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Research Article

Inverse Problem for Determination of An Unknown Coefficient in the Time Fractional Diffusion Equation

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Abstract. The fundamental concern of this article is to apply the *residual power series method* (RPSM) effectively to determine of the unknown coefficient in the time fractional diffusion equation in the Caputo sense with over measured data. First, the fractional power series solution of inverse problem of unknown coefficient is obtained by residual power series method. Finally, efficiency and accuracy of the present method is illustrated by numerical examples.

Keywords. Inverse problem; Diffusion equation; Unknown coefficient; Fractional derivative; Residual power series

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1. Introduction

In this paper, we consider the inverse problem of finding $u(x,t)$ and $k(x)$ in the following problem

$$D_t^\alpha u = (k(x)u_x)_x, \quad 0 < x < l,\ 0 \le t \le T, \tag{1}$$

$$u(x,0) = \varphi_1(x), \quad 0 < x < l, \tag{2}$$

$$k(0)u_x(0,t) = \mu(t), \quad 0 \le t \le T, \tag{3}$$

under the additional condition

$$D_t^\alpha u(x,0) = \varphi_2(x), \quad 0 < x < l, \tag{4}$$

where $\alpha \in (0,1)$ is the fractional order, $D_t^\alpha u$ denotes the αth order of Caputo fractional derivative with respect to t.

Physically speaking, this model describes the diffusion procedure with memory. The coefficient $k(x)$ represents a diffusion coefficient. In various problems in science, determination of the coefficients in the diffusion equation requires some additional information. These kinds of problems are called as inverse problems [7, 8, 15, 17, 23, 28].

Recently, fractional calculus has been considerable popularity. Indeed, fractional calculus plays a central role in numerous applications in nanotechnology, control theory, viscoplasticity flow, biology, signal and image processing and so on fractional calculus [6, 9, 16, 18, 19, 24, 26]. The mathematical and numerical analysis of the direct problem of the time-fractional diffusion has gained much attention [10, 20, 21, 25, 27]. However, the investigation of the inverse problems for the fractional diffusion equation remain rarely.

In the present work, by making use of residual power series (RPS) technique, the coefficients of $u(x,t)$ and $k(x)$ are determined [1–5, 11–14]. The advantage of RPS technique is that it can be employed for inverse problems without linearization, perturbation, or discretization.

2. Preliminaries

In this section, the main definitions and various features of the fractional calculus theory are given.

Definition 1. The Riemann-Liouville time fractional integral of order α of $u(x,t)$ is described as

$$I_t^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_s^t (t-\xi)^{\alpha-1} u(x,\xi)d\xi, & \alpha > 0,\ x \in I,\ t > \xi > s \geq 0 \\ u(x,t), & \alpha = 0. \end{cases}$$

Definition 2. The Caputo's time fractional derivative of order α of $u(x,t)$ is defined as

$$D_t^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_s^t (t-\xi)^{m-\alpha-1} \frac{\partial^m u(x,\xi)}{\partial \xi^m} d\xi, & 0 \leq m-1 < \alpha < m,\ t > \xi > s \geq 0,\ x \in I \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \alpha = m \in N. \end{cases}$$

Definition 3. If $m-1 < \alpha \leq m$, $m \in N$, then

(i) $D_t^\alpha I_t^\alpha u(x,t) = u(x,t)$,

(ii) $I_t^\alpha D_t^\alpha u(x,t) = u(x,t) - \sum_{i=0}^{n-1} \frac{\partial u^i(x,s^+)}{\partial t^i} \frac{t^i}{i!}$.

For more information on fractional derivatives, see [16, 18, 26]. Some essential results of RPSM are given as follows [14]:

Definition 4. A power series expansion of the form

$$\sum_{k=0}^{\infty} f_k(x)(t-t_0)^{k\alpha}, \quad 0 \le m-1 < \alpha \le m,\ t \ge t_0$$

is called multiple fractional power series about $t = t_0$.

Definition 5. The two parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by [16, 26]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \quad z \in C.$$

The Mittag-Leffler function $E_{\alpha,\beta}(z)$ generalizes the exponential function e^z in that $E_{1,1}(z) = e^z$. It is an entire function in z with order $\frac{1}{\alpha}$ and type one [16].

