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Research Article

Solutions of Integral Nonclassical Ordinary Differential Equations Via Contractor Maps

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Abstract. Existence of a unique and bounded stochastic solution of integral nonclassical ordinary differential equation is studied using the method of integral contractor operators.

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1. Introduction

This research is concerned with the existence of solution of integral *Nonclassical Ordinary Differential Equation* (NODE) also known as *Quantum Stochastic Differential Equation* (QSDE). QSDE is a simplified form of the Hudson-Parthasarathy formulation of quantum stochastic integrals of simple adapted processes (see [2] and the references therein for details.

Nonclassical ordinary differential equation has applications in quantum stochastic control theory and quantum dynamical systems. Also see [2, 4–6]. There exist some interesting results on integral solutions of differential equations of various forms using integral contractors in literature [1, 3, 7–17]. The authors in [8], studied a general integral equation of mixed type using the techniques of random contractors. Padgett and Rao [11], used the method in [8] to establish similar result for a class of stochastic integral equation. The results due to [11], is a motivation for extending the method of random contractor operator to this class of nonclassical ODE. The concept of a bounded integral vector contractor was used to obtain some general conditions for the existence of solutions of a class of stochastic integral equation (see [11]) and the references therein. The method was then used to establish some qualitative properties of solutions. Similar results were established in [13] by using random operators and the Volterra-fredholm integral equations.

In this present work, the concept of integral bounded contractor of NODE is introduced and studied using similar methods in the literature (see [7, 10]). We show how this method can be applied to Ito type stochastic differential equations while, real life applications of this method will be considered in subsequent works (see [7, 10, 11, 16]). Considering QSDE formulated by Hudson and Parthasarathy in [2, 4, 6], given by

$$x(t) = x_0 + \int_0^t (E(s,x(s))d\wedge_\pi(s) + F(s,x(s))dA_f(s) + G(s,x(s))dA_g^+(s) + H(s,x(s))ds),$$
$$x(0) = x_0, \quad t \in [0,T]. \tag{1}$$

Integral equation (1) is a noncommutative generalization of the following classical Ito stochastic integral equation:

$$x(t,w) = x_0 + \int_0^t H(s,x(s))d(s) + F(s,x(s))dQ(s),$$
$$x(0) = x_0, \quad t \in [0,T], \tag{2}$$

where $Q(t)$ is a Martingale and H, F are non anticipated Brownian functions. If we assume $Q(t)$ be defined as follows:

$$Q(t) = A(t) + A'(t) = w(t),$$

where $w(t)$ is the evaluation of the Brownian path w at time t. Then equation (2) becomes

$$x(t,w) = x_0 + \int_0^t H(s,x(s))d(s) + F(s,x(s))dw(s),$$
$$x(0) = x_0, \quad t \in [0,T]. \tag{3}$$

It has been established in [2, 4–6], that integral equation (1) is equivalent to the *Nonclassical Ordinary Differential Equation* (NODE) in integral form given by

$$\langle \eta, x(t)\xi \rangle = \langle \eta, x_0\xi \rangle + \int_0^t P(s,x(s))(\eta,\xi)ds, \tag{4}$$

where the map $(\eta,\xi) \to P(s,x(s))(\eta,\xi)$ is a sesquilinear form well defined in [2, 4]. $x \in \widetilde{\mathcal{A}}$, $\eta,\xi \in sesq(\mathbb{D} \underline{\otimes} \mathbb{E})$.

The paper is organized as follows: In Section 2, we establish some preliminaries. In Section 3, the major result is established. Conclusion of the paper is presented in Section 4.

