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Research Article

Generalized Szász-Kantorovich Type Operators

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Abstract. In this note, we present Kantorovich modification of the operators introduced by V. Miheşan [*Creative Math. Inf.* **17** (2008), 466 – 472]. First, we derive some indispensable auxiliary results in the second section. We present a quantitative Voronovskaja type theorem, local approximation theorem by means of second order modulus of continuity and weighted approximation for these operators. Furthermore, we show the rate of convergence of these operators to certain functions with the help of the illustrations using Maple algorithms.

Keywords. Positive approximation process; Rate of convergence; Modulus of continuity; Steklov mean

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1. Introduction

In 1950, Szász [29] constructed the following linear positive operators

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \tag{1.1}$$

where $x \in [0,\infty)$ and $f(x)$ is a continuous function on $[0,\infty)$ whenever the above sum converges uniformly.

The Szász-Mirakyan operators and their modifications have been intensively studied in recent years. Sucu [28] introduced linear positive operators generated by Dunkl generalization of exponential function and obtained some approximation results of these operators. In 2014, Aral et al. [7] defined a generalization of Szász-Mirakyan operators and gave the quantitative type theorems in order to obtain the degree of weighted convergence by using the weighted modulus of continuity. Agrwal et al. [4] presented a Stancu type Kantorovich modification of q-Bernstein-Schurer operators and established a convergence theorem by using the well known Bohman-Korovkin criterion and find the estimate of the rate of convergence by means of modulus of continuity and Lipschitz function for these operators. Atakut and Büyükyazici [8] introduced Kantorovich-Szász type operators involving Brenke polynomials and studied convergence properties of these operators by using Bohman-Korovkin's theorem. Very recently, Acar et al. [1] introduced a new general class of operators which have the classical Szász-Mirakyan ones as a basis, and fix the functions e^{ax} and e^{2ax} with $a > 0$ and obtained Voronovskaja type theorem of these operators. Several authors also introduced different types of generalizations of these operators and studied their approximation properties, we refer the reader to e.g. (cf. [2,3,5,6,9,13–15,17,19,20,24,25,27,30]).

Miheşan [23] presented an important generalization of the well-known Szász operators depending on $\alpha \in \mathbb{R}$ as

$$\mathscr{G}_n^{(\alpha)}(f;x) = \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right), \quad x \in [0,\infty), \tag{1.2}$$

where $\alpha + nx > 0$, $m_{n,k}^{(\alpha)}(x) = \frac{(\alpha)_k}{k!}\frac{(\frac{nx}{\alpha})^k}{(1+\frac{nx}{\alpha})^{\alpha+k}}$ and $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1), (\alpha)_0 = 1$, is the rising factorial. The operators $\mathscr{G}_n^{(\alpha)}$ preserve the linear functions and they reduce to the following well-known operators in special cases:

(i) If $\alpha = -n$, we get the Bernstein operators [11].

(ii) If $\alpha = n$, we obtain Baskakov operators [10].

(iii) If $\alpha = nx$, $x > 0$, we get the Lupaş operators [22].

(iv) If $\alpha \to \infty$, we obtain Szász-Mirakjan operators [29].

For $C_\gamma[0,\infty) := \{f \in C[0,\infty) : |f(t)| \le Me^{\gamma t}$ for some $\gamma > 0$, $M > 0$ and $t \in [0,\infty)\}$, we propose the following Kantorovich type modification of the operators (1.2) as:

$$\mathscr{K}_n^{(\alpha)}(f;x) = (n+1)\sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt. \tag{1.3}$$

In this paper, we prove the basic convergence theorem for the operators (1.3) by using Bohman-Korovkin criterion. We also find estimates of the rate of convergence involving modulus of continuity and Lipschitz function. Furthermore, we study weighted approximation and pointwise convergence of the operators in terms of quantitative Voronovskaja type theorem. Lastly, we show the rate of convergence of these operators to certain functions with the help of the illustrations using Maple algorithms.

