Communications in Mathematics and Applications
Vol. 9, No. 3, pp. 249–264, 2018
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v9i3.719

http://www.rgnpublications.com

***Proceedings*:**
3rd International Conference on Pure and Applied Mathematics
Department of Mathematics, University of Sargodha, Sargodha, Pakistan
November 10-11, 2017

Shape Preservation of the Stationary 4-Point Quaternary Subdivision Schemes

Research Article

Khurram Pervez

Department of Mathematics, Balochistan University of Information Technology, Engineering & Management Sciences, Quetta, Pakistan
thekp007@gmail.com

Abstract. In this paper, the shape preserving properties of the stationary 4-point quaternary approximating and interpolating subdivision schemes of Ko [9] are fully investigated. We will analyzed what conditions should be introduced on the initial control points so that the limit curve achieved by the subdivision schemes presented in [9] are both monotonicity and convexity preserving. Conclusively the whole discussion is followed by examples.

Keywords. Quaternary; Approximating; Interpolating; Monotonicity; Convexity

MSC. 41A05; 41A25; 41A30; 65D17; 65D10; 40A05; 52B55

Received: November 27, 2017 **Revised:** February 15, 2018 **Accepted:** April 6, 2018

1. Introduction

The subdivision schemes are the main pillar in the field of computer graphics, reverse engineering, computer animation and *Computer Aided Geometric Design* (CAGD). Now a days subdivision schemes are widely used in CAGD with lots of interesting applications in industry because of their high efficiency. The subdivision scheme outlines a curve out of an initial control mesh by subdividing them according to some refining rule.

Numerous research papers have been published during the past couple of years in the field of convexity preservation. In 1994, Cai *et al*. [14] derived the convexity preserving algorithm for four-point interpolating subdivision scheme of Dyn *et al*. [7], which is regarded as a pioneer paper in this field. In 1999, Dyn *et al*. [6] deduced convexity preservation of the four-point interpolatory scheme [7]. In 2009, Cai [5] presented convexity preservation of the ternary interpolating four-point C^2 stationary subdivision scheme of Hassan *et al*. [8] in 2002. In 2013, Amat *et al*. [2] introduce a new approach towards proving convexity preserving properties for interpolatory subdivision scheme. In 2014, Tan *et al*. [13] presented convexity preservation of five-point binary subdivision scheme with a parameter. In 2015, Siddiqi and Noreen [11] discussed convexity preservation of six point C^2 ternary interpolating subdivision scheme which was presented by Mustafa and Pakeeza [10] in 2010. In 2017, Tan *et al*. [12] analyzed the shape-preserving properties of the four-point binary subdivision scheme. Recently, in 2017, Akram *et al*. [1] analyzed shape preservation of 4-point interpolating non-stationary subdivision scheme of Beccari *et al*. [4]. In this paper we shall focus on shape preservation properties of stationary 4-point quarternary Approximating and interpolatory subdivision schemes of C^2 continuity presented by Ko [9] in 2009.

The remainder of the paper is organized as follows. In Section 2 we recall the refinement rules of the 4-point quaternary approximating and interpolating subdivision schemes introduced in [9]. In Section 3 and Section 4 we investigated the monotonicity preserving and convexity preserving of the approximating and interpolating subdivision schemes respectively. Numerical examples and conclusion are drawn in Section 5 and Section 6, respectively.

2. Quaternary 4-Point Approximating and Interpolating Subdivision Schemes

Based on literature [9], first of all, we give the definition of a stationary 4-point quaternary approximating and interpolating subdivision schemes.

Definition 2.1. A polygon $p^k = \{p_i^k\}_{i\in\mathbb{Z}}$ is mapped to a refined polygon $p^{k+1} = \{p_i^{k+1}\}_{i\in\mathbb{Z}}$ by applying the following four subdivision rules for approximating scheme:

$$\begin{cases} p_{4i}^{k+1} = -\frac{35}{1024}p_{i-1}^k + \frac{945}{1024}p_i^k + \frac{135}{1024}p_{i+1}^k - \frac{21}{1024}p_{i+2}^k, \\ p_{4i+1}^{k+1} = -\frac{65}{1024}p_{i-1}^k + \frac{715}{1024}p_i^k + \frac{429}{1024}p_{i+1}^k - \frac{55}{1024}p_{i+2}^k, \\ p_{4i+2}^{k+1} = -\frac{55}{1024}p_{i-1}^k + \frac{429}{1024}p_i^k + \frac{715}{1024}p_{i+1}^k - \frac{65}{1024}p_{i+2}^k, \\ p_{4i+3}^{k+1} = -\frac{21}{1024}p_{i-1}^k + \frac{135}{1024}p_i^k + \frac{945}{1024}p_{i+1}^k - \frac{35}{1024}p_{i+2}^k \end{cases} \tag{2.1}$$

and for interpolating scheme:

$$\begin{cases} p_{4i}^{k+1} = p_i^k, \\ p_{4i+1}^{k+1} = -\frac{7}{128}p_{i-1}^k + \frac{105}{128}p_i^k + \frac{35}{128}p_{i+1}^k - \frac{5}{128}p_{i+2}^k, \\ p_{4i+2}^{k+1} = -\frac{1}{16}p_{i-1}^k + \frac{9}{16}p_i^k + \frac{9}{16}p_{i+1}^k - \frac{1}{16}p_{i+2}^k, \\ p_{4i+3}^{k+1} = -\frac{5}{128}3p_{i-1}^k + \frac{35}{128}p_i^k + \frac{105}{128}1p_{i+1}^k - \frac{7}{128}p_{i+2}^k. \end{cases} \tag{2.2}$$

3. Monotonicity Preservation

In the field of subdivision scheme, shape preservation is an important research topic. In this section, we will drive monotonicity property for the subdivision schemes (2.1) and (2.2).