2. Preliminaries

We adopt the definitions and notations of some of the spaces $L^2_{loc}(\widetilde{\mathcal{A}})_{mvs}$, $\widetilde{\mathcal{A}}$, $sesq(\mathbb{D} \underline{\otimes} \mathbb{E})$ from the references [4, 5] and the references therein. $\widetilde{\mathcal{A}}$ is a locally convex space whose topology is generated by a family of seminorm well defined in the references [2, 4–6]. $\mathbb{D}$ is an inner product space, $\mathbb{E}$ is a linear space generated by the exponential vectors in some Hilbert space $\Gamma(L^2_\gamma(\mathbb{R}_+))$. The spaces $L^p_{loc}(\widetilde{\mathcal{A}})_{mvs} \times [0,T]$ and $Ad(\widetilde{\mathcal{A}})_{wac}$ denote the spaces of bounded linear operators that are locally p-integrable and weakly absolutely continuous on $[0,T]$. In a similar manner, we define the spaces $L^2_{loc}(\widetilde{\mathcal{A}})(\eta,\xi)$ and $wac(\widetilde{\mathcal{A}})(\eta,\xi)$ as the complex valued equivalents of $L^2_{loc}(\widetilde{\mathcal{A}})_{mvs}$ and $Ad(\widetilde{\mathcal{A}})_{wac}$. The space $wac(\widetilde{\mathcal{A}})(\eta,\xi)$ of complex valued functions on $[0,T]$ is equipped with the supremum norm defined by

$$\|x(t)\|_{\eta\xi} = \sup\{|x(t)(\eta,\xi)|; t \in [0,T]\}.$$

Definition 1. Let $K_{\eta\xi} > 0$ be a constant and let the map $N : [0,T] \times wac(\widetilde{\mathcal{A}})(\eta,\xi) \to L^p_{loc}(\widetilde{\mathcal{A}})(\eta,\xi)$ be bounded and continuous on $[0,T]$, such that

$$\left\| P(t,x(t)+y(t)+\int_0^t N(s,x(s))y(s)d(s)) - P(t,x(t)) - N(t,x(t))y(t) \right\|_{\eta\xi} \le K_{\eta\xi}\|y(t)\|_{\eta\xi}, \quad t \in [0,T]. \quad (5)$$

$(x,y \in \widetilde{\mathcal{A}})$. Then we say that the complex-valued map $P(s,x(s))(\eta,\xi)$ has a bounded stochastic integral contractor $\{I + \int_0^t N ds\}$.

The next definition in [11], is recast within the context of this work.

Definition 2. A bounded stochastic integral contractor is said to be regular if the integral equation

$$x_0(\eta,\xi) = \langle \eta, y(t)\xi \rangle + \int_0^t N(s,x(s))(\eta,\xi)y(s)(\eta,\xi)d(s) \quad (6)$$

has a solution $y \in L^p_{loc}(\widetilde{\mathcal{A}})$, $t \in [0,T]$ for any $x \in L^p_{loc}(\widetilde{\mathcal{A}})$.

Remark 3. If the map $(\eta,\xi) \to P(s,x(s))(\eta,\xi)$ is Lipschitzian with the Lipschitz function defined by $K^p_{\eta\xi} : [0,T] \to \mathbb{R}_+$ with $N \le 1$, then we obtain some of the results in [6] and the references therein.

Next, we present the main results in this work.

3. Main Result

Theorem 4. *Let the map P have a bounded integral contractor, then the equation* (4) *respectively* (1) *has an absolutely continuous solution. Also a unique continuous solution exists if the bounded integral contractor is regular.*

Proof. Define the Cauchy sequences $\{x_{n+1}\}_{n\ge0}$ and $\{y_n\}_{n\ge0}$ in $\widetilde{\mathcal{A}}$ as

$$x_{n+1}(t)(\eta,\xi) = x_n(t)(\eta,\xi) - \left[y_n(t)(\eta,\xi) + \int_0^t N(s,x_n(s))(\eta,\xi)y_n(s)(\eta,\xi)d(s) \right] \quad (7)$$

and

$$y_n(t)(\eta,\xi) = x_n(t)(\eta,\xi) - \int_0^t P(s,x_n(s))(\eta,\xi)d(s) - x_0(\eta,\xi). \qquad \square \quad (8)$$

Note. In what follows; $\eta,\xi \in sesq(\mathbb{D} \underline{\otimes} \mathbb{E})$ is arbitrary.
Using (7) and (8), we obtain the following:

$$\begin{aligned} y_{n+1}(t)(\eta,\xi) &= x_{n+1}(t)(\eta,\xi) - \int_0^t P(s,x_{n+1}(s))(\eta,\xi)ds - \langle \eta, x_0\xi \rangle \\ &= x_n(t)(\eta,\xi) - y_n(t)(\eta,\xi) - \int_0^t N(s,x_n(s))(\eta,\xi)y_n(s)(\eta,\xi)ds \end{aligned}$$