This paper is organized as follows: In section 2, we prove certain results for generalized Szász-Kantorovich type operators which will be useful in deriving the main results of this paper. In section 3, we obtain local approximation and degree of approximation. In section 4, we derive the rate of convergence of these operators to certain functions with the help of the illustrations using Maple algorithms.

2. Auxiliary Results

In this section we prove certain results, which are necessary to derive the main results.

Let $e_i(t) = t^i$, $i = \overline{0,6}$.

Lemma 1. *For the operators $\mathscr{K}_n^{(\alpha)}(f;x)$, we have*

1. $\mathscr{K}_n^{(\alpha)}(e_0;x) = 1$,
2. $\mathscr{K}_n^{(\alpha)}(e_1;x) = \dfrac{nx}{n+1} + \dfrac{1}{2(n+1)}$,
3. $\mathscr{K}_n^{(\alpha)}(e_2;x) = \dfrac{n^2x^2(1+\alpha)}{\alpha(n+1)^2} + \dfrac{2nx}{(n+1)^2} + \dfrac{1}{3(n+1)^2}$,
4. $\mathscr{K}_n^{(\alpha)}(e_3;x) = \dfrac{n^3x^3(1+\alpha)(2+\alpha)}{\alpha^2(n+1)^3} + \dfrac{9n^2x^2(1+\alpha)}{2n\alpha(n+1)^3} + \dfrac{7nx}{2(n+1)^3} + \dfrac{1}{4(n+1)^3}$,
5. $\mathscr{K}_n^{(\alpha)}(e_4;x) = \dfrac{n^4x^4(1+\alpha)(2+\alpha)(3+\alpha)}{(n+1)^4\alpha^3} + \dfrac{8n^3x^3(1+\alpha)(2+\alpha)}{(n+1)^4\alpha^2} + \dfrac{15n^2x^2(1+\alpha)}{(n+1)^4\alpha} + \dfrac{6nx}{(n+1)^4} + \dfrac{1}{5(n+1)^4}$,
6. $\mathscr{K}_n^{(\alpha)}(e_5;x) = \dfrac{n^5x^5(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)}{\alpha^4(n+1)^5} + \dfrac{25n^4x^4(1+\alpha)(2+\alpha)(3+\alpha)}{2(n+1)^5\alpha^3} + \dfrac{130n^3x^3(1+\alpha)(2+\alpha)}{3(n+1)^5\alpha^2} + \dfrac{45n^2x^2(1+\alpha)}{(n+1)^5\alpha} + \dfrac{31nx}{3(n+1)^5} + \dfrac{1}{5(n+1)^4}$,
7. $\mathscr{K}_n^{(\alpha)}(e_6;x) = \dfrac{n^6x^6(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)(5+\alpha)}{(n+1)^6\alpha^5} + \dfrac{18n^5x^5(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)}{(n+1)^6\alpha^4} + \dfrac{100n^4x^4(1+\alpha)(2+\alpha)(3+\alpha)}{(n+1)^6\alpha^3} + \dfrac{200n^3x^3(1+\alpha)(2+\alpha)}{(n+1)^6\alpha^2} + \dfrac{129n^2x^2(1+\alpha)}{(n+1)^6\alpha} + \dfrac{18nx}{(n+1)^6} + \dfrac{1}{7(n+1)^6}$.