Lemma 1. *Suppose we have a set of initial control points* $\{p_i^0\}_{i\in\mathbb{Z}}$,

$$\dots p_{-2}^0 < p_{-1}^0 < p_0^0 < p_1^0 < \dots < p_{n-1}^0 < p_n^0 \dots .$$

Define first order divided difference by $D_i^k = p_{i+1}^k - p_i^k$, *taking*

$$q_i^k = \frac{D_{i+1}^k}{D_i^k}, \quad Q^k = \max_i \left\{ q_i^k, \frac{1}{q_i^k} \right\}, \qquad \text{for all } k \geq 0,\ i,k \in \mathbb{Z}.$$

Furthermore, let, for:

(i) *Approximating scheme* (2.1): $1 \leq \rho \leq \frac{59+2\sqrt{849}}{5}$, $\rho \in \mathbb{R}$,

(ii) *Interpolating scheme* (2.2): $1 \leq \rho \leq \frac{14+\sqrt{201}}{5}$, $\rho \in \mathbb{R}$.

If $\frac{1}{\rho} \leq Q^0 \leq \rho$, $\{p_i^k\}$ *is defined by the subdivision scheme* (2.1) *and* (2.2), *then:*

$$D_i^k > 0, \ \frac{1}{\rho} \leq Q^k \leq \rho, \quad \text{for all } k \geq 0,\ i,k \in \mathbb{Z}. \tag{3.1}$$

Proof. **(i)** To prove Lemma 1, we use mathematical induction on k.

(I) By hypothesis, when $k = 0$,

$$D_i^0 = p_{i+1}^0 - p_i^0 > 0, \ \frac{1}{\rho} \leq Q^0 \leq \rho,$$

then (3.1) is satisfied.

(II) Suppose that (3.1) satisfies for some $k \geq 1$, then we will verify it also holds for $k+1$.

We first prove

$$D_i^k > 0, \quad \text{for all } k \geq 0,\ i \in \mathbb{Z},\ k \in \mathbb{Z}.$$

Assume, $D_i^k > 0$, for all $i \in \mathbb{Z}$, holds for some $k \geq 1$. Then, it follows: for all $i \in \mathbb{Z}$,

$$\begin{aligned}
D_{4i}^{k+1} &= p_{4i+1}^{k+1} - p_{4i}^{k+1} \\
&= \frac{1}{1024}\left[-30p_{i-1}^k - 230p_i^k + 294p_{i+1}^k - 34p_{i+2}^k\right] \\
&= \frac{1}{1024}\left[30\left(p_i^k - p_{i-1}^k\right) + 260\left(p_{i+1}^k - p_i^k\right) - 34\left(p_{i+2}^k - p_{i+1}^k\right)\right] \\
&= \frac{1}{1024}\left[30D_{i-1}^k + 260D_i^k - 34D_{i+1}^k\right] \\
&= \frac{D_i^k}{1024}\left[\frac{30}{q_{i-1}^k} + 260 - 34q_i^k\right] \\
&= \frac{D_i^k}{128}\left[\frac{30}{\rho} + 260 - 34\rho\right] > 0
\end{aligned} \tag{3.2}$$

and

$$D_{4i+1}^{k+1} = p_{4i+2}^{k+1} - p_{4i+1}^{k+1}$$

$$= \frac{1}{1024}\left[10p_{i-1}^k - 286p_i^k + 286p_{i+1}^k - 10p_{i+2}^k\right]$$
$$= \frac{1}{1024}\left[-10\left(p_i^k - p_{i-1}^k\right) + 276\left(p_{i+1}^k - p_i^k\right) - 10\left(p_{i+2}^k - p_{i+1}^k\right)\right]$$
$$= \frac{1}{1024}\left[-10D_{i-1}^k + 276D_i^k - 10D_{i+1}^k\right]$$
$$= \frac{D_i^k}{1024}\left[-\frac{10}{q_{i-1}^k} + 276 - 10q_i^k\right]$$
$$= \frac{D_i^k}{1024}\left[-\frac{10}{\rho} + 276 - 10\rho\right] > 0. \tag{3.3}$$

Similarly we can show that,

$$D_{4i+2}^{k+1} > 0 \quad \text{and} \quad D_{4i+3}^{k+1} > 0$$

which implies that

$$D_i^{k+1} > 0, \quad \text{for all } i \in \mathbb{Z}.$$

Therefore, by induction

$$D_i^k > 0, \quad \text{for all } k \geq 0,\ i \in \mathbb{Z},\ k \in \mathbb{Z}.$$

(III) To prove $\frac{1}{\rho} \leq Q^k \leq \rho$, for all $k \geq 0$, $k \in \mathbb{Z}$.

Since

$$q_{4i}^k = \frac{D_{4i+1}^{k+1}}{D_{4i}^{k+1}} = \frac{\frac{D_i^k}{1024}\left[-\frac{10}{q_{i-1}^k} + 276 - 10q_i^k\right]}{\frac{D_i^k}{1024}\left[\frac{30}{q_{i-1}^k} + 260 - 34q_i^k\right]},$$

$$q_{4i}^k - \rho = \frac{\left[-\frac{10}{q_{i-1}^k} + 276 - 10q_i^k\right] - \rho\left[\frac{30}{q_{i-1}^k} + 260 - 34q_i^k\right]}{\left[\frac{30}{q_{i-1}^k} + 260 - 34q_i^k\right]},$$

$$q_{4i}^k - \rho = \frac{-\frac{10}{q_{i-1}^k} + 276 - 10q_i^k - \frac{30\rho}{q_{i-1}^k} - 260\rho + 34\rho q_i^k}{\frac{30}{q_{i-1}^k} + 260 - 34q_i^k},$$

$$q_{4i}^k - \rho = \frac{A}{B}. \tag{3.4}$$

As denominator in (3.4), i.e., $B > 0$ and numerator satisfies:

$$A \leq (-10 + 34\rho)q_i^k - \frac{10}{q_{i-1}^k} - \frac{30\rho}{q_{i_1}^k} + 276 - 260\rho$$
$$\leq (-10 + 34\rho)\rho - \frac{10}{\rho} - 30 + 276 - 260\rho$$
$$= \frac{1}{\rho}\left(34\rho^3 - 270\rho^2 + 216\rho - 10\right) \leq 0.$$

Therefore,

$$q^k_{4i} \le \rho.$$

Similarly, in the same way, we can get

$$q^k_{4i+1} \le \rho,\ q^k_{4i+2} \le \rho,\ q^k_{4i+3} \le \rho$$

and

$$\frac{1}{q^k_{4i}} \le \rho,\ \frac{1}{q^k_{4i+1}} \le \rho,\ \frac{1}{q^k_{4i+2}} \le \rho,\ \frac{1}{q^k_{4i+3}} \le \rho$$

which implies

$$\frac{1}{\rho} \le Q^{k+1} \le \rho.$$

Therefore, by induction, we have

$$\frac{1}{\rho} \le Q^k \le \rho, \quad \text{for all } k \ge 0,\ k \in \mathbb{Z}.$$

This completes the proof.

