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Research Article

Some Infinite Sums Related to the k-Fibonacci Numbers

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Abstract. In this study, some infinite sums related to the k-Fibonacci numbers have been obtained by using infinite sums related to classic Fibonacci numbers in literature.

Keywords. Infinite sums; Fibonacci numbers

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1. Introduction

The well-known Fibonacci sequence and the golden ratio with the many interesting features [2, 11, 13], have been attracted attention of theoretical physics [1, 5, 6], engineerings [4, 14], architects [3, 8], orthodontics [12] as much as mathematicians. Numerous features of this interesting number sequence have been found over time [9]. Different number sequences, such as the Pell and Lucas number sequences that relate to Fibonacci sequence, have been discussed along with studies on Fibonacci sequence, and their different generalizations have been mentioned [10]. Similarly, Falcon and Plaza introduced the k-Fibonacci sequence, which is a generalization of these number sequences, giving the classic Fibonacci sequence and the classic Pell sequence for $k = 1$ and $k = 2$, respectively. For any integer number $k \geqslant 1$, the kth Fibonacci sequence $\{F_{n,k}\}_{n\in\mathbb{N}}$ is defined recurrently by

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad (n \geqslant 1) \tag{1.1}$$

where $F_{k,0}=0$, $F_{k,1}=1$. The solution of the equation (1.1) is

$$F_{k,n}=\frac{r_1^n-r_2^n}{r_1-r_2}, \tag{1.2}$$

where the roots of characteristic equation of (1.1) are $r_1=\frac{k+\sqrt{k^2+4}}{2}$, $r_2=\frac{k-\sqrt{k^2+4}}{2}$ [7].

In this study, based on the some infinite sums of the Fibonacci numbers [15] is investigated counterparts in the k-Fibonacci numbers.

2. Main Results

In this section, we obtain some results related to the k-Fibonacci numbers by using [15]. Also, the equalities given for the infinite sums in the theorems corresponds to the limit phrase of the sums.

Theorem 2.1. *For k-Fibonacci numbers, the equality*

$$\sum_{n=1}^{\infty}\frac{1}{F_{k,n}}=\frac{k^2+k+1}{k^2}+\sum_{n=2}^{\infty}\frac{(-1)^n}{F_{k,n-1}F_{k,n}F_{k,n+1}} \tag{2.1}$$

holds.

Proof. We can write the equality

$$\sum_{s=2}^{n}\left(\frac{1}{F_{k,s}}-\frac{F_{k,s}}{F_{k,s-1}F_{k,s+1}}\right)=\sum_{s=2}^{n}\left(\frac{F_{k,s-1}F_{k,s+1}-F_{k,s}^2}{F_{k,s-1}F_{k,s}F_{k,s+1}}\right).$$

By using Cassini formula [7]

$$F_{k,n-1}F_{k,n+1}-F_{k,n}^2=(-1)^n$$

for k-Fibonacci numbers in above equality, we have

$$\sum_{s=2}^{n}\left(\frac{1}{F_{k,s}}-\frac{F_{k,s}}{F_{k,s-1}F_{k,s+1}}\right)=\sum_{s=2}^{n}\left(\frac{(-1)^s}{F_{k,s-1}F_{k,s}F_{k,s+1}}\right). \tag{2.2}$$

On the other hand, it is obvious from equation (1.1)

$$\begin{aligned}\sum_{s=2}^{n}\frac{F_{k,s}}{F_{k,s-1}F_{k,s+1}}&=\frac{1}{k}\sum_{s=2}^{n}\frac{F_{k,s+1}-F_{k,s-1}}{F_{k,s-1}F_{k,s+1}}\\&=\frac{1}{k}\sum_{s=2}^{n}\left(\frac{1}{F_{k,s-1}}-\frac{1}{F_{k,s+1}}\right)\\&=\frac{1}{k}\left[\left(\frac{1}{F_{k,1}}-\frac{1}{F_{k,3}}\right)+\left(\frac{1}{F_{k,2}}-\frac{1}{F_{k,4}}\right)+\left(\frac{1}{F_{k,3}}-\frac{1}{F_{k,5}}\right)+\dots\right.\\&\quad\left.+\left(\frac{1}{F_{k,n-3}}-\frac{1}{F_{k,n-1}}\right)+\left(\frac{1}{F_{k,n-2}}-\frac{1}{F_{k,n}}\right)+\left(\frac{1}{F_{k,n-1}}-\frac{1}{F_{k,n+1}}\right)\right]\\&=\frac{1}{k}\left[\left(1+\frac{1}{k}\right)-\left(\frac{1}{F_{k,n}}+\frac{1}{F_{k,n+1}}\right)\right].\end{aligned}$$

