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Numerical Modeling of SEIR Measles Dynamics with Diffusion

Research Article

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Abstract. A novel unconditionally positive *finite difference* (FD) scheme is developed to solve numerically SEIR measles epidemic model with diffusion. In population dynamics, positivity of subpopulations is an essential requirement. The proposed FD scheme preserves the positivity of the solution of the model. The consistency and unconditional stability is proved. The proposed FD scheme is explicit in nature which is an extra feature of this scheme. Comparisons are also made with forward Euler explicit FD scheme and Crank Nicolson implicit FD scheme. Simulations of a numerical test are also presented to verify all the attributes of the proposed scheme.

Keywords. SEIR Measles epidemic model with diffusion; Finite difference scheme; Positivity; Consistency; Stability

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1. Introduction

Measles is an acute, high contagious viral disease characterized by final stage maculopapular rash erupting successively over the neck, face, body, arms and legs and accompanied by high fever. Measles is an RNA virus. Virus is present in the nasopharyngeal secretions, blood and urine. It is worldwide disease both epidemic and endemic. Highest incidence is in winter. Mode of transmission is by direct or indirect contact and droplet spray. The period of infectivity is 4 days before and 5 days after the appearance of rash. The essential lesion is in the skin, respiratory tract, intestinal tract and conjunctivae. Koplik's spot appear on skin which consist serous exudate and proliferation of endothelial cells. Measles is a communicable disease due to which 200 million people are died worldwide in the last 150 years. Mathematical modeling becomes the important tool to study the spread, control and optimal control of communicable diseases [1, 3, 5, 6].

In this paper, we deal with the numerical solution of SEIR reaction diffusion epidemic model [2]. The main theme of this work is to develop an unconditional, positivity preserving explicit FD scheme [4, 7] for SEIR reaction diffusion epidemic model. Jansen and Twizell [9] developed an unconditional, positivity preserving finite difference scheme for SEIR measles epidemic model containing ordinary differential equations. Chinviriyasit *et al*. [8] proposed an unconditionally convergent implicit FD scheme for the solution of diffusive SIR whooping cough model. Al-Showikh and Twizell [2] presented an implicit FD scheme for the numerical solution of SEIR measles reaction diffusion epidemic model. *Nonstandard Finite Difference* (NSFD) schemes are also positivity preserving, unconditionally convergent FD schemes. NSFD schemes are introduced by Mickens [10]. NSFD schemes are commonly used for the numerical solutions of epidemic models containing ordinary differential equations [11–13].

Here, a spatially-structured system will be studied and three numerical methods are used for the numerical solution of the spatial spread of Measles in one space dimension.

$$\left.\begin{aligned}
\frac{\partial S}{\partial t} &= \mu N - \mu S - \beta I S + d_s \frac{\partial^2 S}{\partial x^2} \\
\frac{\partial E}{\partial t} &= \beta I S - \mu E - \sigma E + d_E \frac{\partial^2 E}{\partial x^2} \\
\frac{\partial I}{\partial t} &= \sigma E - \mu I - \gamma I + d_I \frac{\partial^2 I}{\partial x^2} \\
\frac{\partial R}{\partial t} &= \gamma I - \mu R + d_R \frac{\partial^2 R}{\partial x^2}
\end{aligned}\right\} \tag{1.1}$$

Since R is not present in first three equations, so system (1.1) can be written as

$$\left.\begin{aligned}
\frac{\partial S}{\partial t} &= \mu N - \mu S - \beta I S + d_s \frac{\partial^2 S}{\partial x^2} \\
\frac{\partial E}{\partial t} &= \beta I S - \mu E - \sigma E + d_E \frac{\partial^2 E}{\partial x^2} \\
\frac{\partial I}{\partial t} &= \sigma E - \mu I - \gamma I + d_I \frac{\partial^2 I}{\partial x^2}
\end{aligned}\right\} \tag{1.2}$$

The initial conditions are of the form

$$S(x,0) = S_0,\ E(x,0) = E_0,\ I(x,0) = I_0, \quad 0 \le x \le L$$

and the boundary conditions are

$$\frac{\partial S(0,t)}{\partial x} = \frac{\partial E(0,t)}{\partial x} = \frac{\partial I(0,t)}{\partial x} = 0; \quad t > 0,$$
$$\frac{\partial S(L,t)}{\partial x} = \frac{\partial E(L,t)}{\partial x} = \frac{\partial I(L,t)}{\partial x} = 0; \quad t > 0,$$

where

S = Susceptible individuals

E = Exposed individuals

I = Infected individuals

R = Recovered individuals

μ = Birth rate and death rate

β = the rate at which susceptible individuals are infected by those who are infectious

σ = the rate at which exposed individuals become infected

γ = the rate at which infected individuals recover

μ, β, σ, γ are considered as positive parameters

2. Equilibrium Points of the System

There are two equilibrium points of SEIR epidemic model, *Disease Free Equilibrium* (DFE) and *Endemic Equilibrium* (EE).