Proof. This lemma follows easily applying ([23], Lemma 4.1) and the definition of the generalized Szász-Kantorovich operators (1.3). Hence the details are omitted. □

Lemma 2. *For $m = 1,2,4,6$, the m^{th} order central moments of $\mathscr{K}_n^{(\alpha)}$ defined as $\vartheta_{n,\alpha,m}(x) = \mathscr{K}_n^{(\alpha)}((t-x)^m;x)$, we have*

1. $\vartheta_{n,\alpha,1}(x) = \dfrac{1}{2(n+1)} - \dfrac{x}{(n+1)}$;
2. $\vartheta_{n,\alpha,2}(x) = \dfrac{x^2(n^2+\alpha)}{(n+1)^2\alpha} + \dfrac{x(n-1)}{(n+1)^2} + \dfrac{1}{3(n+1)^2}$;

3. $$\vartheta_{n,\alpha,4}(x)=\frac{x^4\left(3n^4(2+\alpha)+6n^2\alpha^2-8n^3\alpha+\alpha^3\right)}{(n+1)^4\alpha^3}+\frac{x^3\left(16n^3+6(n-3)n^2\alpha+2(3n-1)\alpha^2\right)}{(n+1)^4\alpha^2}+\frac{x^2\left(3n^2(\alpha+5)-10n\alpha+2\alpha\right)}{(n+1)^4\alpha}+\frac{x(5n-1)}{(n+1)^4}+\frac{1}{5(n+1)^4};$$
4. $$\vartheta_{n,\alpha,6}(x)=\frac{x^6\left(5n^6(24+\alpha(3\alpha+26))-24n^5\alpha(5\alpha+6)+45n^4\alpha^2(\alpha+2)+15n^2\alpha^4-40n^3\alpha^3\right)}{(n+1)^6\alpha^5}+\frac{\left(\begin{array}{c}x^5(432n^5+450(n-1)n^4\alpha+15n^3(16+n(3n-23))\alpha^2\\+90(n-1)n^2\alpha^3+3(5n-1)\alpha^4)\end{array}\right)}{(n+1)^6\alpha^4}+\frac{5x^4\left(n^4(120+\alpha(116+9\alpha))-2n^3\alpha(52+33\alpha)+9n^2\alpha^2(\alpha+5)-10n\alpha^3+\alpha^3\right)}{(n+1)^6\alpha^3}+\frac{5x^3\left(80n^3+6n^2(11n-9)\alpha+(-1+3n(5+(n-7)n))\alpha^2\right)}{(n+1)^6\alpha^3}+\frac{x^2\left(3\alpha-56n\alpha+n^2(70\alpha+129)\right)}{(n+1)^6\alpha}+\frac{x(17n-1)}{(n+1)^6}+\frac{1}{7(n+1)^6}.$$

Remark 1. If $\alpha=\alpha(n)\to\infty$, as $n\to\infty$ and $\lim\limits_{n\to\infty}\frac{n}{\alpha(n)}=l\in\mathbb{R}$, then

$$\lim_{n\to\infty} n\vartheta_{n,\alpha,1}(x)=\frac{(1-2x)}{2};$$
$$\lim_{n\to\infty} n\vartheta_{n,\alpha,2}(x)=x(lx+1);$$
$$\lim_{n\to\infty} n^2\vartheta_{n,\alpha,4}(x)=3x^2(1+lx)^2;$$
$$\lim_{n\to\infty} n^3\vartheta_{n,\alpha,6}(x)=15x^3(1+lx)^3.$$

3. Direct Results

Theorem 1. *Let $f\in C_\gamma[0,\infty)$ and $\alpha=\alpha(n)\to\infty$ as $n\to\infty$. Then $\lim\limits_{n\to\infty}\mathscr{K}_n^{(\alpha)}(f;x)=f(x)$, uniformly in each compact subset of $[0,\infty)$.*

Proof. From Lemma 1, $\mathscr{K}_n^{(\alpha)}(e_0;x)=1$, $\mathscr{K}_n^{(\alpha)}(e_1;x)\to x$, $\mathscr{K}_n^{(\alpha)}(e_2;x)\to x^2$, as $n\to\infty$ uniformly in each compact subset of $[0,\infty)$. By Bohman-Korovkin Theorem, it follows that $\mathscr{K}_n^{(\alpha)}(f;x)\to f(x)$, as $n\to\infty$ uniformly in each compact subset of $[0,\infty)$. □