3. RPSM for Time Fractional Heat Equation

In order to get the RPS solution, the following principal steps are applied:

Step 1. The fractional power series expansion for the solutions of eqns. (1)-(4) about $t = 0$ is established in the following form:

$$u(x,t) = \sum_{k=0}^{\infty} f_k(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, \quad 0 < \alpha \le 1,\ x \in I,\ 0 \le t < R. \tag{5}$$

By applying RPS technique, the mth truncated series of $u(x,t)$, $u_m(x,t)$ is obtained in the following form:

$$u_m(x,t) = \sum_{k=0}^{m} f_k(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, \quad 0 < \alpha \le 1,\ x \in I,\ 0 \le t < R. \tag{6}$$

The 0th RPS approximate solution is assumed to be the initial condition:

$$u_0(x,t) = f_0(x) = u(x,0) = \varphi_1(x). \tag{7}$$

Hence, we have

$$u_m(x,t) = \varphi_1(x) + \sum_{k=1}^{m} f_k(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, \quad 0 < \alpha \le 1,\ x \in I,\ 0 \le t,\ m = 1,2,3,\dots. \tag{8}$$

When determined $f_k(x)$, $k = 1,2,3,\dots,m$, the mth RPS approximate solution will be constructed.

Step 2. Let the residual function for eqns. (1)-(4) be defined in the following form:

$$Res(x,t) = D_t^{\alpha} u - (k(x)u_x)_x. \tag{9}$$

Hence, the mth residual function has the following form

$$Res_m(x,t) = D_t^{\alpha} u_m - (k(x)(u_m)_x)_x. \tag{10}$$

From [16, 18, 26], some results of $Res_m(x,t)$ which satisfy the following expressions $Res(x,t) = 0$, $\lim_{m\to\infty} Res_m(x,t) = Res(x,t)$ for each $x \in I$ and $t \ge 0$ and

$$D_t^{(i)\alpha} Res(x,0) = D_t^{(i)\alpha} Res_m(x,0) = 0, \quad i = 0,1,2,\dots,m. \tag{11}$$

Step 3. Replacing in eqn. (10) by $u_m(x,t)$ and take in the fractional derivative of $Res_m(x,t)$, $m = 1,2,3,\dots$ at $t = 0$ with eqn. (11), we obtain the following algebraic system of equations:

$$D_t^{(m-1)\alpha} Res_m(x,0) = 0, \quad 0 < \alpha \le 1,\ m = 1,2,3,\dots \tag{12}$$

Step 4. The coefficients $f_k(x)$, $k = 1,2,3,\dots,m$ are determined by solving the system (12). Thus, $u_m(x,t)$ is constructed.

In the next step, illustrating the above processes, $u_m(x,t)$ are obtained for $m = 1,2,3$.

For $\mathbf{m} = \mathbf{1}$, substituting

$$u_1(x,t) = \varphi_1(x) + f_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)} \tag{13}$$

in to

$$Res_1(x,t) = D_t^\alpha u_1(x,t) - (k(x)(u_1)_x)_x \tag{14}$$

we obtained the following:

$$Res_1(x,t) = f_1(x) - k'(x)\left[\varphi_1'(x) + f_1'(x)\frac{t^\alpha}{\Gamma(1+\alpha)}\right] - k(x)\left[\varphi_1''(x) + f_1''(x)\frac{t^{\alpha)}}{\Gamma(1+\alpha}\right]. \tag{15}$$

Hence, from eqn. (12) and (15), we have $f_1(x) = k'(x)\varphi_1'(x) + k(x)\varphi_1''(x)$.