DFE point is $(S_{DFE}, E_{DFE}, I_{DFE}) = (N, 0, 0)$

EE point is $(S_{EE}, E_{EE}, I_{EE}) = \left(\frac{N}{R_0}, \frac{\mu N}{\mu+\sigma}\left(1 - \frac{1}{R_0}\right), \frac{\mu}{\beta}(R_0 - 1)\right)$

whereas $R_0 = \frac{\sigma\beta N}{(\mu+\sigma)(\mu+\gamma)}$, when $d_s = d_E = d_I = 0$ is the basic reproductive number with the conditions that

if $R_0 < 1$, No epidemics

if $R_0 > 1$, Epidemics occurs

3. Numerical Methods

In this section, we rewrite system (1.2)

$$\frac{\partial S}{\partial t} = \mu N - \mu S - \beta I S + d_s \frac{\partial^2 S}{\partial x^2}, \tag{3.1}$$

$$\frac{\partial E}{\partial t} = \beta I S - \mu E - \sigma E + d_E \frac{\partial^2 E}{\partial x^2}, \tag{3.2}$$

$$\frac{\partial I}{\partial t} = \sigma E - \mu I - \gamma I + d_I \frac{\partial^2 I}{\partial x^2}. \tag{3.3}$$

The initial conditions are of the form

$$S(x,0)=S_0, E(x,0)=E_0,\ I(x,0)=I_0, \quad 0\le x\le L$$

and the boundary conditions are

$$\frac{\partial S(0,t)}{\partial x}=\frac{\partial E(0,t)}{\partial x}=\frac{\partial I(0,t)}{\partial x}=0; \quad t>0,$$
$$\frac{\partial S(L,t)}{\partial x}=\frac{\partial E(L,t)}{\partial x}=\frac{\partial I(L,t)}{\partial x}=0; \quad t>0.$$

Divide $[0,L]\times[0,T]$ into $M\times N$ with step sizes $h=\frac{L}{M}$ and $\tau=\frac{T}{N}$.
Grid points are

$$x_i=ih, \quad i=0,1,2,\dots,M,$$
$$t_n=n\tau, \quad n=0,1,2,\dots,N,$$

S_i^n, E_i^n and $I_i^n$are denoted as FD approximations of $S(ih,n\tau)$, $E(ih,n\tau)$ and $I(ih,n\tau)$, respectively.

To solve the above system, we use three different finite difference schemes, two classical schemes and a newly developed finite difference scheme.

The classical schemes are forward Euler scheme and Crank Nicolson scheme. Third FD scheme is an unconditionally positivity preserving scheme proposed by Chen-Charpentier *et al*. [7] and Appadu *et al*. [4], which is developed with the help of rules defined by Mickens [10].

Forward Euler explicit scheme for the system is

$$S_i^{n+1}=S_i^n+\tau\mu N-\tau\mu S_i^n-\tau\beta S_i^nI_i^n+R_1(S_{i-1}^n-2S_i^n+S_{i+1}^n), \tag{3.4}$$
$$E_i^{n+1}=E_i^n-\tau(\mu+\sigma)E_i^n+\tau\beta S_i^nI_i^n+R_2(E_{i-1}^n-2E_i^n+E_{i+1}^n), \tag{3.5}$$
$$I_i^{n+1}=I_i^n-\tau(\mu+\gamma)I_i^n+\tau\sigma E_i^n+R_3(I_{i-1}^n-2I_i^n+I_{i+1}^n). \tag{3.6}$$

Crank Nicolson scheme for the system is

$$(1+R_1)S_i^{n+1}-\frac{R_1}{2}\left(S_{i-1}^{n+1}+S_{i+1}^{n+1}\right)=(1-R_1)S_i^n+\frac{R_1}{2}(S_{i-1}^n+S_{i+1}^n)+\tau\mu N-\tau\mu S_i^n-\tau\beta S_i^nI_i^n, \tag{3.7}$$
$$(1+R_2)E_i^{n+1}-\frac{R_2}{2}\left(E_{i-1}^{n+1}+E_{i+1}^{n+1}\right)=(1-R_2)E_i^n+\frac{R_2}{2}(E_{i-1}^n+E_{i+1}^n)-\tau(\mu+\sigma)I_i^n+\tau\beta S_i^nI_i^n, \tag{3.8}$$
$$(1+R_3)I_i^{n+1}-\frac{R_3}{2}\left(I_{i-1}^{n+1}+I_{i+1}^{n+1}\right)=(1-R_3)I_i^n+\frac{R_3}{2}(I_{i-1}^n+I_{i+1}^n)-\tau(\mu+\gamma)I_i^n+\tau\sigma E_i^n. \tag{3.9}$$

Here,

$$R_1=d_S\frac{\tau}{h^2}, R_2=d_E\frac{\tau}{h^2} \text{ and } R_3=d_I\frac{\tau}{h^2}.$$

Now the proposed FD scheme for (3.1) is constructed as follows

$$S_i^{n+1}=S_i^n+R_1\left(S_{i-1}^n+S_{i+1}^n\right)-2R_1S_i^{n+1}+\tau\mu N-\tau\beta I_i^nS_i^{n+1}-\tau\mu S_i^{n+1}. \tag{3.10}$$

