy'(t_n) + \frac{(jh)^2}{2!}y''(t_n) + \frac{(jh)^3}{3!}f_n + \frac{(jh)^4}{4!}f'_n + \frac{(jh)^5}{5!}f''_n,$$

$$y'_{n+j} \equiv y'(t_n + jh) \cong y'(t_n) + jhy''(t_n) + \frac{(jh)^2}{2!}f_n + \frac{(jh)^3}{3!}f'_n + \frac{(jh)^4}{4!}f''_n + \frac{(jh)^5}{5!}f'''_n,$$

$$y''_{n+j} \equiv y''(t_n + jh) \cong y''(t_n) + jhf_n + \frac{(jh)^2}{2!}f'_n + \frac{(jh)^3}{3!}f''_n + \frac{(jh)^4}{4!}f'''_n + \frac{(jh)^5}{5!}f^{iv}_n,$$

$$y'''_{n+j} \equiv y'''(t_n + jh) \cong f_n + jhf'_n + \frac{(jh)^2}{2!}f''_n + \frac{(jh)^3}{3!}f'''_n + \frac{(jh)^4}{4!}f^{iv}_n + \frac{(jh)^5}{5!}f^{v}_n.$$

We proceed with the implementation by substituting the known values of t_n and y_n into the differential equations. Then, the differential equation is differentiated to obtain the expression for higher derivatives via partial differentiation as follows:

$$\begin{aligned}
y''' &= f(t, y, y', y'') = f_j, \\
y^{iv} &= f_t + y'f_y + y''f_{y'} + ff_{y''} = \left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + f\frac{\partial}{\partial y''}\right) = Df_j \\
y^{v} &= f_{tt} + (y')^2 f_{yy} + (y'')^2 f_{y'y'} + f^2 f_{y''y''} + 2y'ff_{ty} + 2y''f_{ty'} \\
&\quad + 2y'y''f_{yy'} + 2y'ff_{yy''} + 2y''ff_{y'y''} + Df_j(f_{y'}) + f_j(y'' + f_{y'}) \\
&= D^2 f_j + (f_{y''})Df_j + f_j(y'' + f_{y'})_j \\
&\ \vdots \\
&D^P f_j
\end{aligned}$$

where p is the order of the computational method. Also, note that

$$D = \left(\frac{\partial}{\partial t} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + f\frac{\partial}{\partial y''}\right) \quad \text{and} \quad D^2 = D(D).$$

6. Results

6.1 Numerical Experiments

In this section, we shall approximate some modeled third order oscillatory problems of the form (1) using the computational method formulated.

Problem 6.1. Consider the third order oscillatory differential equation,

$$y'''(t) = -y'(t), \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 2 \tag{20}$$

with the exact solution is given by

$$y(t) = 2(1 - \cos t) + \sin t. \tag{21}$$

Source: [2]

Table 1. Showing the result for Problem 6.1

t	Exact solution	Computed solution	Error	Error in [2]	Eval t/sec
0.1000	0.109825086090777	0.109825086090776	3.7470e-016	1.6613 e-12	0.1421
0.2000	0.238536175112578	0.238536175112577	8.3267e-016	7.5411 e-12	0.1861
0.3000	0.384847228410128	0.384847228410126	1.3878e-015	1.3843 e-09	0.3092
0.4000	0.547296354302881	0.547296354302879	1.4433e-015	4.5006 e-09	0.8164
0.5000	0.724260414823458	0.724260414823457	1.5543e-015	1.0520 e-08	1.2584
0.6000	0.913971243575679	0.913971243575677	1.9984e-015	1.9715 e-08	1.4651
0.7000	1.114533312668715	1.114533312668713	2.8866e-015	3.2968 e-08	1.6622
0.8000	1.323942672205193	1.323942672205189	4.4409e-015	5.0419 e-08	1.8853
0.9000	1.540106973086156	1.540106973086152	3.5527e-015	7.2608 e-08	1.9926
1.0000	1.760866373071619	1.760866373071613	5.3291e-015	9.9511 e-08	2.1447

Problem 6.2. Consider the third order oscillatory differential equation,

$$y'''(t) = y''(t) - y'(t) + y(t), \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1, \quad h = 0.01 \tag{22}$$

with the exact solution is given by,

$$y(t) = \cos t \tag{23}$$

Source: [18]