3.1 Local approximation

Let $\widetilde{C}_B[0,\infty)$ be the space of all real valued bounded and uniformly continuous functions f on $[0,\infty)$, endowed with the norm

$$\|f\|_{\widetilde{C}_B[0,\infty)}=\sup_{x\in[0,\infty)}|f(x)|.$$

For $f\in\widetilde{C}_B[0,\infty)$, the Steklov mean is defined as

$$f_h(x)=\frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}[2f(x+u+v)-f(x+2(u+v))]\,du\,dv. \tag{3.1}$$

By simple computation, it is observed that

(a) $\|f_h - f\|_{\widetilde{C}_B[0,\infty)} \le \omega_2(f,h)$.

(b) $f_h', f_h'' \in \widetilde{C}_B[0,\infty)$ and $\|f_h'\|_{\widetilde{C}_B[0,\infty)} \le \frac{5}{h}\omega(f,h)$, $\|f_h''\|_{\widetilde{C}_B[0,\infty)} \le \frac{9}{h^2}\omega_2(f,h)$,

where the second order modulus of continuity is defined as

$$\omega_2(f,\delta) = \sup_{x,u,v\ge 0}\ \sup_{|u-v|\le\delta} |f(x+2u)-2f(x+u+v)+f(x+2v)|, \quad \delta \ge 0.$$

The usual modulus of continuity of $f \in \widetilde{C}_B[0,\infty)$ is given by

$$\omega(f,\delta) = \sup_{x,u,v\ge 0}\ \sup_{|u-v|\le\delta} |f(x+u)-f(x+v)|.$$

Theorem 2. *Let $f \in \widetilde{C}_B[0,\infty)$. Then for every $x \ge 0$, the following inequality holds*

$$\left|\mathcal{K}_n^{(\alpha)}(f;x) - f(x)\right| \le 5\omega\left(f, \sqrt{\vartheta_{n,\alpha,2}(x)}\right) + \frac{13}{2}\omega_2\left(f, \sqrt{\vartheta_{n,\alpha,2}(x)}\right).$$

Proof. For $x \ge 0$, and applying the Steklov mean f_h that is given by (3.1), we can write

$$\left|\mathcal{K}_n^{(\alpha)}(f;x)-f(x)\right| \le \mathcal{K}_n^{(\alpha)}(|f-f_h|;x) + |\mathcal{K}_n^{(\alpha)}(f_h - f_h(x);x)| + |f_h(x)-f(x)|. \tag{3.2}$$

From (1.3), for every $f \in \tilde{C}_B[0,\infty)$, we have

$$\left|\mathcal{K}_n^{(\alpha)}(f;x)\right| \le \|f\|_{\widetilde{C}_B[0,\infty)}. \tag{3.3}$$

Using property (a) of Steklov mean and (3.3), we get

$$\mathcal{K}_n^{(\alpha)}(|f-f_h|;x) \le \|\mathcal{K}_n^{(\alpha)}(f-f_h)\|_{\widetilde{C}_B[0,\infty)} \le \|f-f_h\|_{\widetilde{C}_B[0,\infty)} \le \omega_2(f,h).$$

By Taylor's expansion and Cauchy-Schwarz inequality, we have

$$\left|\mathcal{K}_n^{(\alpha)}(f_h - f_h(x);x)\right| \le \|f_h'\|_{\widetilde{C}_B[0,\infty)} \sqrt{\mathcal{K}_n^{(\alpha)}\left((t-x)^2;x\right)} + \frac{1}{2}\|f_h''\|_{\widetilde{C}_B[0,\infty)} \mathcal{K}_n^{(\alpha)}\left((t-x)^2;x\right).$$

By Lemma 2 and property (b) of Steklov mean, we obtain

$$\left|\mathcal{K}_n^{(\alpha)}(f_h - f_h(x);x)\right| \le \frac{5}{h}\omega(f,h)\sqrt{\vartheta_{n,\alpha,2}(x)} + \frac{9}{2h^2}\omega_2(f,h)\vartheta_{n,\alpha,2}(x).$$