(ii) To prove Lemma 1, we proceeds by mathematical induction.

(I) By hypothesis, when $k = 0$, $D^0_i = p^0_{i+1} - p^0_i > 0$, $\frac{1}{\rho} \le Q^0 \le \rho$, then (3.1) is satisfied.

(II) Suppose that (3.1) satisfies for some $k \ge 1$, then we will verify it also holds for $k+1$.

We first prove $D^k_i > 0$, for all $k \ge 0$, $i \in \mathbb{Z}$, $k \in \mathbb{Z}$.

Assume, $D^k_i > 0$, for all $i \in \mathbb{Z}$, true for some $k \ge 1$. Then, it follows: for all $i \in \mathbb{Z}$,

$$\begin{aligned} D^{k+1}_{4i} &= p^{k+1}_{4i+1} - p^{k+1}_{4i} \\ &= \frac{1}{128}\left[-7p^k_{i-1} - 23p^k_i + 35p^k_{i+1} - 5p^k_{i+2}\right] \\ &= \frac{1}{128}\left[7\left(p^k_i - p^k_{i-1}\right) + 30\left(p^k_{i+1} - p^k_i\right) - 5\left(p^k_{i+2} - p^k_{i+1}\right)\right] \\ &= \frac{1}{128}\left[7D^k_{i-1} + 30D^k_i - 5D^k_{i+1}\right] \\ &= \frac{D^k_i}{128}\left[\frac{7}{q^k_{i-1}} + 30 - 5q^k_i\right] \\ &= \frac{D^k_i}{128}\left[\frac{7}{\rho} + 30 - 5\rho\right] > 0 \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} D^{k+1}_{4i+1} &= p^{k+1}_{4i+2} - p^{k+1}_{4i+1} \\ &= \frac{1}{128}\left[-1p^k_{i-1} - 33p^k_i + 37f^k_{i+1} - 3p^k_{i+2}\right] \\ &= \frac{1}{128}\left[\left(p^k_i - p^k_{i-1}\right) + 34\left(p^k_{i+1} - p^k_i\right) - 3\left(p^k_{i+2} - p^k_{i+1}\right)\right] \end{aligned}$$

$$= \frac{1}{128}\left[D_{i-1}^k + 34D_i^k - 3D_{i+1}^k\right]$$
$$= \frac{D_i^k}{128}\left[\frac{1}{q_{i-1}^k} + 34 - 3q_i^k\right]$$
$$= \frac{D_i^k}{128}\left[\frac{1}{\rho} + 34 - 3\rho\right] > 0. \tag{3.6}$$

Similarly, we can show that

$$D_{4i+2}^{k+1} > 0 \quad \text{and} \quad D_{4i+3}^{k+1} > 0$$

which implies that

$$D_i^{k+1} > 0, \quad \text{for all } i \in \mathbb{Z}.$$

Therefore, by induction

$$D_i^k > 0, \quad \text{for all } k \geq 0,\ i,k \in \mathbb{Z}.$$

(III) To prove $\frac{1}{\rho} \leq Q^k \leq \rho$, for all $k \geq 0$, $k \in \mathbb{Z}$.

Since

$$q_{4i}^k = \frac{D_{4i+1}^{k+1}}{D_{4i}^{k+1}} = \frac{\frac{D_i^k}{128}\left[\frac{1}{q_{i-1}^k} + 34 - 3q_i^k\right]}{\frac{D_i^k}{128}\left[\frac{7}{q_{i-1}^k} + 30 - 5q_i^k\right]},$$

$$q_{4i}^k - \rho = \frac{\left[\frac{1}{q_{i-1}^k} + 34 - 3q_i^k\right] - \rho\left[\frac{7}{q_{i-1}^k} + 30 - 5q_i^k\right]}{\left[\frac{7}{q_{i-1}^k} + 30 - 5q_i^k\right]},$$

$$q_{4i}^k - \rho = \frac{\frac{1}{q_{i-1}^k} + 34 - 3q_i^k - \frac{7\rho}{q_{i-1}^k} - 30\rho + 5\rho q_i^k}{\frac{7}{q_{i-1}^k} + 30 - 5q_i^k},$$

$$q_{4i}^k - \rho = \frac{A}{B}. \tag{3.7}$$

As denominator in (3.7), i.e., $B > 0$ and numerator satisfies:

$$A \leq (-3+5\rho)q_i^k + \frac{1}{q_{i-1}^k} - \frac{7\rho}{q_{i_1}^k} + 34 - 30\rho$$
$$\leq (-3+5\rho)\rho + \frac{1}{\rho} - 7 + 34 - 30\rho$$
$$= \frac{1}{\rho}\left(5\rho^3 - 33\rho^2 + 27\rho + 1\right)$$
$$= \frac{1}{\rho}(\rho-1)(5\rho^2 - 28\rho - 1) \leq 0.$$

Therefore,

$$q_{4i}^k \leq \rho.$$

Similarly, in the same way, we can get

$$q^k_{4i+1} \le \rho,\ q^k_{4i+2} \le \rho,\ q^k_{4i+3} \le \rho$$

and

$$\frac{1}{q^k_{4i}} \le \rho,\ \frac{1}{q^k_{4i+1}} \le \rho,\ \frac{1}{q^k_{4i+2}} \le \rho,\ \frac{1}{q^k_{4i+3}} \le \rho$$

which implies

$$\frac{1}{\rho} \le Q^{k+1} \le \rho\,.$$

Therefore, by induction, we have

$$\frac{1}{\rho} \le Q^k \le \rho, \quad \text{for all } k \ge 0,\ k \in \mathbb{Z}.$$

This completes the proof. □

A direct consequence of Lemma 1 is Theorem 2.