If the limits of both sides of the last equation are taken for $n\to\infty$, then we have

$$\lim_{n\to\infty}\sum_{s=2}^{n}\frac{F_{k,s}}{F_{k,s-1}F_{k,s+1}}=\lim_{n\to\infty}\frac{1}{k}\left[\left(1+\frac{1}{k}\right)-\left(\frac{1}{F_{k,n}}+\frac{1}{F_{k,n+1}}\right)\right].$$

From equation (1.2), we write

$$\lim_{n\to\infty}\sum_{s=2}^{n}\frac{F_{k,s}}{F_{k,s-1}F_{k,s+1}} = \lim_{n\to\infty}\frac{1}{k}\left[\left(1+\frac{1}{k}\right)-\frac{r_1-r_2}{r_1^n-r_2^n}-\frac{r_1-r_2}{r_1^{n+1}-r_2^{n+1}}\right]$$
$$= \lim_{n\to\infty}\frac{1}{k}\left[\left(1+\frac{1}{k}\right)-\frac{r_1-r_2}{r_1^n\left[1-\left(\frac{r_2}{r_1}\right)^n\right]}-\frac{r_1-r_2}{r_1^{n+1}\left[1-\left(\frac{r_2}{r_1}\right)^{n+1}\right]}\right].$$

It is obvious that $\lim_{n\to\infty}\left(\frac{r_2}{r_1}\right)^n=0$ from $r_2<r_1$. Thus we have

$$\sum_{n=2}^{\infty}\frac{F_{k,n}}{F_{k,n-1}F_{k,n+1}}=\frac{k+1}{k^2}.$$

If the limits of both sides of equation (2.2) are taken for $n\to\infty$, then we have

$$\lim_{n\to\infty}\sum_{s=2}^{n}\left(\frac{1}{F_{k,s}}-\frac{F_{k,s}}{F_{k,s-1}F_{k,s+1}}\right)=\lim_{n\to\infty}\sum_{s=2}^{n}\left(\frac{(-1)^s}{F_{k,s-1}F_{k,s}F_{k,s+1}}\right).$$

From the last equation, we can write

$$\sum_{n=2}^{\infty}\frac{1}{F_{k,n}}=\sum_{n=2}^{\infty}\left(\frac{F_{k,n}}{F_{k,n-1}F_{k,n+1}}+\frac{(-1)^n}{F_{k,n-1}F_{k,n}F_{k,n+1}}\right).$$

Thus, we obtain

$$\sum_{n=1}^{\infty}\frac{1}{F_{k,n}}=\frac{1}{F_{k,1}}+\sum_{n=2}^{\infty}\left(\frac{F_{k,n}}{F_{k,n-1}F_{k,n+1}}+\frac{(-1)^n}{F_{k,n-1}F_{k,n}F_{k,n+1}}\right)$$
$$=1+\frac{k+1}{k^2}+\sum_{n=2}^{\infty}\frac{(-1)^n}{F_{k,n-1}F_{k,n}F_{k,n+1}}$$
$$=\frac{k^2+k+1}{k^2}+\sum_{n=2}^{\infty}\frac{(-1)^n}{F_{k,n-1}F_{k,n}F_{k,n+1}}.$$

□

Theorem 2.2. *For $n\geq 2$, the equality*

$$\sum_{n=2}^{\infty}\frac{1}{F_{k,n-1}F_{k,n+1}}=\frac{1}{k^2}$$

holds.