Choosing $h = \sqrt{\vartheta_{n,\alpha,2}(x)}$, and substituting the values of the above estimates in (3.2), we get the desired relation. □

3.2 Degree of Approximation

Let $c_1 \ge 0$, $c_2 > 0$ be fixed. We consider the following Lipschitz-type space (see [26]):

$$Lip_M^{(c_1,c_2)}(\eta) := \left\{ f \in \tilde{C}_B[0,\infty) : |f(t)-f(x)| \le M\frac{|t-x|^\eta}{(t+c_1x^2+c_2x)^{\frac{\eta}{2}}}; x,t \in (0,\infty)\right\},$$

where M is a positive constant and $0 < s \le 1$.

Theorem 3. *Let $f \in Lip_M^{(c_1,c_2)}(\eta)$ and $s \in (0,1]$. Then, for all $x \in (0,\infty)$, we have*

$$|\mathcal{K}_n^{(\alpha)}(f;x) - f(x)| \le M\left(\frac{\vartheta_{n,\alpha,2}(x)}{c_1x^2+c_2x}\right)^{\frac{s}{2}}.$$

Proof. By Hölder's inequality with $p = \frac{2}{s}$, $q = \frac{2}{2-s}$, we have

$$\begin{aligned}\mathcal{K}_n^{(\alpha)}(f;x) &= (n+1)\sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)-f(x)|dt \\ &\le \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)-f(x)|^{\frac{2}{s}}dt\right)^{\frac{s}{2}}\end{aligned}$$

$$\le \left\{(n+1)\sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)-f(x)|^{\frac{2}{s}}dt\right\}^{\frac{s}{2}} \left(\sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\right)^{\frac{2-s}{2}}$$
$$= \left\{(n+1)\sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)-f(x)|^{\frac{2}{s}}dt\right\}^{\frac{s}{2}}$$
$$\le M\left((n+1)\sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \frac{(t-x)^2}{(t+c_1x^2+c_2x)}dt\right)^{\frac{s}{2}}$$
$$\le \frac{M}{(c_1x^2+c_2x)^{\frac{s}{2}}}\left((n+1)\sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)^2dt\right)^{\frac{s}{2}}$$
$$= \frac{M}{(c_1x^2+c_2x)^{\frac{s}{2}}}\left(\mathscr{K}_n^{(\alpha)}((t-x)^2;x)\right)^{\frac{s}{2}}$$
$$= \frac{M}{(c_1x^2+c_2x)^{\frac{s}{2}}}(\vartheta_{n,\alpha,2}^{c}(x))^{\frac{s}{2}}.$$

Thus the theorem is proved. □

Theorem 4. *For any $f \in \widetilde{C}_B^1[0,\infty)$ and $x \in [0,\infty)$, we have*

$$\left|\mathscr{K}_n^{(\alpha)}(f;x)-f(x)\right| \le |f'(x)|\left|\vartheta_{n,\alpha,1}(x)\right| + 2\sqrt{\vartheta_{n,\alpha,2}(x)}\ \omega\left(f',\sqrt{\vartheta_{n,\alpha,2}(x)}\right). \tag{3.4}$$

Proof. Let $f \in \widetilde{C}_B^1[0,\infty)$. For any $t,x \in [0,\infty)$, we have

$$f(t)-f(x) = f'(x)(t-x)+\int_x^t \left(f'(u)-f'(x)\right)du.$$

Applying $\mathscr{K}_n^{(\alpha)}(\cdot;x)$ on both sides of the above relation, we get

$$\mathscr{K}_n^{(\alpha)}(f(t)-f(x);x) = f'(x)\mathscr{K}_n^{(\alpha)}(t-x;x)+\mathscr{K}_n^{(\alpha)}\left(\int_x^t \left(f'(u)-f'(x)\right)du;x\right).$$