Theorem 2. *Suppose the initial control points* $\{p^0_i\}_{i\in\mathbb{Z}}$ *with* $p^0_i = \left(x^0_i, f^0_i\right)$ *are strictly monotone decreasing (strictly monotone increasing). Denote:*

$$X^0 = \max_i \left\{ \frac{x^0_{i+2} - x^0_{i+1}}{x^0_{i+1} - x^0_i}, \frac{x^0_{i+1} - x^0_i}{x^0_{i+2} - x^0_{i+1}} \right\}, \quad Q^0 = \max_i \left\{ q^0_i, \frac{1}{q^0_i} \right\}.$$

Then, for $\frac{1}{\rho} \le X^0 \le \rho$, *and* $\frac{1}{\rho} \le Q^0 \le \rho$,

(i) *Approximating scheme* (2.1)*:* $1 \le \rho \le \frac{59+2\sqrt{849}}{5}$, $\rho \in \mathbb{R}$,

(ii) *Interpolating scheme* (2.2)*:* $1 \le \rho \le \frac{14+\sqrt{201}}{5}$, $\rho \in \mathbb{R}$,

the limit functions generated by the subdivision schemes (2.1) *and* (2.2) *are strictly monotone decreasing (strictly monotone increasing).*

4. Convexity Preservation

Definition 4.1. Given a set of initial control points $\{P^0_i\}_{i\in\mathbb{Z}}$ with $p^0_i = \left(x^0_i, p^0_i\right)$ are strictly convex, where $\{x^0_i\}_{i\in\mathbb{Z}}$ are equidistant points. For convenience, we make $\Delta x^0_i = x^0_{i+1} - x^0_i = 1$. By the subdivision schemes (2.1) and (2.2), we have $\Delta x^{k+1}_i = x^{k+1}_{i+1} - x^{k+1}_i = \frac{1}{2}\Delta x^k_i = \frac{1}{2^{k+1}}$.

Definition 4.2. Let $d^k_i = 2^k(p^k_{i-1} - 2p^k_i + p^k_{i+1})$ denote the second order divided differences. In the following, we will prove $d^0_i > 0$, for all $k \ge 0$, $k, i \in \mathbb{Z}$. The subdivision scheme (2.1) can thus, be written in terms of second order divided differences as follows:

$$d^{k+1}_{4i} = \frac{1}{64}\left[-21d^k_{i-1} + 98d^k_i - 13d^k_{i+1}\right],$$

$$d^{k+1}_{4i+1} = \frac{1}{64}\left[40d^k_i + 24d^k_{i+1}\right],$$

$$d^{k+1}_{4i+2} = \frac{1}{64}\left[24d^k_i + 40d^k_{i+1}\right],$$

$$d^{k+1}_{4i+3} = \frac{1}{64}\left[-13d^k_i + 98d^k_{i+1} - 21d^k_{i+2}\right].$$

Similarly, the subdivision scheme (2.2) can be written as follows:

$$d_{4i}^{k+1} = \frac{1}{32}\left[-5d_{i-1}^k + 18d_i^k - 5d_{i+1}^k\right],$$

$$d_{4i+1}^{k+1} = \frac{1}{16}\left[3d_i^k + d_{i+1}^k\right],$$

$$d_{4i+2}^{k+1} = \frac{1}{8}\left[d_i^k + d_{i+1}^k\right],$$

$$d_{4i+3}^{k+1} = \frac{1}{16}\left[d_i^k + 3d_{i+1}^k\right].$$

Theorem 3. *Suppose the initial control points* $\{P_i^0\}_{i\in\mathbb{Z}}$, $P_i^0 = (x_i^0, p_i^0)$, *which are strictly convex i.e.,*

$$d_i^0 > 0, \quad \textit{for all } i \in \mathbb{Z},$$

Given that

$$\Gamma^k = \max_i \left\{ r_i^k, \frac{1}{r_i^k} \right\}, \quad \textit{where } r_i^k = \frac{d_{i+1}^k}{d_i^k}, \textit{ for all } k \ge 0,\ k \in \mathbb{Z}.$$

Furthermore, let for,

(i) *Approximating scheme* (2.1): $1 \le \lambda \le \frac{20}{17}$, $\lambda \in \mathbb{R}$.

(ii) *Interpolating scheme* (2.2): $1 \le \lambda \le \frac{9+\sqrt{21}}{6}$, $\lambda \in \mathbb{R}$.

Then for $\frac{1}{\lambda} \le \Gamma^0 \le \lambda$,

$$d_i^0 > 0,\ \frac{1}{\lambda} \ge \Gamma^k < \lambda, \quad \textit{for all } k \ge 0,\ i \in \mathbb{Z},\ k \in \mathbb{Z}. \tag{4.1}$$

Namely, the limit functions generated by the 4*-point quaternary approximating and interpolating schemes defined in* (2.1) *and* (2.2) *preserve convexity.*

Proof. **(i)** To verify Theorem 3, we use mathematical induction on k.

(I) By hypothesis, (4.1) holds true for $k = 0$, $d_i^0 > 0$, $\frac{1}{\lambda} \le \Gamma^0 < \lambda$.

(II) Suppose that (4.1) holds for some $k \ge 1$.