Proof. We can write the equality

$$\sum_{s=2}^{n}\frac{1}{F_{k,s-1}F_{k,s+1}}=\sum_{s=2}^{n}\frac{F_{k,s}}{F_{k,s-1}F_{k,s}F_{k,s+1}}.$$

From the equation (1.1), we have

$$\sum_{s=2}^{n}\frac{1}{F_{k,s-1}F_{k,s+1}}=\sum_{s=2}^{n}\frac{1}{k}\left[\frac{F_{k,s+1}-F_{k,s-1}}{F_{k,s-1}F_{k,s}F_{k,s+1}}\right]$$
$$=\frac{1}{k}\sum_{s=2}^{n}\left(\frac{1}{F_{k,s-1}F_{k,s}}-\frac{1}{F_{k,s}F_{k,s+1}}\right)$$
$$=\frac{1}{k}\left[\left(\frac{1}{F_{k,1}F_{k,2}}-\frac{1}{F_{k,2}F_{k,3}}\right)+\left(\frac{1}{F_{k,2}F_{k,3}}-\frac{1}{F_{k,3}F_{k,4}}\right)+\dots\right.$$
$$\left.+\left(\frac{1}{F_{k,n-2}F_{k,n-1}}-\frac{1}{F_{k,n-1}F_{k,n}}\right)+\left(\frac{1}{F_{k,n-1}F_{k,n}}-\frac{1}{F_{k,n}F_{k,n+1}}\right)\right]$$

$$= \frac{1}{k}\left[\frac{1}{F_{k,1}F_{k,2}} - \frac{1}{F_{k,n}F_{k,n+1}}\right] = \frac{1}{k}\left[\frac{1}{k} - \frac{1}{F_{k,n}F_{k,n+1}}\right].$$

If the limits of both sides of the last equation are taken for $n\to\infty$, then we can write

$$\lim_{n\to\infty}\sum_{s=2}^{n}\frac{1}{F_{k,s-1}F_{k,s+1}} = \lim_{n\to\infty}\frac{1}{k}\left[\frac{1}{k} - \frac{1}{F_{k,n}F_{k,n+1}}\right].$$

By considering $\lim_{n\to\infty}\frac{1}{F_{k,n}F_{k,n+1}} = 0$, we obtain

$$\sum_{n=2}^{\infty}\frac{1}{F_{k,n-1}F_{k,n+1}} = \frac{1}{k^2}.$$

□

Theorem 2.3. *For $n \geq 1$, the equality*

$$\sum_{n=1}^{\infty}\frac{1}{F_{k,n}F_{k,n+2}^2F_{k,n+3}} + \sum_{n=1}^{\infty}\frac{1}{F_{k,n}F_{k,n+1}^2F_{k,n+3}} = \frac{1}{k^3(k^2+1)}$$

holds.

Proof.

$$\sum_{s=1}^{n}\frac{1}{F_{k,s}F_{k,s+2}^2F_{k,s+3}} + \sum_{s=1}^{n}\frac{1}{F_{k,s}F_{k,s+1}^2F_{k,s+3}}$$

$$= \sum_{s=1}^{n}\left(\frac{F_{k,s+1}}{F_{k,s}F_{k,s+1}F_{k,s+2}^2F_{k,s+3}} + \frac{F_{k,s+2}}{F_{k,s}F_{k,s+1}^2F_{k,s+2}F_{k,s+3}}\right).$$