After some computation, we have

$$S_i^{n+1}=\frac{S_i^n+R_1(S_{i-1}^n+S_{i+1}^n)+\mu\tau N}{1+2R_1+\tau\mu+\tau\beta I_i^n}. \tag{3.11}$$

In similar way, we have

$$E_i^{n+1} = \frac{E_i^n + R_2(E_{i-1}^n + E_{i+1}^n) + \tau\beta S_i^n I_i^n}{1 + 2R_2 + \tau(\mu+\sigma)} \tag{3.12}$$

and

$$I_i^{n+1} = \frac{I_i^n + R_3(I_{i-1}^n + I_{i+1}^n) + \tau\sigma E_i^n}{1 + 2R_3 + \tau(\mu+\gamma)}. \tag{3.13}$$

Here,

$$R_1 = d_S\frac{\tau}{h^2},\ R_2 = d_E\frac{\tau}{h^2} \ \text{ and } \ R_3 = d_I\frac{\tau}{h^2}.$$

Positivity of the solution provides that

$$S_i^n \ge 0,\ E_i^n \ge 0,\ I_i^n \ge 0 \implies S_i^{n+1} \ge 0,\ E_i^{n+1} \ge 0,\ I_i^{n+1} \ge 0.$$

So, the proposed FD scheme is unconditionally positivity preserving [4, 7].

4. Stability

The stability range of all the schemes is determined by Von Neumann stability method.

The stability range of Forward Euler explicit scheme is $R_1 \le \frac{2-\tau\mu}{4}$, $R_2 \le \frac{2-\tau(\mu+\sigma)}{4}$ and $R_3 \le \frac{2-\tau(\mu+\gamma)}{4}$.

For the stability of Crank Nicolson FD scheme, we substitute $\zeta(t+\Delta t)e^{i\omega x}$, $\zeta(t)e^{i\omega x}$, $\zeta(t)e^{i\omega(x-\Delta x)}$, $\zeta(t)e^{i\omega(x+\Delta x)}$, $\zeta(t+\Delta t)e^{i\omega(x-\Delta x)}$ and $\zeta(t+\Delta t)e^{i\omega(x+\Delta x)}$ for S_i^{n+1}, S_i^n, S_{i-1}^n, S_{i+1}^n, S_{i-1}^{n+1} and S_{i+1}^{n+1} in (3.7) and after linearizing, we get

$$(1+R_1)\zeta(t+\Delta t)e^{i\omega x} - \frac{R_1}{2}\left(e^{i\omega(x-\Delta x)} + e^{i\omega(x+\Delta x)}\right)\zeta(t+\Delta t)$$
$$= (1-R_1)\zeta(t)e^{i\omega x} + \frac{R_1}{2}\left(e^{i\omega(x-\Delta x)} + e^{i\omega(x+\Delta x)}\right)\zeta(t) - \tau\mu\zeta(t)e^{i\omega x}.$$

Dividing both sides by $e^{i\omega x}$, we get

$$(1+R_1)\zeta(t+\Delta t) - \frac{R_1}{2}\left(e^{i\omega(-\Delta x)} + e^{i\omega(\Delta x)}\right)\zeta(t+\Delta t) = (1-R_1)\zeta(t) + \frac{R_1}{2}\left(e^{i\omega(-\Delta x)} + e^{i\omega(\Delta x)}\right)\zeta(t) - \tau\mu\zeta(t),$$

$$(1+R_1)\zeta(t+\Delta t) - \frac{R_1}{2}(2\cos(\omega\Delta x))\zeta(t+\Delta t) = (1-R_1)\zeta(t) + \frac{R_1}{2}(2\cos(\omega\Delta x))\zeta(t) - \tau\mu\zeta(t),$$

$$(1+R_1)\zeta(t+\Delta t) - R_1\left(1-\sin^2(\omega\Delta x/2)\right)\zeta(t+\Delta t) = (1-R_1)\zeta(t) + R_1\left(1-\sin^2(\omega\Delta x/2)\right)\zeta(t) - \tau\mu\zeta(t),$$

$$\left(1+2R_1\sin^2(\omega\Delta x/2)\right)\zeta(t+\Delta t) = \left(1-2R_1\sin^2(\omega\Delta x/2) - \tau\mu\right)\zeta(t),$$

$$\left|\frac{\zeta(t+\Delta t)}{\zeta(t)}\right| = \left|\frac{\left(1-2R_1\sin^2(\omega\Delta x/2) - \tau\mu\right)}{\left(1+2R_1\sin^2(\omega\Delta x/2)\right)}\right| < 1.$$

The amplification factor is less than 1 in above case, which guarantees the unconditional stability of Crank Nicolson implicit scheme. By adopting the same procedure for (3.8) and (3.9), we can verify the unconditional stability of Crank Nicolson FD scheme for all cases.