We verify it is also holds for $k+1$. We first show that

$$d_i^k > 0, \quad \text{for all } k \ge 0,\ i \in \mathbb{Z},\ k \in \mathbb{Z}.$$

From the assumption that

$$d_i^k > 0, \quad \text{for all } i \in \mathbb{Z},$$

it follows: for all $i \in \mathbb{Z}$

$$\begin{aligned} d_{4i}^{k+1} &= \frac{1}{64}\left[-21d_{i-1}^k + 98d_i^k - 13d_{i+1}^k\right] \\ &= \frac{d_i^k}{64}\left[-21\frac{d_{i-1}^k}{d_i^k} + 98 - 13\frac{d_{i+1}^k}{d_i^k}\right] \\ &= \frac{d_i^k}{64}\left[-21\frac{1}{r_{i-1}^k} + 98 - 13r_i^k\right] \end{aligned}$$

$$\geq \frac{d_i^k}{64}\left[-21\frac{1}{\lambda}+98-13\lambda\right] \geq 0$$

and

$$\begin{aligned} d_{4i+1}^{k+1} &= \frac{1}{64}\left[40d_i^k + 24d_{i+1}^k\right] \\ &= \frac{d_i^k}{64}\left[40 + 24\frac{d_{i+1}^k}{d_i^k}\right] \\ &= \frac{d_i^k}{64}\left[40 + 24r_i^k\right] \\ &\geq \frac{d_i^k}{64}[40 + 24\lambda] \geq 0. \end{aligned}$$

Similarly, we can show that

$$d_{4i+2}^{k+1} \geq 0 \quad \text{and} \quad d_{4i+2}^{k+1} \geq 0$$

which implies that

$$d_i^{k+1} > 0, \quad \text{for all } i \in \mathbb{Z}.$$

Therefore, by induction, we have

$$d_i^k > 0, \quad \text{for all } k \geq 0,\ i \in \mathbb{Z},\ k \in \mathbb{Z}.$$

(III) To prove $\frac{1}{\lambda} \leq \Gamma^{k+1} < \lambda,\ k \geq 0,\ i \in \mathbb{Z},\ k \in \mathbb{Z}$.

Since:

$$\begin{aligned} r_{4i}^{k+1} &= \frac{d_{4i+1}^{k+1}}{d_{4i}^{k+1}} \\ &= \frac{\frac{1}{64}\left[40d_i^k + 24d_{i+1}^k\right]}{\frac{1}{64}\left[-21d_{i-1}^k + 98d_i^k - 13d_{i+1}^k\right]} \\ &= \frac{\frac{d_i^k}{64}\left[40 + 24\frac{d_{i+1}^k}{d_i^k}\right]}{\frac{d_i^k}{64}\left[-21\frac{d_{i-1}^k}{d_i^k} + 98 - 13\frac{d_{i+1}^k}{d_i^k}\right]} \\ &= \frac{\left[40 + 24r_i^k\right]}{\left[-21\frac{1}{r_{i-1}^k} + 98 - 13r_i^k\right]} \end{aligned}$$

then

$$r_{4i}^{k+1} - \lambda = \frac{40 + 24r_i^k + 21\lambda\frac{1}{r_{i-1}^k} - 98\lambda + 13\lambda r_i^k}{-21\frac{1}{r_{i-1}^k} + 98 - 13r_i^k}.$$

By $d_{4i}^{k+1} \geq 0$, the numerator of the above expression satisfies:

$$\text{Numerator} \leq 40 + 24r_i^k + 21\lambda\frac{1}{r_{i-1}^k} - 98\lambda + 13\lambda r_i^k$$

$$= (24+13\lambda)r_i^k + 40 + 21\lambda \frac{1}{r_{i-1}^k} - 98\lambda$$
$$= (24+13\lambda)\lambda + 40 + 21\lambda^2 - 98\lambda$$
$$= 34\lambda^2 - 74\lambda + 40$$
$$= 2(\lambda-1)\left(10\lambda - \frac{20}{17}\right) \le 0.$$

Therefore,

$$r_{4i}^{k+1} \le \lambda.$$

By the similar way, we get

$$r_{4i+1}^{k+1} \le \lambda,\ r_{4i+2}^{k+1} \le \lambda,\ r_{4i+3}^{k+1} \le \lambda.$$

Since

$$\frac{1}{r_{4i}^{k+1}} = \frac{d_{4i}^{k+1}}{d_{4i+1}^{k+1}}$$
$$= \frac{\frac{1}{64}\left[-21d_{i-1}^k + 98d_i^k - 13d_{i+1}^k\right]}{\frac{1}{64}\left[40d_i^k + 24d_{i+1}^k\right]}$$
$$= \frac{\frac{d_i^k}{64}\left[-21\frac{d_{i-1}^k}{d_i^k} + 98 - 13\frac{d_{i+1}^k}{d_i^k}\right]}{\frac{d_i^k}{64}\left[40 + 24\frac{d_{i+1}^k}{d_i^k}\right]}$$
$$= \frac{\left[-21\frac{1}{r_{i-1}^k} + 98 - 13r_i^k\right]}{\left[40 + 24r_i^k\right]}$$

then

$$\frac{1}{r_{4i}^{k+1}} - \lambda = \frac{-21\frac{1}{r_{i-1}^k} + 98 - 13r_i^k - 40\lambda - 24\lambda r_i^k}{40 + 24r_i^k}.$$

As denominator $d_{4i+1}^{k+1} \ge 0$, the numerator of the above expression satisfies:

$$\text{Numerator} \le -21\frac{1}{r_{i-1}^k} + 98 - 13r_i^k - 40\lambda - 24\lambda r_i^k$$
$$= (-13-24\lambda)r_i^k + 98 - 21\frac{1}{r_{i-1}^k} - 40\lambda$$
$$= -24\lambda^2 - 74\lambda + 98$$
$$= -2(\lambda-1)\left(\lambda + \frac{49}{12}\right) \le 0$$

and similarly,

$$\frac{1}{r_{4i+1}^{k+1}} \le \lambda,\ \frac{1}{r_{4i+2}^{k+1}} \le \lambda,\ \frac{1}{r_{4i+3}^{k+1}} \le \lambda$$

which implies

$$\frac{1}{\lambda} \le \Gamma^{k+1} < \lambda.$$

Therefore, by mathematical induction, we have

$$\frac{1}{\lambda} \le \Gamma^{k} < \lambda, \quad \text{for all } k \ge 0,\ k \in \mathbb{Z}.$$

This completes the proof.

