Communications in Mathematics and Applications
Vol. 10, No. 4, pp. 673–679, 2019
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v10i4.804

Research Article

Gaussian Pell-Lucas Polynomials

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Abstract. In this paper, we first define the Gaussian Pell-Lucas polynomial sequence. We then obtain Binet formula, generating function and determinantal representation of this sequence. Also, some properties of the Gaussian Pell-Lucas polynomials are investigated.

Keywords. Pell-Lucas numbers; Gaussian Pell-Lucas numbers; Gaussian Pell-Lucas polynomials

MSC. 11B37; 11B39; 11B75; 11B83

Received: February 4, 2018 **Accepted:** March 25, 2019

1. Introduction

In 1963, the complex Fibonacci numbers are introduced by Horadam [6]. After this seminal paper, Gaussian Fibonacci, Lucas, Pell and Pell-Lucas numbers are studied by many authors [2, 3, 5, 8]. The Gaussian Fibonacci and Lucas numbers are defined recursively by the relations $GF_{n+1} = GF_n + GF_{n-1}$, where $GF_0 = i$, $GF_1 = 1$, and $GL_{n+1} = GL_n + GL_{n-1}$ with initial conditions $GL_0 = 2-i$, $GL_1 = 1+2i$, respectively. Also, the Gaussian Pell numbers are defined recursively by $GP_{n+1} = 2GP_n + GP_{n-1}$ with initial conditions $GP_0 = i$, $GP_1 = 1$, and the Gaussian Pell-Lucas numbers are defined as $GQ_{n+1} = 2GQ_n + GQ_{n-1}$, where $GQ_0 = 2-2i$, $GQ_1 = 2+2i$.

On the other hand, the Pell polynomial sequence is defined by the recurrence relation $P_{n+1}(x) = 2xP_n(x) + P_{n-1}(x)$, where $P_0(x) = 0$, $P_1(x) = 1$. Similarly, the Pell-Lucas polynomial sequence is defined as $Q_0(x) = 2$, $Q_1(x) = 2x$, and $Q_{n+1}(x) = 2xQ_n(x) + Q_{n-1}(x)$. Moreover, some properties related with these sequences are studied by Horadam and Mahon [7].

In [4], Halici and Oz introduced the Gaussian Pell polynomials satisfied the recurrence relation $GP_{n+1}(x) = 2xGP_n(x) + GP_{n-1}(x)$, where $GP_0(x) = i$ and $GP_1(x) = 1$. In a similar way, the Gaussian Jacobsthal and Jacobsthal-Lucas polynomials are studied in [1] by Asci and Gurel.

The main objective of this paper is to define the Gaussian Pell-Lucas polynomials, and to investigate some properties of these polynomials.

In Section 2, we define the Gaussian Pell-Lucas polynomial sequence that generalize the Gaussian Pell-Lucas number sequence given in [3]. Moreover, we give the generating function and Binet formula for the Gaussian Pell-Lucas polynomial sequence. We also obtain summation formula and determinantal representation of this sequence. In the rest of Section 2, by using Binet formula, we give well-known identities such as Catalan's and d'Ocagne's identities involving the Gaussian Pell-Lucas polynomial sequence.

2. Main Results

In this section, we first define the Gaussian Pell-Lucas polynomial sequence. Then we give generating function, Binet formula, determinantal representation and some properties of this sequence.

Definition 2.1. The Gaussian Pell-Lucas polynomial sequence $\{GQ_n(x)\}_{n=0}^{\infty}$ is defined, for $n \geq 1$, recursively by

$$GQ_{n+1}(x) = 2xGQ_n(x) + GQ_{n-1}(x)$$

with initial conditions $GQ_0(x) = 2 - 2xi$ and $GQ_1(x) = 2x + 2i$.

Clearly, if we take $x = 1$, we obtain the Gaussian Pell-Lucas numbers. Also, it is easy to see that

$$GQ_n(x) = Q_n(x) + iQ_{n-1}(x),$$

where $Q_n(x)$ is the nth Pell-Lucas polynomial.

The first few terms of the Gaussian Pell-Lucas polynomials are: $2-2xi$, $2x+2i$, $4x^2+2+2xi$, $8x^3+6x+(4x^2+2)i$, $16x^4+16x^2+2+(8x^3+6x)i$.