Now after applying Von Neumann method to proposed FD scheme (3.10) for (3.1) as discussed in the Crank Nicolson FD scheme and then linearizing, we have

$$\zeta(t+\Delta t)e^{i\omega x} = \zeta(t)e^{i\omega x} + R_1\left(e^{i\omega(x-\Delta x)} + e^{i\omega(x+\Delta x)}\right)\zeta(t) - 2R_1\zeta(t+\Delta t)e^{i\omega x} - \tau\mu\zeta(t+\Delta t)e^{i\omega x}.$$

Dividing both sides by $e^{i\omega x}$, we get

$$\zeta(t+\Delta t) = \zeta(t) + R_1\left(e^{i\omega(-\Delta x)} + e^{i\omega(\Delta x)}\right)\zeta(t) - 2R_1\zeta(t+\Delta t) - \tau\mu\zeta(t+\Delta t),$$

$$\zeta(t+\Delta t) = \zeta(t) + R_1(2\cos(\omega\Delta x))\zeta(t) - 2R_1\zeta(t+\Delta t) - \tau\mu\zeta(t+\Delta t),$$

$$(1+2R_1+\tau\mu)\zeta(t+\Delta t) = (1+2R_1\cos(\omega\Delta x))\zeta(t),$$

$$\left|\frac{\zeta(t+\Delta t)}{\zeta(t)}\right| = \left|\frac{1+2R_1-4R_1 sin^2(\omega\Delta x/2)}{1+2R_1+\tau\mu}\right| \le \left|\frac{1-2R_1}{1+2R_1+\tau\mu}\right| < 1.$$

Similarly, we have,

$$\left|\frac{\zeta(t+\Delta t)}{\zeta(t)}\right| = \left|\frac{1+2R_2-4R_2 sin^2(\omega\Delta x/2)}{1+2R_2+\tau(\mu+\sigma)}\right| \le \left|\frac{1-2R_2}{1+2R_2+\tau(\mu+\sigma)}\right| < 1,$$

$$\left|\frac{\zeta(t+\Delta t)}{\zeta(t)}\right| = \left|\frac{1+2R_3-4R_3 sin^2(\omega\Delta x/2)}{1+2R_3+\tau(\mu+\gamma)}\right| \le \left|\frac{1-2R_3}{1+2R_3+\tau(\mu+\gamma)}\right| < 1.$$

It is clear that proposed FD scheme is unconditionally stable [4, 7].

5. Consistency of Proposed Scheme

To check the consistency of proposed FD scheme, we use Taylor series expansion of S_i^{n+1}, S_{i+1}^n and S_{i-1}^n

$$S_i^{n+1} = S_i^n + \tau\frac{\partial S}{\partial t} + \frac{\tau^2}{2!}\frac{\partial^2 S}{\partial t^2} + \frac{\tau^3}{3!}\frac{\partial^3 S}{\partial t^3} + \dots$$

$$S_{i+1}^n = S_i^n + h\frac{\partial S}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 S}{\partial x^2} + \frac{h^3}{3!}\frac{\partial^3 S}{\partial x^3} + \dots$$

$$S_{i-1}^n = S_i^n - h\frac{\partial S}{\partial ??} + \frac{h^2}{2!}\frac{\partial^2 S}{\partial x^2} - \frac{h^3}{3!}\frac{\partial^3 S}{\partial x^3} + \dots$$

Proposed scheme for (3.1) is

$$S_i^{n+1} = S_i^n + R_1(S_{i-1}^n + S_{i+1}^n) - 2R_1 S_i^{n+1} + \mu\tau N - \mu\tau S_i^{n+1} - \tau\mu\beta S_i^{n+1} I_i^n.$$

Put the values of S_i^{n+1}, S_{i+1}^n and S_{i-1}^n in above equation and after simplification we get,

$$\left(\frac{\partial S}{\partial t} + \frac{\tau}{2!}\frac{\partial^2 S}{\partial t^2} + \frac{\tau^2}{3!}\frac{\partial^3 S}{\partial t^3} + \dots\right)\left(1 + \frac{d_S\tau}{h^2} + \tau\mu + \tau\beta I_n^i\right)$$
$$= 2d_S\left(\frac{1}{2!}\frac{\partial^2 S}{\partial x^2} + \frac{h^2}{4!}\frac{\partial^4 S}{\partial x^4} + \dots\right) + S_i^n(-\mu - \beta I_n^i) + \mu N.$$

Put $\tau = h^3$ and $h \to 0$, the above equation becomes (3.1) [4, 7].