(ii) To verify Theorem 3, we use mathematical induction on k.

(I) By hypothesis, (4.1) holds true when $k = 0$, $d_i^0 > 0$, $\frac{1}{\lambda} \le \Gamma^0 < \lambda$.

(II) Suppose that (4.1) holds for some $k \ge 1$.

We verify it is also holds for $k+1$. We first show that

$$d_i^k > 0, \quad \text{for all } k \ge 0,\ k, i \in \mathbb{Z}.$$

From the assumption that

$$d_i^k > 0, \quad \text{for all } i \in \mathbb{Z},$$

it follows: for all $i \in \mathbb{Z}$

$$\begin{aligned} d_{4i}^{k+1} &= \frac{1}{32}\left[-5d_{i-1}^k + 18d_i^k - 5d_{i+1}^k\right] \\ &= \frac{d_i^k}{32}\left[-5\frac{d_{i-1}^k}{d_i^k} + 18 - 5\frac{d_{i+1}^k}{d_i^k}\right] \\ &= \frac{d_i^k}{32}\left[-5\frac{1}{r_{i-1}^k} + 18 - 5r_i^k\right] \\ &\ge \frac{d_i^k}{32}\left[-5\frac{1}{\lambda} + 18 - 5\lambda\right] \ge 0 \end{aligned}$$

and

$$\begin{aligned} d_{4i+1}^{k+1} &= \frac{1}{16}\left[3d_i^k + d_{i+1}^k\right] \\ &= \frac{d_i^k}{16}\left[3 + \frac{d_{i+1}^k}{d_i^k}\right] \\ &= \frac{d_i^k}{16}\left[3 + r_i^k\right] \\ &\ge \frac{d_i^k}{16}[3+\lambda] \ge 0. \end{aligned}$$

Similarly, we can show that

$$d_{4i+2}^{k+1} \ge 0 \quad \text{and} \quad d_{4i+2}^{k+1} \ge 0$$

which implies that

$$d_i^{k+1} > 0, \quad \text{for all } i \in \mathbb{Z}.$$

Therefore, by induction, we have

$$d_i^k > 0, \quad \text{for all } k \geq 0,\ i \in \mathbb{Z},\ k \in \mathbb{Z}.$$

(III) To prove $\frac{1}{\lambda} \leq \Gamma^{k+1} < \lambda$, $i \in \mathbb{Z}$, $k \in \mathbb{Z}$.

Since:

$$\begin{aligned}
r_{4i}^{k+1} &= \frac{d_{4i+1}^{k+1}}{d_{4i}^{k+1}} \\
&= \frac{\frac{1}{16}\left[3d_i^k + d_{i+1}^k\right]}{\frac{1}{32}\left[-5d_{i-1}^k + 18d_i^k - 5d_{i+1}^k\right]} \\
&= \frac{\frac{d_i^k}{16}\left[3 + \frac{d_{i+1}^k}{d_i^k}\right]}{\frac{d_i^k}{32}\left[-5\frac{d_{i-1}^k}{d_i^k} + 18 - 5\frac{d_{i+1}^k}{d_i^k}\right]} \\
&= \frac{2\left[3 + r_i^k\right]}{\left[-5\frac{1}{r_{i-1}^k} + 18 - 5r_i^k\right]}
\end{aligned}$$

then

$$r_{4i}^{k+1} - \lambda = \frac{6 + 2r_i^k + 5\lambda\frac{1}{r_{i-1}^k} - 18\lambda + 5\lambda r_i^k}{-5\frac{1}{r_{i-1}^k} + 18 - 5r_i^k}.$$

By $d_{4i}^{k+1} \geq 0$, the numerator of the above expression satisfies:

$$\begin{aligned}
\text{Numerator} &\leq 6 + 2r_i^k + 5\lambda\frac{1}{r_{i-1}^k} - 18\lambda + 5\lambda r_i^k \\
&= (2+5\lambda)r_i^k + 6 + 5\lambda\frac{1}{r_{i-1}^k} - 18\lambda \\
&= (2+5\lambda)\lambda + 6 + 5\lambda^2 - 18\lambda \\
&= 10\lambda^2 - 16\lambda + 6 = (\lambda-1)(10\lambda-6) \leq 0.
\end{aligned}$$

Therefore,

$$r_{4i}^{k+1} \leq \lambda.$$

By the similar way, we get

$$r_{4i+1}^{k+1} \leq \lambda,\ r_{4i+2}^{k+1} \leq \lambda,\ r_{4i+3}^{k+1} \leq \lambda.$$

Since

$$\begin{aligned}
\frac{1}{r_{4i}^{k+1}} &= \frac{d_{4i}^{k+1}}{d_{4i+1}^{k+1}} \\
&= \frac{\frac{1}{32}\left[-5d_{i-1}^k + 18d_i^k - 5d_{i+1}^{k+1}\right]}{\frac{1}{16}\left[3d_i^k + d_{i+1}^k\right]}
\end{aligned}$$

$$= \frac{\frac{d_i^k}{32}\left[-5\frac{d_{i-1}^k}{d_i^k}+18-5\frac{d_{i+1}^k}{d_i^k}\right]}{\frac{d_i^k}{16}\left[3+\frac{d_{i+1}^k}{d_i^k}\right]}$$

$$= \frac{\left[-5\frac{1}{r_{i-1}^k}+18-5r_i^k\right]}{\left[6+2r_i^k\right]}$$

then,

$$\frac{1}{r_{4i}^{k+1}}-\lambda = \frac{-5\frac{1}{r_{i-1}^k}+18-5r_i^k-6\lambda-2\lambda+5\lambda r_i^k}{6+2r_i^k}.$$