We now give the generating function for the Gaussian Pell-Lucas polynomials by the following:

Theorem 2.2. *The generating function of the Gaussian Pell-Lucas polynomial sequence $\{GQ_n(x)\}_{n=0}^{\infty}$ denoted by $g(t,x)$ is*

$$g(t,x) = \frac{2-2xt+(4x^2t+2t-2x)i}{1-2xt-t^2}.$$

Proof. The generating function for the sequence $\{GQ_n(x)\}_{n=0}^{\infty}$ can be written in power series. Then, we have

$$g(t,x) = \sum_{n=0}^{\infty} GQ_n(x)t^n = GQ_0(x) + GQ_1(x)t + GQ_2(x)t^2 + GQ_3(x)t^3 + GQ_4(x)t^4 \dots,$$

$$2xtg(t,x) = 2xGQ_0(x)t + 2xGQ_1(x)t^2 + 2xGQ_2(x)t^3 + 2xGQ_3(x)t^4 + \dots,$$

and

$$t^2 g(t,x) = GQ_0(x)t^2 + GQ_1(x)t^3 + GQ_2(x)t^4 + \dots .$$

Hence, we obtain

$$(1 - 2xt - t^2)g(t,x) = 2 - 2xi + 4x^2 ti - 2xt + 2ti.$$

Thus, we get

$$g(t,x) = \frac{2 - 2xt + (4x^2 t + 2t - 2x)i}{1 - 2xt - t^2}.$$

This completes the proof. □

The next theorem gives us the Binet formula for the sequence $\{GQ_n(x)\}_{n=0}^{\infty}$.

Theorem 2.3. *The nth term of the Gaussian Pell-Lucas polynomial sequence is given by*

$$GQ_n(x) = \alpha^n(x) + \beta^n(x) - [\beta(x)\alpha^n(x) + \alpha(x)\beta^n(x)]i,$$

where $\alpha(x) = x + \sqrt{1+x^2}$ *and* $\beta(x) = x - \sqrt{1+x^2}$ *are the roots of the equation* $r^2 - 2xr - 1 = 0$.

Proof. It is known that the general solution for the recurrence relation is given by $GQ_n(x) = c_1\alpha^n(x) + c_2\beta^n(x)$, where c_1 and c_2 are any constants.
Plugging the general solution in the initial conditions gives the system

$$c_1 + c_2 = 2 - 2xi, \quad c_1(x + \sqrt{1+x^2}) + c_2(x - \sqrt{1+x^2}) = 2x + 2i.$$

Then we obtain $c_1 = 1 - \beta(x)i$ and $c_2 = 1 - \alpha(x)i$. Therefore, we get

$$GQ_n(x) = \alpha^n(x) + \beta^n(x) - \beta(x)\alpha^n(x)i - \alpha(x)\beta^n(x)i$$

which completes the proof. □

Theorem 2.4. *For* $n \geq 1$*, the sum of the Gausian Pell-Lucas polynomials is*

$$\sum_{k=1}^{n} GQ_k(x) = \frac{1}{2x}[GQ_{n+1}(x) + GQ_n(x) - 2x - 2 + (2x-2)i].$$

Proof. From the recurrence relation of the Gaussian Pell-Lucas polynomial sequence, we have

$$GQ_n(x) = \frac{1}{2x}(GQ_{n+1}(x) - GQ_{n-1}(x)).$$

Then, we get

$$\begin{aligned}
GQ_1(x) &= \frac{1}{2x}(GQ_2(x) - GQ_0(x)) \\
GQ_2(x) &= \frac{1}{2x}(GQ_3(x) - GQ_1(x)) \\
GQ_3(x) &= \frac{1}{2x}(GQ_4(x) - GQ_2(x)) \\
&\vdots \\
GQ_{n-1}(x) &= \frac{1}{2x}(GQ_n(x) - GQ_{n-2}(x))
\end{aligned}$$

$$GQ_n(x) = \frac{1}{2x}(GQ_{n+1}(x) - GQ_{n-1}(x))$$

Thus, we obtain

$$\sum_{k=1}^{n} GQ_k(x) = \frac{1}{2x}[GQ_{n+1}(x) + GQ_n(x) - GQ_1(x) - GQ_0(x)]$$
$$= \frac{1}{2x}[GQ_{n+1}(x) + GQ_n(x) - 2x - 2 + (2x-2)i].$$

This completes the proof. □

The following corollary follows from the above theorem.