Taylor series expansion of E_i^{n+1}, E_{i+1}^n and E_{i-1}^n

$$E_i^{n+1} = E_i^n + \tau\frac{\partial E}{\partial t} + \frac{\tau^2}{2!}\frac{\partial^2 E}{\partial t^2} + \frac{\tau^3}{3!}\frac{\partial^3 E}{\partial t^3} + \dots$$

$$E_{i+1}^{n} = E_i^n + h\frac{\partial E}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 E}{\partial x^2} + \frac{h^3}{3!}\frac{\partial^3 E}{\partial x^3} + \dots$$
$$E_{i-1}^{n} = E_i^n - h\frac{\partial E}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 E}{\partial x^2} - \frac{h^3}{3!}\frac{\partial^3 E}{\partial x^3} + \dots$$

Proposed scheme for (3.2) is

$$E_i^{n+1} = E_i^n + R_2\left(E_{i-1}^n + E_{i+1}^n\right) + \tau\beta S_i^n I_i^n - 2R_2 E_i^{n+1} - \tau(\mu+\sigma)E_i^n .$$

Put the values of E_i^{n+1}, E_{i+1}^n and E_{i-1}^n in above equation and after simplification we get,

$$\left(\frac{\partial E}{\partial t} + \frac{\tau}{2!}\frac{\partial^2 E}{\partial t^2} + \frac{\tau^2}{3!}\frac{\partial^3 E}{\partial t^3} + \dots\right)\left(1 + 2\frac{d_E\tau}{h^2} + \tau(\mu+\sigma)\right)$$
$$= 2d_E\left(\frac{1}{2!}\frac{\partial^2 E}{\partial x^2} + \frac{h^2}{4!}\frac{\partial^4 E}{\partial x^4} + \dots\right) + \beta S_i^n I_i^n + E_i^n\left(-(\mu+\sigma)\right).$$

Put $\tau = h^3$ and $h \to 0$, the above equation becomes (3.2).

Taylor series expansion of I_i^{n+1},I_{i+1}^n and I_{i-1}^n

$$I_i^{n+1} = I_i^n + \tau\frac{\partial I}{\partial t} + \frac{\tau^2}{2!}\frac{\partial^2 I}{\partial t^2} + \frac{\tau^3}{3!}\frac{\partial^3 I}{\partial t^3} + \dots$$
$$I_{i+1}^n = I_i^n + h\frac{\partial I}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 I}{\partial x^2} + \frac{h^3}{3!}\frac{\partial^3 I}{\partial x^3} + \dots$$
$$I_{i-1}^n = I_i^n - h\frac{\partial I}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 I}{\partial x^2} - \frac{h^3}{3!}\frac{\partial^3 I}{\partial x^3} + \dots$$

Proposed scheme for (3.3) is

$$I_i^{n+1} = I_i^n + R_3\left(I_{i-1}^n + I_{i+1}^n\right) - 2R_3 I_i^{n+1} - \tau\left(\mu+\gamma\right)I_i^n + \tau\sigma E_i^n .$$

Put the values of I_i^{n+1},I_{i+1}^n and I_{i-1}^n in above equation and after simplification we get,

$$\left(\frac{\partial I}{\partial t} + \frac{\tau}{2!}\frac{\partial^2 I}{\partial t^2} + \frac{\tau^2}{3!}\frac{\partial^3 I}{??t^3} + \dots\right)\left(1 + 2\frac{d_I\tau}{h^2} + \tau\left(\mu+\nu\right)\right)$$
$$= 2d_I\left(\frac{1}{2!}\frac{\partial^2 I}{\partial x^2} + \frac{h^2}{4!}\frac{\partial^4 I}{\partial x^4} + \dots\right) + I_i^n\left(-(\mu+\nu)\right) + \sigma E_i^n .$$

Put $\tau = h^3$ and $h \to 0$, the above equation becomes (3.3).

We see that forward Euler finite difference scheme is

(i) Explicit in nature, which is easy to solve and take less CPU time.

(ii) It is conditionally stable.

(iii) It could not preserve positivity property as it has negative term.

Crank Nicolson scheme is

(i) Implicit in nature which has complex computation and take more CPU time than explicit scheme.

(ii) It is unconditionally stable.

(iii) It also could not preserve positivity property.

The Proposed FD scheme is

(i) Explicit in nature.

(ii) It is unconditionally stable.

(iii) It preserves positivity property.

Next simulations of a test problem are presented to verify the results of all the schemes.

6. Numerical Experiment

The following values of parameters [9] are used in Numerical Experiments.

Table 1. Parameter values of Measles Model

Parameters	N	μ	σ	γ
	5×10^7 year^{-1}	0.02	45.6 year^{-1}	73 year^{-1}
Disease Free Equilibrium	$(S_o,E_o,I_o)=(N,0,0)$ $\beta=0.1\times10^{-5}$			
Endemic Equilibrium	$(S_e,E_e,I_e)=(2.435\times10^7, 1.125\times10^4,\ 7022.67)$, $\beta=0.3\times10^{-5}$			

6.1 Experiment 1

In the first experiment, the following initial condition is supposed

$$S(x,0)=\begin{cases}12500000x, & 0\le x<0.5\\ 12500000(1-x), & 0.5\le x\le 1\end{cases}, \qquad E(x,0)=\begin{cases}50000x, & 0\le x<0.5\\ 50000(1-x), & 0.5\le x\le 1\end{cases}$$

$$I(x,0)=\begin{cases}30000x, & 0\le x<0.5\\ 30000(1-x), & 0.5\le x\le 1\end{cases}$$

Figure 1 reveals that maximum value of susceptible and infected class is concentrated at the middle of domain $[0,1]$ and the value decreases linearly to zero at the boundaries $x=0$ and $x=1$.