From the equation (1.1), we can write the following equality

$$= \frac{1}{k}\sum_{s=1}^{n}\left(\frac{F_{k,s+2}-F_{k,s}}{F_{k,s}F_{k,s+1}F_{k,s+2}^2F_{k,s+3}} + \frac{F_{k,s+3}-F_{k,s+1}}{F_{k,s}F_{k,s+1}^2F_{k,s+2}F_{k,s+3}}\right)$$

$$= \frac{1}{k}\sum_{s=1}^{n}\left[\frac{1}{F_{k,s}F_{k,s+1}^2F_{k,s+2}} - \frac{1}{F_{k,s+1}F_{k,s+2}^2F_{k,s+3}}\right]$$

$$= \frac{1}{k}\left[\left(\frac{1}{F_{k,1}F_{k,2}^2F_{k,3}} - \frac{1}{F_{k,2}F_{k,3}^2F_{k,4}}\right) + \left(\frac{1}{F_{k,2}F_{k,3}^2F_{k,4}} - \frac{1}{F_{k,3}F_{k,4}^2F_{k,5}}\right) + \dots\right.$$

$$\left. + \frac{1}{F_{k,n}F_{k,n+1}^2F_{k,n+2}} - \frac{1}{F_{k,n+1}F_{k,n+2}^2F_{k,n+3}}\right].$$

Thus, we obtain

$$\sum_{s=1}^{n}\frac{1}{F_{k,s}F_{k,s+2}^2F_{k,s+3}} + \sum_{s=1}^{n}\frac{1}{F_{k,s}F_{k,s+1}^2F_{k,s+3}} = \frac{1}{k}\left[\frac{1}{k^2(k^2+1)} - \frac{1}{F_{k,n+1}F_{k,n+2}^2F_{k,n+3}}\right].$$

If the limits of both sides of the last equation are taken for $n\to\infty$, then we can write

$$\lim_{n\to\infty}\left(\sum_{s=1}^{n}\frac{1}{F_{k,s}F_{k,s+2}^2F_{k,s+3}} + \sum_{s=1}^{n}\frac{1}{F_{k,s}F_{k,s+1}^2F_{k,s+3}}\right)$$

$$= \lim_{n\to\infty}\frac{1}{k}\left[\frac{1}{k^2(k^2+1)} - \frac{1}{F_{k,n+1}F_{k,n+2}^2F_{k,n+3}}\right].$$

By considering $\lim_{n\to\infty}\frac{1}{F_{k,n+1}F_{k,n+2}^2F_{k,n+3}}=0$, we obtain

$$\sum_{n=1}^{\infty}\frac{1}{F_{k,n}F_{k,n+2}^2F_{k,n+3}}+\sum_{n=1}^{\infty}\frac{1}{F_{k,n}F_{k,n+1}^2F_{k,n+3}}=\frac{1}{k^3(k^2+1)}.$$ □

Theorem 2.4. *For k-Fibonacci numbers, the equalities*

(a) $F_{k,n+1}=\prod_{s=1}^{n}\left(k+\frac{F_{k,s-1}}{F_{k,s}}\right)$,

(b) $\frac{F_{k,n+1}}{F_{k,n}}=k+\sum_{s=2}^{n}\frac{(-1)^s}{F_{k,s}F_{k,s-1}}$.

hold.

Proof. (a)

$$F_{k,n+1}=\frac{F_{k,n+1}F_{k,n}F_{k,n-1}\dots F_{k,2}}{F_{k,n}F_{k,n-1}\dots F_{k,2}F_{k,1}}=\prod_{s=1}^{n}\left(\frac{F_{k,s+1}}{F_{k,s}}\right).$$

From the equation (1.1), we have

$$F_{k,n+1}=\prod_{s=1}^{n}\left(\frac{kF_{k,s}+F_{k,s-1}}{F_{k,s}}\right)=\prod_{s=1}^{n}\left(k+\frac{F_{k,s-1}}{F_{k,s}}\right).$$

(b)

$$\begin{aligned}\frac{F_{k,n+1}}{F_{k,n}}&=\left(\frac{F_{k,n+1}}{F_{k,n}}-\frac{F_{k,n}}{F_{k,n-1}}\right)+\left(\frac{F_{k,n}}{F_{k,n-1}}-\frac{F_{k,n-1}}{F_{k,n-2}}\right)+\dots+\left(\frac{F_{k,3}}{F_{k,2}}-\frac{F_{k,2}}{F_{k,1}}\right)+\frac{F_{k,2}}{F_{k,1}}\\&=k+\sum_{s=2}^{n}\left(\frac{F_{k,s+1}}{F_{k,s}}-\frac{F_{k,s}}{F_{k,s-1}}\right)\\&=k+\sum_{s=2}^{n}\left(\frac{F_{k,s+1}F_{k,s-1}-F_{k,s}^2}{F_{k,s}F_{k,s-1}}\right).\end{aligned}$$