$$
\begin{aligned}
&-\int_0^t P(s,x_n(s)(\eta,\xi)-y_n(s)(\eta,\xi)\\
&-\int_0^t N(u,x_n(u))(\eta,\xi)y_n(u)(\eta,\xi)du)ds-\langle\eta,x_0\xi\rangle\\
=&\int_0^t\{P(s,x_n(s))(\eta,\xi)d(s)-N(s,x_n(s))(\eta,\xi)y_n(s)(\eta,\xi)\langle\eta,x_0\xi\rangle\\
&-P(s,x_n(s))(\eta,\xi)-y_n(t)(\eta,\xi)-\int_0^t N(u,x_n(u))(\eta,\xi)y_n(u)(\eta,\xi)du\}ds,
\end{aligned}
$$

where $\langle\eta,x_0\xi\rangle=x_0(\eta,\xi)$. By using (5), we get

$$\|y_{n+1}(t)\|_{\eta\xi}\le K_{\eta\xi}\int_0^t\|y_n(s)\|_{\eta\xi}ds,\quad n\ge 0.$$

Proving this by induction, we obtain

$$\|y_n(t)\|_{\eta\xi}\le K_{\eta\xi}\int_0^t(t-s)^n\|P(s,x(s))\|_{\eta\xi}ds,\quad n=0,1,2,\dots \tag{9}$$

Since the map $s\to P(s,x(s))(\eta,\xi)$ is continuous on $[0,T]$, we let $R_{\eta\xi}=\sup\limits_{s\in[0,T]}\|P(t,x(t)\|_{\eta\xi}$, $t\in[0,T]$ and (9) becomes

$$\|y_n(t)\|_{\eta\xi}\le\frac{R_{\eta\xi}K_{\eta\xi}^n t^{n+1}}{(n+1)!},\quad n=0,1,2,\dots \tag{10}$$

$$
\begin{aligned}
\|y_n(t)\|_{\eta\xi}&=\left\|\sum_{m=n+1}^{\infty}y_n(t)\right\|_{\eta\xi}\\
&\le\sum_{m=n+1}^{\infty}\|y_n(t)\|_{\eta\xi}\\
&\le\sum_{m=n+1}^{\infty}\frac{R_{\eta\xi}K_{\eta\xi}^{m-1}T^m}{m!},\quad n=0,1,2,\dots. \qquad (11)
\end{aligned}
$$

Similarly, by substituting (11) in (7), we get

$$x_{n+1}(t)(\eta,\xi)-x_n(t)(\eta,\xi)=-y_n(t)(\eta,\xi)-\int_0^t N(s,x_n(s))(\eta,\xi)y_n(t)(\eta,\xi)ds,$$

$$
\begin{aligned}
\|x_{n+1}(t)-x_n(t)\|_{\eta\xi}&\le\|y_n(t)\|_{\eta\xi}-\int_0^t N(s,x_n(s))(\eta,\xi)\|y_n(t)\|_{\eta\xi}ds\\
&\le\sum_{m=n+1}^{\infty}\frac{R_{\eta\xi}K_{\eta\xi}^{m-1}T^m}{m!}+\sum_{m=n+1}^{\infty}\frac{R_{\eta\xi}K_{\eta\xi}^{m-1}T^m}{m!}\int_0^t N(s,x_n(s))(\eta,\xi)ds\\
&\le\sum_{m=n+1}^{\infty}\frac{R_{\eta\xi}K_{\eta\xi}^{m-1}T^m}{m!}+\sum_{m=n+1}^{\infty}\frac{R_{\eta\xi}M_{\eta\xi}K_{\eta\xi}^{m-1}T^mT}{m!}\\
&\le\sum_{m=n+1}^{\infty}\frac{R_{\eta\xi}K_{\eta\xi}^{m-1}T^m}{m!}[1+M_{\eta\xi}T], \qquad (12)
\end{aligned}
$$

where $M_{\eta\xi}=\sup\{\|N(t,x(t))\|_{\eta\xi}:t\in[0,T],x\in\widetilde{\mathcal{A}}\}$.

Remark 5. If $M_{\eta\xi}T=0$ in (12), we obtain similar results in [6].