Using the well known property of modulus of continuity

$$|f(t)-f(x)| \le \omega(f,\delta)\left(\frac{|t-x|}{\delta}+1\right), \quad \delta>0,$$

we obtain

$$\left|\int_x^t \left(f'(u)-f'(x)\right)du\right| \le \omega(f',\delta)\left(\frac{(t-x)^2}{\delta}+|t-x|\right).$$

Therefore, it follows

$$\left|\mathscr{K}_n^{(\alpha)}(f;x)-f(x)\right| \le |f'(x)|\ |\mathscr{K}_n^{(\alpha)}(t-x;x)|+\omega(f',\delta)\left\{\frac{1}{\delta}\mathscr{K}_n^{(\alpha)}((t-x)^2;x)+\mathscr{K}_n^{(\alpha)}(|t-x|;x)\right\}.$$

Using Cauchy-Schwarz inequality, we have

$$\left|\mathscr{K}_n^{(\alpha)}(f;x)-f(x)\right| \le |f'(x)|\ |\mathscr{K}_n^{(\alpha)}(t-x;x)|+\omega(f',\delta)\left\{\frac{1}{\delta}\sqrt{\mathscr{K}_n^{(\alpha)}((t-x)^2;x)}+1\right\}\sqrt{\mathscr{K}_n^{(\alpha)}((t-x)^2;x)}.$$

Choosing $\delta = \sqrt{\vartheta_{n,\alpha,2}(x)}$, the required result follows. □

Let $H_\rho[0,\infty)$ be the space of all real valued functions on $[0,\infty)$ satisfying the condition $|f(x)| \le N_f\rho(x)$, where N_f is a positive constant depending only on f and $\rho(x)=1+x^2$ is a weight function. Let $C_\rho[0,\infty)$ be the space of all continuous functions in $H_\rho[0,\infty)$ endowed with the norm

$$\|f\|_\rho := \sup_{x\in[0,\infty)} \frac{|f(x)|}{\rho(x)}$$

and

$$C_\rho^0[0,\infty) := \left\{ f \in C_\rho[0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{\rho(x)} \text{ exists and is finite} \right\}.$$

The usual modulus of continuity of f on $[0,b]$ is defined as

$$\omega_a(f,\delta) = \sup_{0<|t-x|\le\delta}\ \sup_{x,t\in[0,a]} |f(t)-f(x)|.$$

Theorem 5. *Let $f \in C_\rho[0,\infty)$. Then, we have*

$$|\mathscr{K}_n^{(\alpha)}(f;x) - f(x)| \le 4N_f(1+x^2)\vartheta_{n,\alpha,2}(x) + 2\omega_{a+1}\left(f, \sqrt{\vartheta_{n,\alpha,2}(x)}\right). \tag{3.5}$$

Proof. From [18], for $x\in[0,a]$ and $t\ge 0$, we have

$$|f(t)-f(x)| \le 4N_f(1+x^2)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta), \quad \delta>0.$$

By Cauchy-Schwarz inequality, we may write

$$\begin{aligned}|\mathscr{K}_n^{(\alpha)}(f;x)-f(x)| &\le 4N_f(1+x^2)\mathscr{K}_n^{(\alpha)}((t-x)^2;x)+\omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}\mathscr{K}_n^{(\alpha)}(|t-x|;x)\right)\\ &\le 4N_f(1+x^2)\vartheta_{n,\alpha,2}(x)+\omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\vartheta_{n,\alpha,2}(x)}\right).\end{aligned}$$

Now, choosing $\delta = \sqrt{\vartheta_{n,\alpha,2}(x)}$, we obtain (3.5). □

4. Weighted Approximation

Theorem 6. *Let $f \in C_\rho^0[0,\infty)$ and $\alpha = \alpha(n) \to \infty$ as $n\to\infty$. Then, we have*

$$\lim_{n\to\infty} \|\mathscr{K}_n^{(\alpha)}(f) - f\|_\rho = 0. \tag{4.1}$$

Proof. In order to prove this result it is sufficient to verify the following three relations (see [16])

$$\lim_{n\to\infty} \|\mathscr{K}_n^{(\alpha)}(e_i;\cdot) - e_i\|_\rho = 0, \quad i=0,1,2. \tag{4.2}$$

Since $\mathscr{K}_n^{(\alpha)}(1;x)=1$, the condition in (4.2) holds true for $i=0$.