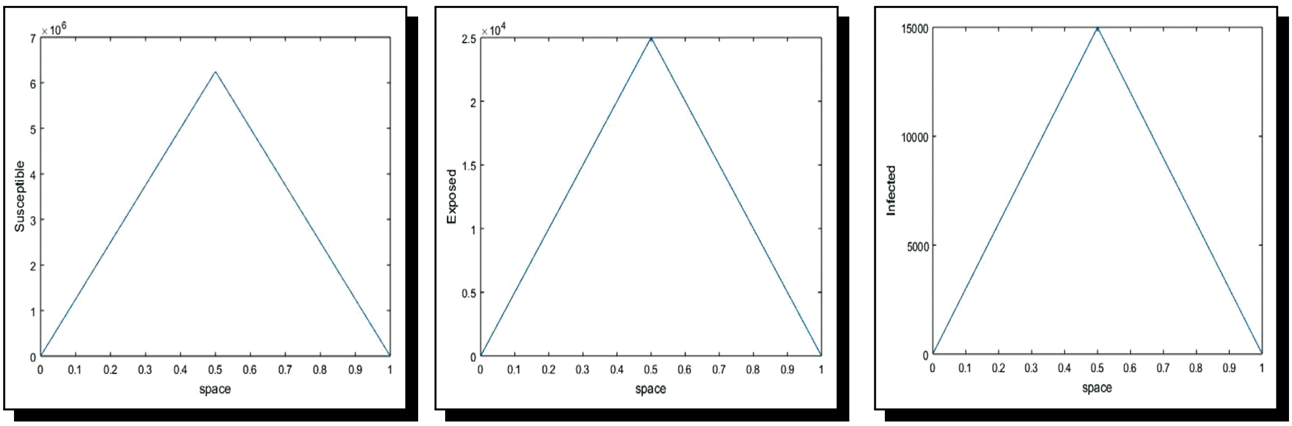

Figure 1. The initial distribution of the value of susceptible, exposed and infected class

6.2 Experiment 2

For this experiment, we take $S_0=12500000$, $E_0=50000$, $I_0=30000$.

6.2.1 Disease Free Equilibrium

Now we express the various schemes graphically which are already discussed.