From

$$D_t^\alpha u(x,0) = \varphi_2(x) \tag{16}$$

and equating we have

$$k'(x) + \frac{\varphi_1''(x)}{\varphi_1'(x)}k(x) = \frac{\varphi_2(x)}{\varphi_1'(x)}. \tag{17}$$

Then we solve the obtained ordinary differential equation, we obtain

$$k(x) = \frac{\left[\int \varphi_2(x)dx + C\right]}{\varphi_1'(x)},$$

where the constant C is obtained by using the boundary condition of the problem, and

$$f_1(x) = \varphi_2(x).$$

Similarly, for $\mathbf{m} = \mathbf{2}$, substituting

$$u_2(x,t) = \varphi_1(x) + f_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + f_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \tag{18}$$

into $Res_2(x,t)$, we obtained:

$$Res_2(x,t) = \varphi_2(x) + f_2(x)\frac{t^\alpha}{\Gamma(1+\alpha)} - k'(x)\left[\varphi_1'(x) + f_1'(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + f_2'(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right]$$
$$- k(x)\left[\varphi_1''(x) + f_1''(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + f_2''(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right]. \tag{19}$$

From eqn. (12) and eqn. (19), we have

$$f_2(x) = k'(x)f_1'(x) + k(x)f_1''(x).$$

For $\mathbf{m}=\mathbf{3}$, substituting,

$$u_3(x,t)=\varphi_1(x)+f_1(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}+f_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+f_3(x)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \tag{20}$$

into $Res_3(x,t)$, now, we have solved the equation $D_t^{2\alpha}Res_3(x,0)=0$, the coefficient $f_3(x)$ is obtained in the following form

$$f_3(x)=k'(x)f_2'(x)+k(x)f_2''(x).$$

Hence, $u_3(x,t)$ can be written as follows:

$$u_3(x,t)=\varphi_1(x)+\varphi_2(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}+(k'(x)f_1'(x)+k(x)f_1''(x))\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$
$$+(k'(x)f_2'(x)+k(x)f_2''(x))\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}. \tag{21}$$

Then, by the recurrence formula, we obtain

$$f_k(x)=k'(x)f_{k-1}'(x)+k(x)f_{k-1}''(x), \quad k=2,3,\dots.$$

4. Illustrative Examples

Example 1. Consider the following time fractional diffusion problem

$$D_t^{\alpha}u=(k(x)u_x)_x, \qquad 0<x<l,\ 0\le t\le T, \tag{22}$$

$$u(x,0)=1+\exp(-x), \quad 0<x<l, \tag{23}$$

$$k(0)u_x(0,t)=-\exp(t), \quad 0\le t\le T \tag{24}$$

and the additional condition

$$D_t^{\alpha}u(x,0)=1+\exp(-x), \quad 0<x<l. \tag{25}$$

According to RPSM, starting with the initial guess approximation, the series solution of eqn. (22) can be written of the form

$$u_1(x,t)=(1+\exp(-x))+f_1(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}. \tag{26}$$

Applying the Caputo derivative according to t in eqn. (26) and equating to eqn. (22), we have $k(x)=1-xe^{-x}+Ce^{x}$. In order to determine the constant C in $k(x)$ the boundary condition is used and $C=0$ is found. Hence, we obtain

$$k(x)=1-xe^{-x}$$

and

$$f_1(x)=1+\exp(-x).$$

Substituting $k(x)$ into eqn. (26), $u_2(x,t)$ can be expressed as follows:

$$u_2(x,t)=(1+\exp(-x))+(1+\exp(-x))\frac{t^{\alpha}}{\Gamma(1+\alpha)}+f_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}. \tag{27}$$

We apply repeating process as in the former application

$$f_k(x)=(1+\exp(-x)), \quad k=2,3,4,\dots.$$

Therefore, the RPS approximate solutions are

$$u(x,t) = (1+\exp(-x)) + (1+\exp(-x))\frac{t^{\alpha}}{\Gamma(1+\alpha)} + (1+\exp(-x))\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \dots . \tag{28}$$

To verify the efficiency and accuracy of the RPS technique, for several values of α, x and t, the absolute error is determined by taking the exact solution into account and they are listed in Table 1.