It can be seen that both (11) and (12) converges to a limit as $n\to\infty$, so that this limit is the solution of the nonclassical differential equation given by

$$\begin{aligned}&\frac{d}{dt}\langle\eta,x(t)\xi\rangle = P(t,x(t))(\eta,\xi),\\ &\quad\langle\eta,x(0)\xi\rangle = \langle\eta,x_0\xi\rangle.\end{aligned} \tag{13}$$

Uniqueness

Suppose that $x_1(t)$ and $x_2(t)$ are two solutions of (13). Then, putting $x_1 = x$, $x_2(t)-x_1(t) = x_0(t)$ in (6), we can find a $y(t) \in [0,T]\times\widetilde{\mathcal{A}}\}$ so that

$$x_2(t)(\eta,\xi) - x_1(t)(\eta,\xi) = y(t)(\eta,\xi) + \int_0^t N(s,x_1(s))(\eta,\xi)y(s)(\eta,\xi)ds. \tag{14}$$

Again by (5), we get

$$\begin{aligned}&\left\|P(t,x_1(t)+y(t)+\int_0^t N(s,x_1(s))y(s)d(s)) - P(s,x_1(t)) - N(t,x(t))y(t)\right\|_{\eta\xi}\\ &\quad\le K_{\eta\xi}\|y(t)\|_{\eta\xi}, \quad t\in[0,T].\end{aligned} \tag{15}$$

Substituting (14) into (15), we get

$$\|P(t,x_2(t)) - P(t,x_1(t)) - N(t,x(t))y(t)\|_{\eta\xi} \le K_{\eta\xi}\|y(t)\|_{\eta\xi}, \tag{16}$$

and by (14) we get

$$\begin{aligned}y(t)(\eta,\xi) &= x_2(t)(\eta,\xi) - x_1(t)(\eta,\xi) - \int_0^t N(s,x_1(s))(\eta,\xi)y(s)(\eta,\xi)ds\\ &= \int_0^t \|P(s,x_2(s)) - P(s,x_1(s)) - N(s,x(s))y(s)\|_{\eta\xi}ds\end{aligned}$$

so that by (16), we obtain

$$\|y(t)\|_{\eta\xi} \le K_{\eta\xi}\int_0^t \|y(s)\|_{\eta\xi}ds.$$

By continuing the process of integration, the following result is obtained for all $n = 0,1,2,\dots$

$$\begin{aligned}&\|y(t)\|_{\eta\xi} \le \|y\|_{\eta\xi}\left[\frac{(K_{\eta\xi}t)^{n+1}}{(n+1)!}\right], \quad t\in[0,T];\\ &\|y(t)\|_{\eta\xi} \le \|y\|_{\eta\xi}\left[\sum_{m=n+1}^{\infty}\frac{(K_{\eta\xi}T)^m}{m!}\right], \quad t\in[0,T].\end{aligned}$$

As $n\to\infty$, y must converge to zero in $\widetilde{\mathcal{A}}$ and so, $x_2(t) = x_1(t)$, $t\in[0,T]$. And the solution is unique.

4. Conclusion

The results in this work, can be extended to Ito integral stochastic differential equation by using the method in [2]. For example, for the exponential vectors $\eta = e(\alpha)$ and $\xi = e(\beta)$, where α and β are purely imaginary-valued functions in $L^2_{\mathbb{C}}(\mathbb{R}_+)$, the equivalent form (4) of the quantum analogue of the classical Ito stochastic differential equation

$$\begin{aligned}&dx(t,w) = -\frac{1}{2}x(t,w)dt - \sqrt{1-x^2(t,w)}\,dw(t),\\ &\quad x(0) = x_0, \quad t\in[0,T]\end{aligned} \tag{17}$$

is given by

$$\frac{d}{dt}E(x(t,w)z(w)) = E\left(-\beta(t)\sqrt{1-x^2(t,w)}z(w)\right) + E\left(-\alpha(t)\sqrt{1-x^2(t,w)}z(w)\right) + E\left(-\frac{1}{2}x(t,w)z(w)\right),$$

$$x(0) = x_0, \quad t \in [0,T], \tag{18}$$

where

$$z(w) = \exp\left\{\int_0^\infty (-\alpha(s)+\beta(s))dw(s) - \frac{1}{2}\int_0^\infty (-\alpha^2(s)+\beta^2(s))ds\right\} \tag{19}$$

(see [2] for details). Therefore, the results here are applicable to Ito type stochastic integral equation.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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