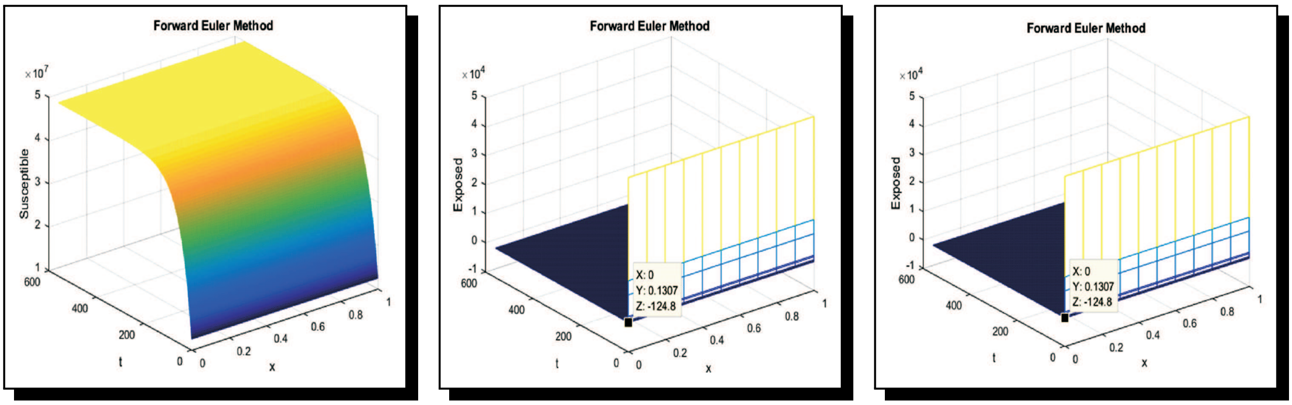

Figure 2. Mesh graphs of forward Euler FD scheme at DFE point

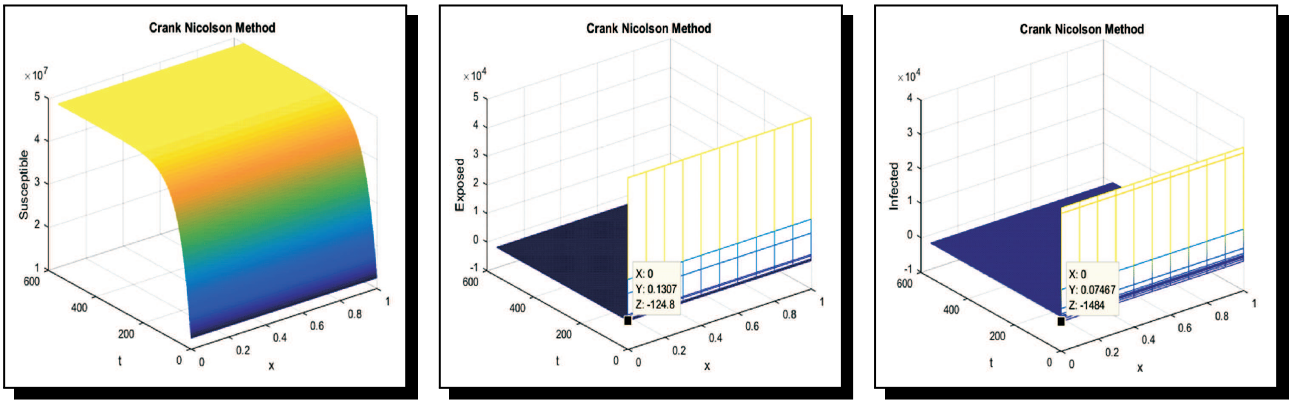

Figure 3. Mesh graphs of Crank Nicolson FD scheme at DFE point

Figures 2 and 3 illustrate the graphs using forward Euler scheme and Crank Nicolson scheme. The graphs reflect that both the schemes failed to preserve positivity property.

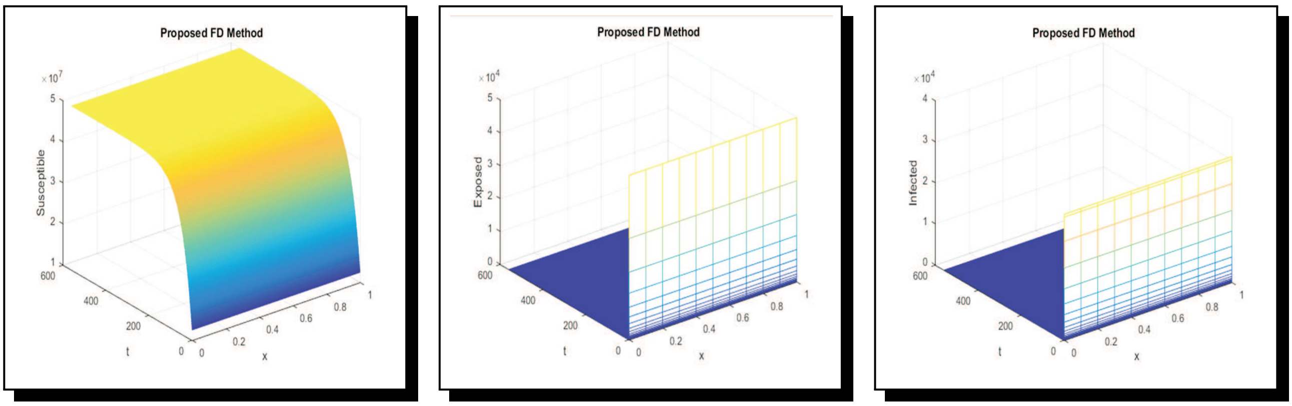

Figure 4. Mesh graphs of proposed FD scheme at DFE point

Figure 4 highlights the graphs using proposed FD scheme. These graphs show the disease free equilibrium. Graphs clearly show that the proposed FD scheme converges to disease free equilibrium points $(N,0,0)$ and preserves positivity property.

6.2.2 Endemic Equilibrium

Next simulation results of EE for all the schemes are presented.

