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Research Article

On Generalized Absolute Riesz Summability Method

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Abstract. This paper presents a generalization of a known theorem dealing with absolute Riesz summability of infinite series to the $\varphi-|\bar{N},p_n;\delta|_k$ summability.

Keywords. Riesz mean; Summability factor; Almost increasing sequences; Infinite series; Hölder inequality; Minkowski inequality

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1. Introduction

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants K and L such that $Kc_n \le b_n \le Lc_n$ (see [1]). Let $\sum a_n$ be a given infinite series with the partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as } n \to \infty, \quad (P_{-i} = p_{-i} = 0,\ i \ge 1). \tag{1.1}$$

The sequence-to-sequence transformation

$$z_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.2}$$

defines the sequence (z_n) of the Riesz mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [9]).

Let (φ_n) be a sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi-|\bar{N},p_n;\delta|_k$, $k\geq 1$ and $\delta\geq 0$, if (see [14])

$$\sum_{n=1}^{\infty}\varphi_n^{\delta k+k-1}|z_n-z_{n-1}|^k<\infty. \tag{1.3}$$

If we take $\varphi_n=\frac{P_n}{p_n}$, then $\varphi-|\bar{N},p_n;\delta|_k$ summability reduces to $|\bar{N},p_n;\delta|_k$ summability (see [5]). If we take $\delta=0$ and $\varphi_n=\frac{P_n}{p_n}$, then $\varphi-|\bar{N},p_n;\delta|_k$ summability reduces to $|\bar{N},p_n|_k$ summability (see [2]).

2. Known Result

In [8], Bor has obtained the following theorem.

Theorem 2.1. *Let (X_n) be an almost increasing sequence and let there be sequences (β_n) and (λ_n) such that*

$$|\Delta\lambda_n|\leq\beta_n, \tag{2.1}$$

$$\beta_n\to 0 \quad \text{as } n\to\infty, \tag{2.2}$$

$$\sum_{n=1}^{\infty}n|\Delta\beta_n|X_n<\infty, \tag{2.3}$$

$$|\lambda_n|X_n=O(1) \tag{2.4}$$

and

$$\sum_{v=1}^{n}\frac{|t_v|^k}{v}=O(X_n) \quad \text{as } n\to\infty, \tag{2.5}$$

where (t_n) is the n-th $(C,1)$ mean of the sequence (na_n). Suppose further, the sequence (p_n) is such that

$$P_n=O(np_n), \tag{2.6}$$

$$P_n\Delta p_n=O(p_np_{n+1}), \tag{2.7}$$

then the series $\sum_{n=1}^{\infty}a_n\frac{P_n\lambda_n}{np_n}$ is summable $|\bar{N},p_n|_k$, $k\geq 1$.

3. Main Result

The purpose of this paper is to generalize above theorem for $\varphi-|\bar{N},p_n;\delta|_k$ summability method in the following form. One can find more applications of generalized absolute summability of infinite series (see [6], [7], [11], [12], [13], [15], [16]).

Theorem 3.1. *Let (X_n) be an almost increasing sequence and $\varphi_np_n=O(P_n)$. If conditions* (2.1)-(2.4)*,* (2.6)-(2.7) *of Theorem* 2.1 *and*

$$\sum_{v=1}^{n}\varphi_v^{\delta k}\frac{1}{v}|t_v|^k=O(X_n) \quad \text{as } n\to\infty, \tag{3.1}$$

$$\sum_{n=v+1}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} = O\left(\varphi_v^{\delta k} \frac{1}{P_v}\right) \quad as\ m \to \infty, \tag{3.2}$$

are satisfied, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ *is summable* $\varphi - |\bar{N}, p_n; \delta|_k$, $k \geq 1$ *and* $0 \leq \delta < 1/k$.

When we take $\delta = 0$ and $\varphi_n = \frac{P_n}{p_n}$ in Theorem 3.1, then we get Theorem 2.1. In this case, the condition (3.1) reduces to the condition (2.5). Also, the condition (3.2) is automatically satisfied.

Remark. It should be noted that under the conditions on the sequence (λ_n), we have that (λ_n) is bounded and $\Delta\lambda_n = O(1/n)$ (see [3]).

Lemma 3.2 ([10])**.** *If* (X_n) *is an almost increasing sequence, then under the conditions* (2.2)-(2.3)*, we have*

$$nX_n\beta_n = O(1) \quad as\ n \to \infty, \tag{3.3}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{3.4}$$

Lemma 3.3 ([4])**.** *If conditions* (2.6) *and* (2.7) *are satisfied, then we have*

$$\Delta\left(\frac{P_n}{n^2 p_n}\right) = O\left(\frac{1}{n^2}\right). \tag{3.5}$$

4. Proof of Theorem 3.1

Let (M_n) be the sequence of $(\bar{N}, p_n)$ mean of the series $\sum \frac{a_n P_n \lambda_n}{n p_n}$. Then, we have

$$M_n = \frac{1}{P_n} \sum_{v=1}^{n} p_v \sum_{r=1}^{v} \frac{a_r P_r \lambda_r}{r p_r} = \frac{1}{P_n} \sum_{v=1}^{n} (P_n - P_{v-1}) \frac{a_v P_v \lambda_v}{v p_v}.$$

Then, for $n \geq 1$, we get

$$M_n - M_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} P_v a_v \lambda_v}{v p_v} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} P_v a_v v \lambda_v}{v^2 p_v}.$$