Table 2. Showing the result for Problem 6.2

t	Exact Solution	Computed Solution	Error	Error in [18]	Eval t/sec
0.0100	0.999950000416665	0.999950000416665	1.1102e-016	0.0000	0.0290
0.0200	0.999800006666578	0.999800006666579	1.3323e-015	0.0000	0.0334
0.0300	0.999550033748988	0.999550033748997	9.6589e-015	0.0000	0.0379
0.0400	0.999200106660978	0.999200106661011	3.2974e-014	1.0e-10	0.0424
0.0500	0.998750260394966	0.998750260395049	8.2379e-014	1.0e-10	0.0469

Problem 6.3. Consider the third order oscillatory differential equation,

$$y'''(t) = 3\sin t, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2 \tag{24}$$

with the exact solution is given by,

$$y(t) = 3\cos t + \frac{t^2}{2} - 2 \tag{25}$$

Source: [2]

Table 3. Showing the result for Problem 6.3

t	Exact solution	Computed Solution	Error	Error in [2]	Eval t/sec
0.1000	0.990012495834077	0.990012495834077	3.3307e-016	2.5934 e-12	0.0093
0.2000	0.960199733523725	0.960199733523725	3.3307e-016	1.1857 e-11	0.0137
0.3000	0.911009467376818	0.911009467376818	3.3307e-016	2.6224 e-11	0.0181
0.4000	0.843182982008655	0.843182982008655	1.1102e-016	4.7034 e-11	0.0233
0.5000	0.757747685671118	0.757747685671118	1.1102e-016	7.2700 e-11	0.0278
0.6000	0.656006844729034	0.656006844729035	4.4409e-016	1.0437 e-10	0.0322
0.7000	0.539526561853465	0.539526561853465	5.5511e-016	1.4049 e-10	0.0366
0.8000	0.410120128041496	0.410120128041496	5.5511e-016	1.8197 e-10	0.0413
0.9000	0.269829904811993	0.269829904811993	7.2164e-016	2.2736 e-10	0.0457
1.0000	0.120906917604418	0.120906917604419	1.0547e-015	2.7729 e-10	0.0501

Problem 6.4. Consider the third order oscillatory differential equation,

$$y'''(t) = -4y'(t) + t, \quad y(0) = \quad y'(0) = 0, \quad y''(0) = 1 \tag{26}$$

with the exact solution is given by,

$$y(t) = \left(\frac{3}{16}\right)(1 - \cos 2t) + \left(\frac{1}{8}\right)t^2 \tag{27}$$

Source: [15]

Table 4. Showing the result for Problem 6.4

t	Exact Solution	Computed Solution	Error	Error in [15]	Eval t/sec
0.1000	0.004987516654767	0.004987516653935	8.3209e-013	9.61000e-10	0.0293
0.2000	0.019801063624459	0.019801063620984	3.4752e-012	6.50000 e-09	0.0341
0.3000	0.043999572204435	0.043999572196618	7.8178e-012	1.59700 e-08	0.0387
0.4000	0.076867491997407	0.076867491983726	1.3681e-011	1.66400 e-08	0.0433
0.5000	0.117443317649724	0.117443317628899	2.0825e-011	2.03000 e-08	0.1052
0.6000	0.164557921035624	0.164557921006662	2.8962e-011	2.66000 e-08	0.1723
0.7000	0.216881160706205	0.216881160668441	3.7764e-011	2.67000 e-08	0.2364
0.8000	0.272974910431492	0.272974910384613	4.6879e-011	2.71000 e-08	0.2984
0.9000	0.331350392754954	0.331350392699013	5.5941e-011	2.77000 e-08	0.3442
1.0000	0.390527531852590	0.390527531787998	6.4592e-011	2.72000 e-08	0.3488

6.2 Discussion of Result

The results obtained in Tables 1-4 clearly show that the computational method developed is computationally reliable and efficient. This is because the computed solution matches the exact solution. In fact, the method obviously performed better than the ones with which we compared our results. The method is also efficient because from the tables, the evaluation times per seconds are very small. This shows that the method generates results very fast (in microseconds).

7. Conclusion

In this paper, the oscillation criteria of third order oscillatory problems have been studied. A highly efficient computational method has also been formulated for the approximation of third order oscillatory problems of the form (1). The basic properties of the method were also analyzed. The stability region of the method was found to A-stable, showing that it can effectively approximate oscillatory problems of third order.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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