By $d_{4i+1}^{k+1} \geq 0$, the numerator of the above expression satisfies:

$$\begin{aligned}\text{Numerator} &\leq -5\frac{1}{r_{i-1}^k}+18-5r_i^k-6\lambda-2\lambda+5\lambda r_i^k\\ &= (-5-2\lambda)r_i^k+18-5\frac{1}{r_{i-1}^k}-6\lambda\\ &= (-5-2\lambda)\lambda+18-5\lambda-6\lambda\\ &= -2\lambda^2-16\lambda+18\\ &= -2(\lambda-1)(\lambda+9)\leq 0.\end{aligned}$$

Therefore,

$$\frac{1}{r_{4i}^{k+1}} \leq \lambda.$$

Similarly,

$$\frac{1}{r_{4i}^{k+1}} \leq \lambda,\ \frac{1}{r_{4i+1}^{k+1}} \leq \lambda,\ \frac{1}{r_{4i+2}^{k+1}} \leq \lambda,\ \frac{1}{r_{4i+3}^{k+1}} \leq \lambda$$

which implies

$$\frac{1}{\lambda} \leq \Gamma^{k+1} < \lambda.$$

Therefore, by mathematical induction, we have

$$\frac{1}{\lambda} \leq \Gamma^k < \lambda, \quad \text{for all } k \geq 0,\ k \in \mathbb{Z}.$$

This completes the proof. □

5. Numerical Examples

In order to examine and certify the efficiency of convexity preserving conditions that are proposed for the schemes (2.1) and (2.2), the following three numerical examples are executed.

Example 5.1. For examining the applications of the scheme (2.1) after three iterations is well demonstrated in Figure 5.1. In the figure, the initial control polygons are shown by the dash-doted lines, and the limit curves are shown by the red solid curves.

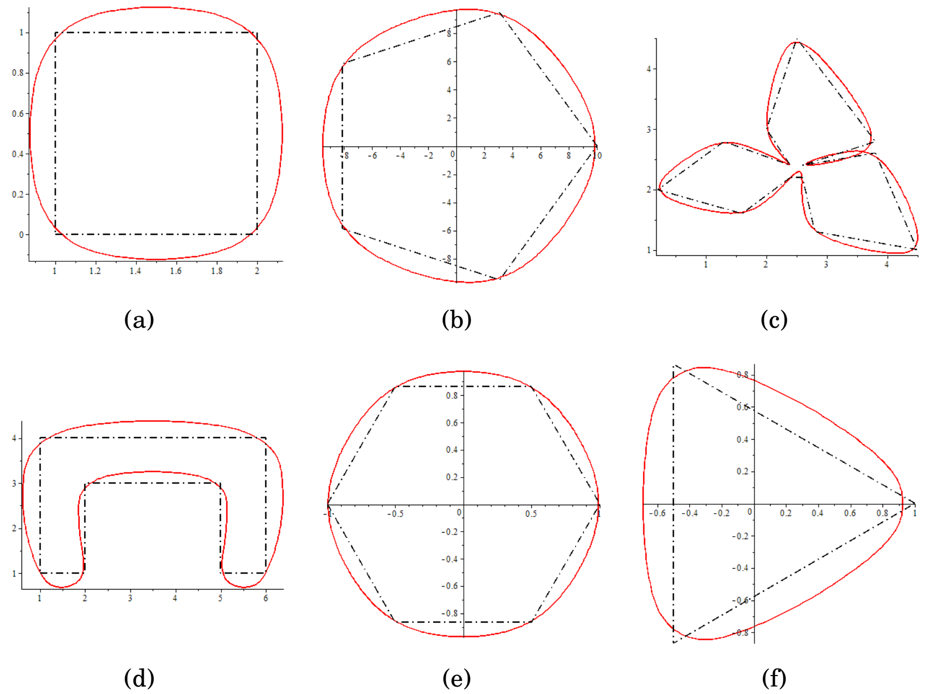


Figure 5.1. Behavior of convexity preserving C^2 limit curves (red full line) and initial polygon (black dash-dotted line) generated by subdivision scheme (2.1).

Example 5.2. For examining the applications of the scheme (2.2) after three iterations is well demonstrated in Figure 5.2. In the figure, the initial control polygons are shown by the doted lines, and the limit curves are shown by the green solid curves.

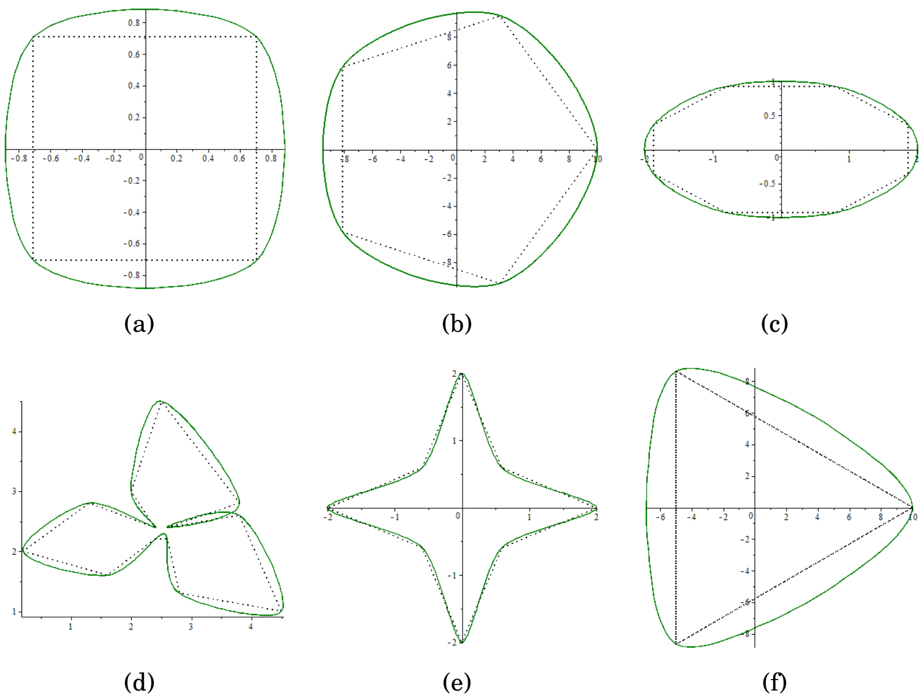
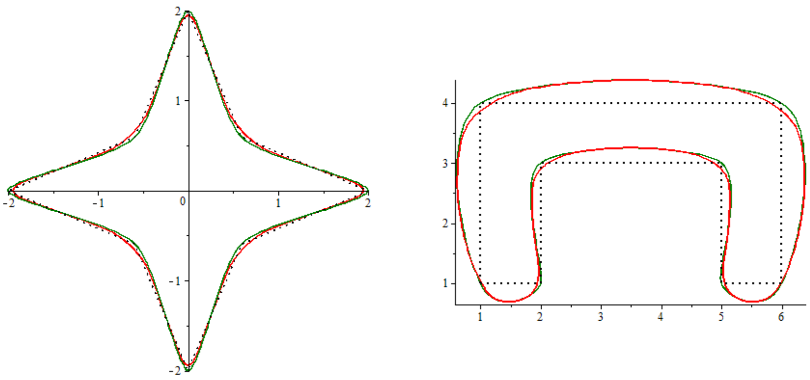