From Abel's transformation, we obtain

$$\begin{aligned} M_n - M_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta\left(\frac{P_{v-1} P_v \lambda_v}{v^2 p_v}\right) \sum_{r=1}^{v} r a_r + \frac{\lambda_n}{n^2} \sum_{v=1}^{n} v a_v \\ &= \frac{(n+1) t_n \lambda_n}{n^2} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{P_v}{p_v} (v+1) t_v p_v \frac{\lambda_v}{v^2} \\ &\quad + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v P_v \Delta\lambda_v (v+1) \frac{t_v}{v^2 p_v} \\ &\quad - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \lambda_{v+1} (v+1) t_v \Delta\left(\frac{P_v}{v^2 p_v}\right) \\ &= M_{n,1} + M_{n,2} + M_{n,3} + M_{n,4}. \end{aligned}$$

To prove Theorem 3.1, we have to show that

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k + k - 1} |M_{n,r}|^k < \infty, \quad \text{for } r = 1,2,3,4.$$

First, from Abel's formula, we have

$$\begin{aligned}
\sum_{n=1}^{m}\varphi_n^{\delta k+k-1}|M_{n,1}|^k &= \sum_{n=1}^{m}\varphi_n^{\delta k+k-1}\left|\frac{(n+1)t_n\lambda_n}{n^2}\right|^k \\
&= O(1)\sum_{n=1}^{m}\varphi_n^{\delta k}\frac{1}{n}|\lambda_n||t_n|^k \\
&= O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{r=1}^{n}\varphi_r^{\delta k}\frac{1}{r}|t_r|^k + O(1)|\lambda_m|\sum_{n=1}^{m}\varphi_n^{\delta k}\frac{1}{n}|t_n|^k \\
&= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n + O(1)|\lambda_m|X_m \\
&= O(1)\sum_{n=1}^{m-1}\beta_n X_n + O(1)|\lambda_m|X_m \\
&= O(1) \quad \text{as } m\to\infty,
\end{aligned}$$

by virtue of (2.1), (2.4), (2.6), (3.1) and (3.4).

From Hölder's inequality, as in $M_{n,1}$, we have

$$\begin{aligned}
\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}|M_{n,2}|^k &= \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\frac{P_v}{p_v}(v+1)t_v p_v\frac{\lambda_v}{v^2}\right|^k \\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k-1}\left(\frac{\varphi_n p_n}{P_n}\right)^k\frac{1}{P_{n-1}^k}\left(\sum_{v=1}^{n-1}\frac{P_v}{p_v}p_v|t_v|\frac{|\lambda_v|}{v}\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k p_v|t_v|^k\frac{|\lambda_v|^k}{v^k}\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k p_v|t_v|^k|\lambda_v|^{k-1}|\lambda_v|\frac{1}{v^k}\sum_{n=v+1}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}} \\
&= O(1)\sum_{v=1}^{m}\varphi_v^{\delta k}|\lambda_v|\frac{|t_v|^k}{v} \\
&= O(1) \quad \text{as } m\to\infty,
\end{aligned}$$

by (2.1), (2.4), (2.6), (3.1), (3.2) and (3.4).

Now, using $\Delta\lambda_n = O(1/n)$, we get

$$\begin{aligned}
\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}|M_{n,3}|^k &= \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_vP_v\Delta\lambda_v(v+1)\frac{t_v}{v^2p_v}\right|^k \\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k p_v|\Delta\lambda_v|^k|t_v|^k\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k p_v|\Delta\lambda_v|^{k-1}|\Delta\lambda_v||t_v|^k\sum_{n=v+1}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}} \\
&= O(1)\sum_{v=1}^{m}\varphi_v^{\delta k}v\beta_v\frac{|t_v|^k}{v}
\end{aligned}$$

$$= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\varphi_r^{\delta k}\frac{1}{r}|t_r|^k + O(1)m\beta_m\sum_{v=1}^{m}\varphi_v^{\delta k}\frac{1}{v}|t_v|^k$$

$$= O(1)\sum_{v=1}^{m-1}|\Delta(v\beta_v)|X_v + O(1)m\beta_m X_m$$

$$= O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|X_v + O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1)m\beta_m X_m$$

$$= O(1) \quad \text{as } m\to\infty,$$

by using (2.1), (2.3), (2.6), (3.1), (3.2), (3.3) and (3.4).

Finally, since $\Delta\left(\frac{P_v}{v^2p_v}\right) = O\left(\frac{1}{v^2}\right)$, as in $M_{n,1}$, we have

$$\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}|M_{n,4}|^k = \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|-\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v\lambda_{v+1}(v+1)t_v\Delta\left(\frac{P_v}{v^2p_v}\right)\right|^k$$

$$= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k-1}\left(\frac{\varphi_n p_n}{P_n}\right)^k\frac{1}{P_{n-1}^k}\left(\sum_{v=1}^{n-1}\frac{P_v}{p_v}p_v|t_v|\frac{|\lambda_{v+1}|}{v}\right)^k$$

$$= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k p_v|t_v|^k\frac{|\lambda_{v+1}|^k}{v^k}\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k p_v|t_v|^k|\lambda_{v+1}|^{k-1}|\lambda_{v+1}|\frac{1}{v^k}\sum_{n=v+1}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}\varphi_v^{\delta k}|\lambda_{v+1}|\frac{|t_v|^k}{v}$$

$$= O(1) \quad \text{as } m\to\infty,$$

by (2.1), (2.4), (2.6), (3.1), (3.2) and (3.4).

Thus, the proof of Theorem 3.1 is completed.

When we take $\delta = 0$, $\varphi_n = \frac{P_n}{p_n}$ and (X_n) as a positive non-decreasing sequence, then we get a theorem dealing with $|\bar{N},p_n|_k$ summability (see [4]).

5. Conclusion

In this study, generalized absolute summability of infinite series has been studied. A theorem concerning absolute summability factors, which generalizes a known theorem dealing with the $|\bar{N},p_n|_k$ summability factors of infinite series, has been proved by using almost increasing sequences.

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Competing Interests

The author declares that she has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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