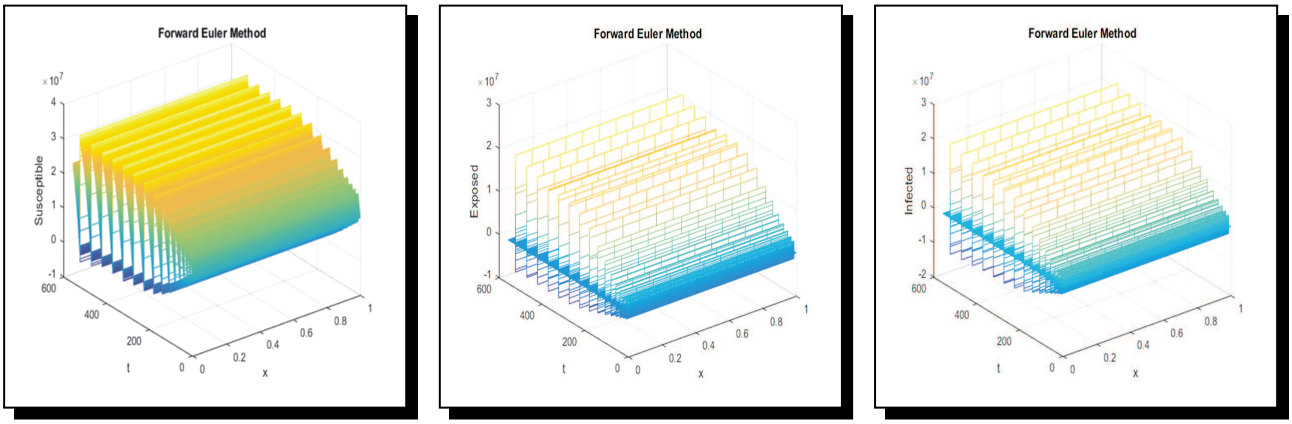

Figure 5. Mesh graphs of forward Euler FD scheme at EE point

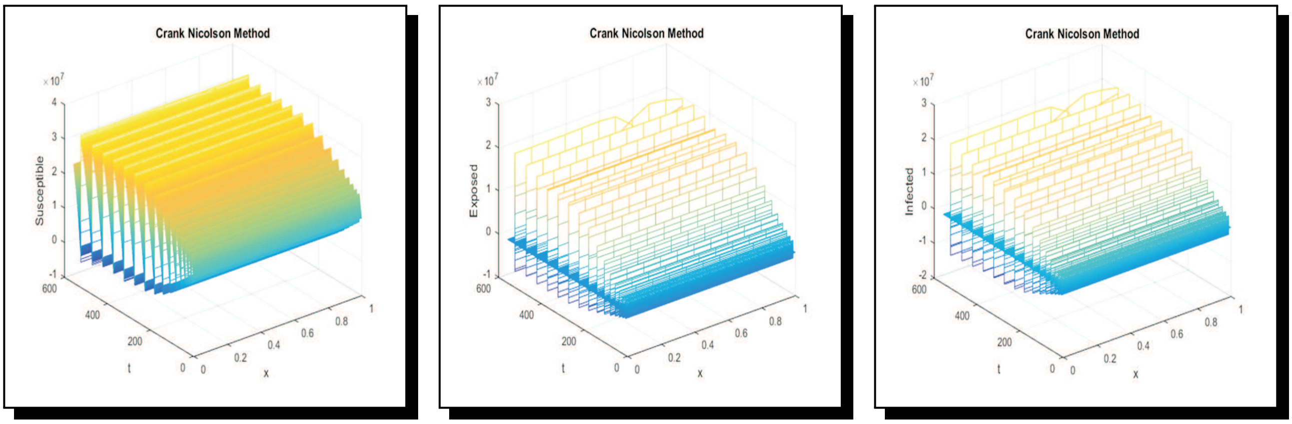

Figure 6. Mesh graphs of Crank Nicolson FD scheme at EE point

Figures 5 and 6 show that forward Euler FD scheme and Crank Nicolson FD scheme produce nonphysical oscillations and hence become unstable.

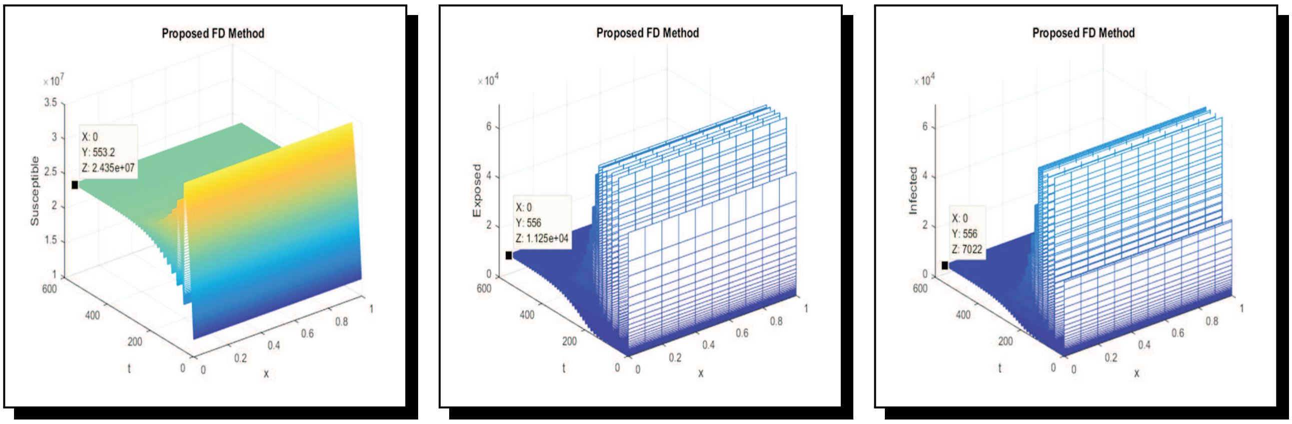

Figure 7. Mesh graphs of proposed FD scheme at EE point

Figures 7 reflects the graphs of endemic equilibrium using proposed FD scheme. Graphs clearly verify that proposed FD scheme converges to endemic equilibrium point $(S_{EE}, E_{EE}, I_{EE}) = (2.435 \times 10^7, 1.125 \times 10^4, 7022.67)$.

7. Conclusion

In this work, SEIR epidemic model with diffusion is solved numerically by forward Euler FD scheme, Crank Nicolson FD scheme and proposed FD scheme. The consistency and unconditional stability of proposed scheme has been proved analytically. Simulations are presented. Graphs show that both existing schemes produce negative values, show unstable behavior, converge to false steady states and diverge while proposed finite difference scheme is unconditionally dynamically consistent with positivity property and unconditionally convergent to true steady states.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

The authors wrote, read and approved the final manuscript.

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