From Cassini formula for k-Fibonacci numbers, we obtain

$$\frac{F_{k,n+1}}{F_{k,n}}=k+\sum_{s=2}^{n}\left(\frac{(-1)^s}{F_{k,s}F_{k,s-1}}\right).$$ □

Theorem 2.5. *For k-Fibonacci numbers, the equality*

$$\sum_{n=1}^{\infty}\frac{F_{k,n}}{F_{k,n+1}F_{k,n+2}}=\frac{1}{k^2}+(1-k)\left(\sum_{n=2}^{\infty}\frac{(-1)^n}{F_{k,n-1}F_{k,n}F_{k,n+1}}\right)$$

holds.

Proof. From the equation (1.1), we can write

$$\begin{aligned}\sum_{s=1}^{n}\frac{F_{k,s}}{F_{k,s+1}F_{k,s+2}}&=\sum_{s=1}^{n}\frac{F_{k,s+2}-kF_{k,s+1}}{F_{k,s+1}F_{k,s+2}}=\sum_{s=1}^{n}\left(\frac{1}{F_{k,s+1}}-\frac{k}{F_{k,s+2}}\right)\\&=\left[\frac{1}{F_{k,2}}-\frac{k}{F_{k,3}}+\frac{1}{F_{k,3}}-\frac{k}{F_{k,4}}+\frac{1}{F_{k,4}}-\frac{k}{F_{k,5}}+\dots\right.\\&\qquad\left.+\frac{1}{F_{k,n}}-\frac{k}{F_{k,n+1}}+\frac{1}{F_{k,n+1}}-\frac{k}{F_{k,n+2}}\right]\end{aligned}$$

$$= \frac{1}{F_{k,2}} - \frac{k}{F_{k,n+2}} + (1-k)\left(\frac{1}{F_{k,3}} + \frac{1}{F_{k,4}} + \ldots + \frac{1}{F_{k,n}} + \frac{1}{F_{k,n+1}}\right)$$
$$= \frac{1}{F_{k,2}} - \frac{k}{F_{k,n+2}} + (1-k)\left[\left(\frac{1}{F_{k,1}} + \frac{1}{F_{k,2}} + \frac{1}{F_{k,3}} + \frac{1}{F_{k,4}} + \ldots + \frac{1}{F_{k,n}}\right) + \left(\frac{1}{F_{k,n+1}} - \frac{1}{F_{k,1}} - \frac{1}{F_{k,2}}\right)\right]$$

From Theorem 2.1, we have

$$\sum_{s=1}^{n} \frac{F_{k,s}}{F_{k,s+1}F_{k,s+2}} = \frac{1}{F_{k,2}} - \frac{k}{F_{k,n+2}} + (1-k)\left[\frac{k^2+k+1}{k^2} + \sum_{s=2}^{n} \frac{(-1)^s}{F_{k,s-1}F_{k,s}F_{k,s+1}} + \left(\frac{1}{F_{k,n+1}} - \frac{1}{F_{k,1}} - \frac{1}{F_{k,2}}\right)\right].$$

If the limits of both sides of the last equation for $n \to \infty$ are taken and necessary arrangements are made, then we write

$$\lim_{n\to\infty} \sum_{s=1}^{n} \frac{F_{k,s}}{F_{k,s+1}F_{k,s+2}} = \lim_{n\to\infty}\left[\frac{1}{F_{k,2}} - \frac{k}{F_{k,n+2}} + (1-k)\left[\frac{k^2+k+1}{k^2} + \sum_{s=2}^{n} \frac{(-1)^s}{F_{k,s-1}F_{k,s}F_{k,s+1}} + \left(\frac{1}{F_{k,n+1}} - \frac{1}{F_{k,1}} - \frac{1}{F_{k,2}}\right)\right]\right].$$