Theorem 2.5. *For $n \geq 1$, we have*

(i) $\sum_{k=1}^{n} GQ_{2k}(x) = \frac{1}{2x}(GQ_{2n+1}(x) - 2x - 2i)$,

(ii) $\sum_{k=1}^{n} GQ_{2k-1}(x) = \frac{1}{2x}(GQ_{2n}(x) - 2 + 2xi)$.

Theorem 2.6. *For $n \geq 1$, let $\mathbf{L_n(x)}$ be an $n \times n$ tridiagonal matrix defined by*

$$\mathbf{L_n(x)} = \begin{pmatrix} 2x+2i & 1 & 0 & 0 & \cdots & 0 \\ -2+2xi & 2x & 1 & 0 & \cdots & 0 \\ 0 & -1 & 2x & 1 & \ddots & 0 \\ 0 & 0 & -1 & 2x & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \ldots & \ldots & 0 & -1 & 2x \end{pmatrix}$$

and let $\mathbf{L_0(x)} = 2 - 2xi$. Then

$$\det \mathbf{L_n(x)} = GQ_n(x).$$

Proof. For the proof we use the mathematical induction on n. For $n = 1$ and $n = 2$, we get

$$\det \mathbf{L_1(x)} = 2x + 2i = GQ_1(x) \quad \text{and} \quad \det \mathbf{L_2(x)} = 4x^2 + 2 + 2xi = GQ_2(x).$$

Let us assume that the equality holds for $n-1$ and $n-2$, that is,

$$\det \mathbf{L_{n-1}(x)} = GQ_{n-1}(x) \quad \text{and} \quad \det \mathbf{L_{n-2}(x)} = GQ_{n-2}(x).$$

Finally, for n, we get

$$\det \mathbf{L_n(x)} = 2x \det \mathbf{L_{n-1}(x)} + \det \mathbf{L_{n-2}(x)} = 2xGQ_{n-1}(x) + GQ_{n-2}(x)$$

which completes the proof. □

Now, we define the matrices $\mathbf{Q}$ and $\mathbf{P}$ as followings:

$$\mathbf{Q} = \begin{pmatrix} 2x & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{P} = \begin{pmatrix} 4x^2+2+2xi & 2x+2i \\ 2x+2i & 2-2xi \end{pmatrix}.$$

Theorem 2.7. *For $n \geq 1$, we have*

$$\mathbf{Q^nP} = \begin{pmatrix} GQ_{n+2}(x) & GQ_{n+1}(x) \\ GQ_{n+1}(x) & GQ_n(x) \end{pmatrix}.$$

Proof. The proof can be done easily by using the mathematical induction on n. □

The consequence of Theorem 2.7 which gives the Cassini's identity for the Gaussian Pell-Lucas polynomial sequence is the following:

Theorem 2.8 (Cassini's Identity)**.** *For positive integer n, we have*

$$GQ_{n-1}(x)GQ_{n+1}(x)-GQ_n^2(x)=8(-1)^{n-1}(1+x^2)(1-xi).$$

Proof. It is obvious that $\det\mathbf{Q^{n-1}}=(-1)^{n-1}$ and $\det\mathbf{P}=8(1+x^2)(1-xi)$. By taking determinant of the matrix

$$\mathbf{Q^{n-1}P}=\begin{pmatrix} GQ_{n+1}(x) & GQ_n(x) \\ GQ_n(x) & GQ_{n-1}(x)\end{pmatrix},$$

we get

$$GQ_{n-1}(x)GQ_{n+1}(x)-GQ_n^2(x)=8(-1)^{n-1}(1+x^2)(1-xi).$$ □

Now, Catalan's and d'Ocagne's identities for the Gaussian Pell-Lucas polynomial sequence are given in the following theorems, respectively.

Theorem 2.9 (Catalan's Identity)**.** *For positive integers n and r, we have*

$$GQ_{n-r}(x)GQ_{n+r}(x)-GQ_n^2(x)=2(-1)^{n-r}(1-xi)(\alpha^r(x)-\beta^r(x))^2.$$