By Lemma 1, we have

$$\begin{aligned}\|\mathscr{K}_n^{(\alpha)}(e_1;\cdot)-e_1\|_\rho &= \sup_{x\ge0}\frac{1}{1+x^2}\left|\frac{nx}{n+1}+\frac{1}{2(n+1)}-x\right|\\ &= \sup_{x\ge0}\frac{x}{1+x^2}\left|\frac{-1}{n+1}\right| + \sup_{x\ge0}\frac{1}{1+x^2}\frac{1}{2(n+1)}.\end{aligned} \tag{4.3}$$

Thus,

$$\lim_{n\to\infty} \|\mathscr{K}_n^{(\alpha)}(e_1;\cdot)-e_1\|_\rho = 0.$$

Finally, we obtain

$$\|\mathscr{K}_n^{(\alpha)}(e_2;\cdot)-e_2\|_\rho = \sup_{x\ge0}\frac{1}{1+x^2}\left|\frac{n^2x^2(1+\alpha)}{\alpha(n+1)^2}+\frac{2nx}{(n+1)^2}+\frac{1}{3(n+1)^2}-x^2\right|$$

$$\leq \sup_{x\geq 0}\frac{x^2}{1+x^2}\left|\frac{(n^2-\alpha-2n\alpha)}{(n+1)^2\alpha}\right|+\sup_{x\geq 0}\frac{x}{1+x^2}\frac{2n}{(n+1)^2}+\sup_{x\geq 0}\frac{1}{1+x^2}\frac{1}{3(n+1)^2}, \tag{4.4}$$

which implies that $\lim_{n\to\infty}\|\mathscr{K}_n^{(\alpha)}(e_2;\cdot)-e_2\|_\rho=0$. □

We apply the weighted modulus of continuity $\Omega(f,\delta)$ defined on $[0,\infty)$ (see [31]). Let

$$\Omega(f,\delta)=\sup_{|h|<\delta,x\in[0,\infty)}\frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}\ \text{for each } C_\rho^0[0,\infty).$$

Lemma 3 ([31]). *Let $f\in C_\rho^0[0,\infty)$, then:*

(i) *$\Omega(f;\delta)$ is a monotone increasing function of δ;*

(ii) $\lim_{\delta\to 0^+}\Omega(f;\delta)=0$;

(iii) *for each $m\in\mathbb{N}$, $\Omega(f,m\delta)\leq m\Omega(f;\delta)$;*

(iv) *for each $\vartheta\in[0,\infty)$, $\Omega(f;\vartheta\delta)\leq(1+\vartheta)\Omega(f;\delta)$.*

In this section we study a quantitative Voronvoskaja type result for the $\mathscr{K}_n^{(\alpha)}$ operators.

Theorem 7. *Suppose $f''\in C_\rho^0[0,\infty)$ and $x>0$. Then, we have*

$$\left|\mathscr{K}_n^{(\alpha)}(f;x)-f(x)-f'(x)\left(\frac{1}{2(n+1)}-\frac{x}{(n+1)}\right)-\frac{f''(x)}{2}\left(\frac{x^2(n^2+\alpha)}{(n+1)^2\alpha}+\frac{x(n-1)}{(n+1)^2}+\frac{1}{3(n+1)^2}\right)\right|$$
$$\leq 8(1+x^2)O(n^{-1})\Omega\left(f'',\frac{1}{\sqrt{n}}\right).$$