Table 1. Approximate third order solution of Example 1 for different value of α and absolute error at $\alpha = 1$

x	t	$\alpha = 0.75$	$\alpha = 0.9$	$\alpha = 1$	Exact	Absolute error
0.25	0.3	2.6678	2.5010	2.4005	2.4011	6×10^{-4}
	0.6	3.5726	3.3734	3.2303	3.2412	0.0109
	0.9	4.7616	4.5143	4.3163	4.3751	0.0588
0.5	0.3	2.4094	2.2587	2.1680	2.1686	6×10^{-4}
	0.6	3.2266	3.0467	2.9175	2.9273	0.0098
	0.9	4.3005	4.0771	3.8982	3.9514	0.0532
0.75	0.3	2.2082	2.0701	1.9870	1.9875	5×10^{-4}
	0.6	2.9571	2.7922	2.6738	2.6828	0.0090
	0.9	3.9413	3.7366	3.5727	3.6214	0.0487
1	0.3	2.0515	1.9232	1.8460	1.8464	4×10^{-4}
	0.6	2.7473	2.5941	2.4841	2.4924	0.0083
	0.9	3.6616	3.4715	3.3192	3.3644	0.0452

Example 2. Consider the following time fractional diffusion problem

$$D_t^{\alpha} u = (k(x)u_x)_x, \qquad 0 < x < l,\ 0 \le t \le T, \tag{29}$$

$$u(x,0) = (x+1)^2, \qquad 0 < x < l, \tag{30}$$

$$k(0)u_x(0,t) = \frac{2}{3}e^{2t}, \quad 0 \le t \le T \tag{31}$$

and the additional condition

$$D_t^{\alpha} u(x,0) = 2(x+1)^2, \quad 0 < x < l. \tag{32}$$

According to RPSM, starting with the initial guess approximation, the series solution of eqn. (29) can be written of the form

$$u_1(x,t) = (x+1)^2 + f_1(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}. \tag{33}$$

Applying the Caputo derivative according to t in eq. (33) and equating to eq. (29), we have $k(x) = \frac{(x+1)^2}{3} + \frac{C}{x+1}$. In order to determine the constant C in $k(x)$ the boundary condition is used and $C = 0$ is found. Hence, we obtain

$$k(x) = \frac{(x+1)^2}{3}$$

and

$$f_1(x) = 2(x+1)^2 .$$

Substituting $k(x)$ into eq. (33), $u_2(x,t)$ can be written as follows:

$$u_2(x,t) = (x+1)^2 + 2(x+1)^2 \frac{t^\alpha}{\Gamma(1+\alpha)} + f_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} . \tag{34}$$

We apply repeating process as in the former application

$$f_k(x) = 2^k(x+1)^2, \quad k = 1,2,3,\dots .$$

Therefore, the RPS approximate solutions are

$$u(x,t) = (x+1)^2 + 2(x+1)^2 \frac{t^\alpha}{\Gamma(1+\alpha)} + 4(x+1)^2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \dots . \tag{35}$$

To verify the efficiency and accuracy of the RPS technique, for several values of α, x and t, the absolute error is determined by taking the exact solution into account and they are listed in Table 2.

Table 2. Approximate third order solution of Example 2 for different value of α and absolute error at $\alpha = 1$

x	t	$\alpha = 0.75$	$\alpha = 0.9$	$\alpha = 1$	Exact	Absolute error
0.25	0.3	3.3575	3.0368	2.8375	2.8471	0.0096
	0.6	5.9443	5.4047	5.0125	5.1877	0.1752
	0.9	10.2415	9.2019	8.4250	9.4526	1.0276
0.5	0.3	4.8347	4.3730	4.0860	4.0998	0.0138
	0.6	8.5598	7.7828	7.2180	7.4703	0.2523
	0.9	14.7478	13.2508	12.1320	13.6117	1.4797
0.75	0.3	6.5806	5.9521	5.5615	5.5802	0.0187
	0.6	11.6508	10.5933	9.8245	10.1679	0.3434
	0.9	20.0733	18.0358	16.5130	18.5270	2.0140
1	0.3	8.5951	7.7742	7.2640	7.2885	0.0245
	0.6	15.2174	13.8361	12.8320	13.2805	0.4485
	0.9	26.2182	23.5569	21.5680	24.1986	2.6306

5. Conclusion

The fundamental aim of this study is to demonstrate the feasibility of the RPSM for solving time-fractional inverse problems in the Caputo sense. The above results and all of the discussed examples reveal that the goal has been achieved successfully with Neumann boundary condition since Dirichlet boundary condition makes inverse problems ill-posed. As a result RPSM can be utilized as a significant method to get analytical solutions of time-fractional inverse problems arising in different branches of science.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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