By considering $\lim\limits_{n\to\infty} \frac{k}{F_{k,n+2}} = 0$ and $\lim\limits_{n\to\infty} \frac{1}{F_{k,n+1}} = 0$, we obtain

$$\sum_{n=1}^{\infty} \frac{F_{k,n}}{F_{k,n+1}F_{k,n+2}} = \frac{1}{k^2} + (1-k)\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{F_{k,n-1}F_{k,n}F_{k,n+1}}\right). \qquad \square$$

Theorem 2.6. *For k-Fibonacci numbers, the equality*

$$\sum_{n=1}^{\infty} \frac{F_{k,n+1}}{F_{k,n}F_{k,n+3}} = \frac{k^4+k^3+2k^2+1}{k^2\left(k^2+1\right)^2} + \frac{1-k}{k^2+1}\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{F_{k,n-1}F_{k,n}F_{k,n+1}}\right)$$

holds.

Proof. From the equation (1.1), we can write

$$\begin{aligned} F_{k,s+3} &= kF_{k,s+2} + F_{k,s+1} \\ &= k\left(kF_{k,s+1} + F_{k,s}\right) + F_{k,s+1} \\ &= \left(k^2+1\right)F_{k,s+1} + kF_{k,s}. \end{aligned}$$

From the last equation, it is obvious

$$F_{k,s+1} = \frac{1}{k^2+1}\left[F_{k,s+3} - kF_{k,s}\right]. \tag{2.3}$$

By using the equaiton (2.3) in the following equation, we have

$$\begin{aligned} \sum_{s=1}^{n} \frac{F_{k,s+1}}{F_{k,s}F_{k,s+3}} &= \frac{1}{k^2+1}\sum_{s=1}^{n}\left(\frac{F_{k,s+3} - kF_{k,s}}{F_{k,s}F_{k,s+3}}\right) \\ &= \frac{1}{k^2+1}\sum_{s=1}^{n}\left(\frac{1}{F_{k,s}} - \frac{k}{F_{k,s+3}}\right). \end{aligned}$$

From the last sum, we can write

$$\begin{aligned}\sum_{s=1}^{n}\frac{F_{k,s+1}}{F_{k,s}F_{k,s+3}} &= \frac{1}{k^2+1}\left[\frac{1}{F_{k,1}}+\frac{1}{F_{k,2}}+\frac{1}{F_{k,3}}-\frac{k}{F_{k,n+1}}-\frac{k}{F_{k,n+2}}-\frac{k}{F_{k,n+3}}+(1-k)\sum_{s=4}^{n}\frac{1}{F_{k,s}}\right]\\ &= \frac{1}{k^2+1}\left[\left(\frac{1}{F_{k,1}}+\frac{1}{F_{k,2}}+\frac{1}{F_{k,3}}-\frac{k}{F_{k,n+1}}-\frac{k}{F_{k,n+2}}-\frac{k}{F_{k,n+3}}\right)\right.\\ &\quad\left.+(1-k)\left(\sum_{s=1}^{n}\frac{1}{F_{k,s}}-\frac{1}{F_{k,1}}-\frac{1}{F_{k,2}}-\frac{1}{F_{k,3}}\right)\right]\\ &= \frac{1}{k^2+1}\left[k\left(\frac{1}{F_{k,1}}+\frac{1}{F_{k,2}}+\frac{1}{F_{k,3}}-\frac{1}{F_{k,n+1}}-\frac{1}{F_{k,n+2}}-\frac{1}{F_{k,n+3}}\right)+(1-k)\sum_{s=1}^{n}\frac{1}{F_{k,s}}\right]\\ &= \frac{1}{k^2+1}\left[k\left(1+\frac{1}{k}+\frac{1}{k^2+1}-\frac{1}{F_{k,n+1}}-\frac{1}{F_{k,n+2}}-\frac{1}{F_{k,n+3}}\right)+(1-k)\sum_{s=1}^{n}\frac{1}{F_{k,s}}\right].\end{aligned}$$