Figure 5.2. Behavior of convexity preserving C^2 limit curves (green full line) and initial polygon (black dotted line) generated by subdivision scheme (2.2).

Example 5.3. Comparison of schemes (2.1) and (2.2) after three iterations is well demonstrated in Figure 5.3. In the figure, the initial control polygons are shown by the doted lines, and the limit curves generated by approximating scheme (2.1) are shown by red full line and limit curves for interpolating scheme (2.2) are shown by the green solid curves.



(a) (b)

Figure 5.3. Comparison of schemes (2.1) and (2.2), doted line shows initial polygon, red solid line shows limit curve generated by scheme (2.1) and green full line generated by scheme (2.2).

6. Conclusion

In CAGD, shape preservation of curves and surfaces is essential tool for modeling curves. Monotonicity and convexity preservation are two major elements in shape preserving. In this paper, a stationary 4-point quaternary approximating and interpolating subdivision schemes are discussed. Successively, the conditions of the initial data guaranteeing monotonicity preservation and convexity preservation are derived. For this purpose it is proved that if initial data is strictly convex and satisfy $\Gamma^0 \in \left(1, \frac{20}{17}\right)$ and $\Gamma^0 \in \left(1, \frac{9+\sqrt{21}}{6}\right)$ then the limit curve generated by the subdivision schemes (2.1) and (2.2) respectively, are also convex.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

References

[1] G. Akram, K. Bibi, K. Rehan and S.S. Siddiqi, Shape preservation of 4-point interpolating non-stationary subdivision scheme, *Journal of Computational and Applied Mathematics* **319** (2017), 480 – 492, DOI: 10.1016/j.cam.2017.01.026.

[2] S. Amat, R. Donat and J.C. Trillo, Proving convexity preserving properties of interpolatory subdivision schemes through reconstruction operators, *Applied Mathematics and Computation* **219** (2013), 7413 – 7421, DOI: 10.1016/j.amc.2013.01.024.

[3] P. Ashraf and G. Mustafa, A generalized non-stationary 4-point b-ary approximating scheme, *British Journal of Mathematics & Computer Science* **4** (2014), 104, DOI: 10.9734/bjmcs/2014/4120.

[4] C. Beccari, G. Casciola and L. Romani, An interpolating 4-point C^2 ternary non-stationary subdivision scheme with tension control, *Computer Aided Geometric Design* **24** (2007), 210 – 219, DOI: 10.1016/j.cagd.2007.02.001.

[5] Z. Cai, Convexity preservation of the interpolating four-point C^2 ternary stationary subdivision scheme, *Computer Aided Geometric Design* **26** (2009), 560 – 565, DOI: 10.1016/j.cagd.2009.02.004.

[6] F. Dyn, F. Kuijt, D. Levin and R. van Damme, Convexity preservation of the four-point interpolatory subdivision scheme, *Computer Aided Geometric Design* **16** (1999), 789 – 792, DOI: 10.1016/s0167-8396(99)00019-9.

[7] N. Dyn, D. Levin and J.A. Gregory, A 4-point interpolatory subdivision scheme for curve design, Computer Aided Geometric Design **4** (1987), 257 – 268, DOI: 10.1016/0167-8396(87)90001-x.

[8] M.F. Hassan, I. Ivrissimitzis, N.A. Dodgson and M.A. Sabin, An interpolating 4-point C^2 ternary stationary subdivision scheme, *Computer Aided Geometric Design* **19** (2002), 1 – 18, DOI: 10.1016/s0167-8396(01)00084-x.

[9] K.-P. Ko, Quatnary approximating 4-point subdivision scheme, *Journal of the Korea Society for Industrial and Applied Mathematics* **13** (2009), 307 – 314, `https://portal.koreascience.or.kr/article/articleresultdetail.jsp?no=E1TAAE_2009_v13n4_307`.

[10] G. Mustafa and P. Ashraf, A new 6-point ternary interpolating subdivision scheme, *Journal of Information and Computing Science* **5** (2010), 199 – 210.

[11] S.S. Siddiqi and T. Noreen, Convexity preservation of six point C^2 interpolating subdivision scheme, *Applied Mathematics and Computation* **265** (2015), 936 – 944, DOI: 10.1016/j.amc.2015.04.024.

[12] J. Tan, B. Wang and J. Shi, A five-point subdivision scheme with two parameters and a four-point shape-preserving scheme, *Mathematical and Computational Applications* **22** (2017), 22, DOI: 10.3390/mca22010022.

[13] J. Tan, Y. Yao, H. Cao and L. Zhang, Convexity preservation of five-point binary subdivision scheme with a parameter, *Applied Mathematics and Computation* **245** (2014), 279 – 288, DOI: 10.1016/j.amc.2014.07.071.

[14] C. Zhijie, Four-point scheme and convex-preserving algorithm, *Journal of Computer Aided Design & Computer Graphics* **6**(1) (1994), 33 – 36 (in Chinese), `http://en.cnki.com.cn/Article_en/CJFDTOTAL-JSJF401.005.htm`.