Proof. From the Binet formula of the sequence $\{GQ_n(x)\}_{n=0}^{\infty}$, we get

$$\begin{aligned} GQ_{n-r}(x)GQ_{n+r}(x)-GQ_n^2(x) &= \{\alpha^{n-r}(x)+\beta^{n-r}(x)-[\beta(x)\alpha^{n-r}(x)+\alpha(x)\beta^{n-r}(x)]i\} \\ &\quad \times\{\alpha^{n+r}(x)+\beta^{n+r}(x)-[\beta(x)\alpha^{n+r}(x)+\alpha(x)\beta^{n+r}(x)]i\} \\ &\quad -\{\alpha^{n}(x)+\beta^{n}(x)-[\beta(x)\alpha^{n}(x)+\alpha(x)\beta^{n}(x)]i\}^2 \\ &= (\alpha(x)\beta(x))^{n-r}[\alpha^{2r}(x)+\beta^{2r}(x)-2\alpha^r(x)\beta^r(x)](1-(\alpha(x)\beta(x))) \\ &\quad -i(\alpha(x)\beta(x))^{n-r}(\alpha(x)+\beta(x))[\alpha^{2r}(x)+\beta^{2r}(x)-2\alpha^r(x)\beta^r(x)] \\ &= (\alpha(x)\beta(x))^{n-r}(\alpha^r(x)-\beta^r(x))^2[1-(\alpha(x)\beta(x))-i(\alpha(x)+\beta(x))]. \end{aligned}$$

Since $\alpha(x)\beta(x)=-1$ and $\alpha(x)+\beta(x)=2x$, we obtain

$$GQ_{n-r}(x)GQ_{n+r}(x)-GQ_n^2(x)=(-1)^{n-r}(\alpha^r(x)-\beta^r(x))^2(2-2xi)$$

which completes the proof. □

Note that if we set $r=1$ in Theorem 2.9, Cassini's identity of the Gaussian Pell-Lucas polynomial sequence, which is given in Theorem 2.8, can be obtained again.

Theorem 2.10 (d'Ocagne's Identity)**.** *Let m and n be any positive integers. Then,*

$$GQ_m(x)GQ_{n+1}(x)-GQ_n(x)GQ_{m+1}(x)=4(-1)^{n+1}\sqrt{1+x^2}(1-xi)(\alpha^{m-n}(x)-\beta^{m-n}(x)).$$

Proof. By using the Binet formula of the sequence $\{GQ_n(x)\}_{n=0}^{\infty}$, we get

$$\begin{aligned} &GQ_m(x)GQ_{n+1}(x)-GQ_n(x)GQ_{m+1}(x) \\ &\quad=\{\alpha^m(x)+\beta^m(x)-[\beta(x)\alpha^m(x)+\alpha(x)\beta^m(x)]i\}\{\alpha^{n+1}(x)+\beta^{n+1}(x)-[\beta(x)\alpha^{n+1}(x)+\alpha(x)\beta^{n+1}(x)]i\} \end{aligned}$$

$$
\begin{aligned}
&-\{\alpha^n(x)+\beta^n(x)-[\beta(x)\alpha^n(x)+\alpha(x)\beta^n(x)]i\}\{\alpha^{m+1}(x)+\beta^{m+1}(x)-[\beta(x)\alpha^{m+1}(x)+\alpha(x)\beta^{m+1}(x)]i\} \\
&=(\alpha(x)-\beta(x))[\alpha^n(x)\beta^m(x)-\alpha^{n+1}(x)\beta^{m+1}(x)-\alpha^m(x)\beta^n(x)+\alpha^{m+1}(x)\beta^{n+1}(x)] \\
&\quad+i(\alpha^2(x)-\beta^2(x))[\alpha^m(x)\beta^n(x)-\alpha^n(x)\beta^m(x)] \\
&=-2(\alpha(x)-\beta(x))[\alpha^m(x)\beta^n(x)-\alpha^n(x)\beta^m(x)]+i(\alpha^2(x)-\beta^2(x))[\alpha^m(x)\beta^n(x)-\alpha^n(x)\beta^m(x)] \\
&=(\alpha(x)-\beta(x))(\alpha(x)\beta(x))^n(\alpha^{m-n}(x)-\beta^{m-n}(x))[-2+i(\alpha(x)+\beta(x))] \\
&=4(-1)^{n+1}\sqrt{1+x^2}(1-xi)(\alpha^{m-n}(x)-\beta^{m-n}(x)).
\end{aligned}
$$

This completes the proof. □

3. Conclusion

In this study, we introduce the concept of the Gaussian Pell-Lucas polynomials. We also give some results including Binet formula, generating function, summation formula and determinantal representation for these polynomials. Moreover, we obtain some well-known identities, such as Catalan's, Cassini's and d'Ocagne's identities, involving the Gaussian Pell-Lucas polynomials. In future, we plan to investigate some others identities and properties for these polynomials.

Acknowledgement

The author would like to thank the anonymous referee for his/her valuable comments and suggestions.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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