If the limits of both sides of the last equation are taken for $n\to\infty$, then we can write

$$\lim_{n\to\infty}\sum_{s=1}^{n}\frac{F_{k,s+1}}{F_{k,s}F_{k,s+3}} = \lim_{n\to\infty}\frac{1}{k^2+1}\left[k\left(1+\frac{1}{k}+\frac{1}{k^2+1}-\frac{1}{F_{k,n+1}}-\frac{1}{F_{k,n+2}}-\frac{1}{F_{k,n+3}}\right)+(1-k)\sum_{s=1}^{n}\frac{1}{F_{k,s}}\right].$$

By considering $\lim_{n\to\infty}\frac{k}{F_{k,n+1}}=0$, $\lim_{n\to\infty}\frac{k}{F_{k,n+2}}=0$, $\lim_{n\to\infty}\frac{k}{F_{k,n+3}}=0$ and from Theorem 2.1, we obtain

$$\sum_{n=1}^{\infty}\frac{F_{k,n+1}}{F_{k,n}F_{k,n+3}} = \frac{k^4+k^3+2k^2+1}{k^2\left(k^2+1\right)^2}+\frac{1-k}{k^2+1}\left(\sum_{n=2}^{\infty}\frac{(-1)^n}{F_{k,n-1}F_{k,n}F_{k,n+1}}\right).$$

□

Theorem 2.7. *For $n\geq 2$, the equality*

$$\sum_{n=2}^{\infty}\frac{F_{k,2n}}{F_{k,n+1}^2F_{k,n-1}^2}=\frac{k^2+1}{k^3}$$

holds.

Proof. From the [7], we can write

$$F_{k,n+m}=F_{k,n}F_{k,m+1}+F_{k,n-1}F_{k,m}.$$

If we get $n=m$ in the last equation, it is obvious

$$\begin{aligned}F_{k,2n} &= F_{k,n}F_{k,n+1}+F_{k,n-1}F_{k,n}\\ &= F_{k,n}\left(F_{k,n+1}+F_{k,n-1}\right).\end{aligned}$$

By using the equation (1.1), we have

$$F_{k,2n}=\frac{1}{k}\left(F_{k,n+1}-F_{k,n-1}\right)\left(F_{k,n+1}+F_{k,n-1}\right)=\frac{1}{k}\left(F_{k,n+1}^2-F_{k,n-1}^2\right). \tag{2.4}$$

By using the equation (2.4) in the following equation, we have

$$\begin{aligned}\sum_{s=2}^{n}\frac{F_{k,2s}}{F_{k,s+1}^2F_{k,s-1}^2} &= \frac{1}{k}\sum_{s=2}^{n}\frac{F_{k,s+1}^2-F_{k,s-1}^2}{F_{k,s+1}^2F_{k,s-1}^2}=\frac{1}{k}\left[\sum_{s=2}^{n}\left(\frac{1}{F_{k,s-1}^2}-\frac{1}{F_{k,s+1}^2}\right)\right]\\ &= \frac{1}{k}\left[1+\frac{1}{k^2}-\frac{1}{F_{k,n}^2}-\frac{1}{F_{k,n+1}^2}\right].\end{aligned}$$

If the limits of both sides of the last equation are taken for $n \to \infty$, then we can write

$$\lim_{n\to\infty} \sum_{s=2}^{n} \frac{F_{k,2s}}{F_{k,s+1}^2 F_{k,s-1}^2} = \lim_{n\to\infty} \frac{1}{k}\left[1 + \frac{1}{k^2} - \frac{1}{F_{k,n}^2} - \frac{1}{F_{k,n+1}^2}\right].$$

By considering $\lim_{n\to\infty} \frac{1}{kF_{k,n}^2} = 0$, $\lim_{n\to\infty} \frac{1}{kF_{k,n+1}^2} = 0$, we have

$$\sum_{n=2}^{\infty} \frac{F_{k,2n}}{F_{k,n+1}^2 F_{k,n-1}^2} = \frac{k^2+1}{k^3}.$$

□

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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