Proof. Applying Taylor's expansion, there exists ξ lying between x and t such that

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+\varepsilon(t,x)(t-x)^2, \tag{4.5}$$

where

$$\varepsilon(t,x)=\frac{f''(\xi)-f''(x)}{2}$$

is a continuous functions that vanishes at 0. Operating the operators $\mathscr{K}_n^{(\alpha)}$ to the above relation and applying Lemma 2, we may write

$$\left|\mathscr{K}_n^{(\alpha)}(f;x)-f(x)-f'(x)\left(\frac{1}{2(n+1)}-\frac{x}{(n+1)}\right)-\frac{f''(x)}{2}\left(\frac{x^2(n^2+\alpha)}{(n+1)^2\alpha}+\frac{x(n-1)}{(n+1)^2}+\frac{1}{3(n+1)^2}\right)\right|$$
$$\leq \mathscr{K}_n^{(\alpha)}(|\varepsilon(t,x)|(t-x)^2,x).$$

Using Lemma 3 and the definition of $\Omega(f,\delta)$, we get

$$|f(t)-f(x)|\leq 2(1+x^2)[1+(t-x)^2]\left(1+\frac{|t-x|}{\delta}\right)(1+\delta^2)\Omega(f,\delta) \tag{4.6}$$

for every $f\in C_\rho^0[0,\infty)$ and $x,t\in[0,\infty)$. By (4.6) and the relation $|\xi-x|\leq|t-x|$, we have

$$\varepsilon(t,x)\leq(1+x^2)[1+(t-x)^2]\left(1+\frac{|t-x|}{\delta}\right)(1+\delta^2)\Omega(f'',\delta).$$

Also,

$$\varepsilon(t,x)=\begin{cases}2(1+x^2)(1+\delta^2)^2\Omega(f'',\delta), & |t-x|<\delta\\ (1+x^2)[1+(t-x)^2]\left(1+\frac{|t-x|}{\delta}\right)(1+\delta^2)\Omega(f'',\delta), & |t-x|\geq\delta.\end{cases}$$

Now choosing $\delta < 1$, we get

$$\varepsilon(t,x) \le 2(1+x^2)\left(1+\frac{(t-x)^4}{\delta^4}\right)(1+\delta^2)^2\Omega(f'',\delta)$$
$$\le 8(1+x^2)\left(1+\frac{(t-x)^4}{\delta^4}\right)\Omega(f'',\delta).$$

By Lemma 2, we find that

$$\mathscr{K}_n^{(\alpha)}(|\varepsilon(t,x)|(t-x)^2,x) = 8(1+x^2)\Omega(f'',\delta)\left(\mathscr{K}_n^{(\alpha)}((t-x)^2,x)+\frac{1}{\delta^4}\mathscr{K}_n^{(\alpha)}((t-x)^6,x)\right)$$
$$= 8(1+x^2)\Omega(f'',\delta)\left(O(n^{-1})+\frac{1}{\delta^4}O(n^{-3})\right).$$

Choosing $\delta = \frac{1}{\sqrt{n}}$, we have

$$\mathscr{K}_n^{(\alpha)}(|\varepsilon(t,x)|(t-x)^2,x) \le 8(1+x^2)O(n^{-1})\Omega\left(f'',\frac{1}{\sqrt{n}}\right).$$

Hence, we get the required result. □

Example 1. The comparison of convergence of $\mathscr{K}_n^{(\alpha)}(f;x)$ (red) and the Szász-Kantorovich $K_n(f;x)$(green) [12] operators for $f(x) = x^{10} - 5x^5 + 6$ (blue), $\alpha = 100$, $n = 100$ and $\alpha = 150$, $n = 150$ is illustrated in Figure 1 and Figure 2. It is notice that Szász-Kantorovich operators [12] give a better approximation to $f(x)$ than $\mathscr{K}_n^{(\alpha)}(f;x)$.

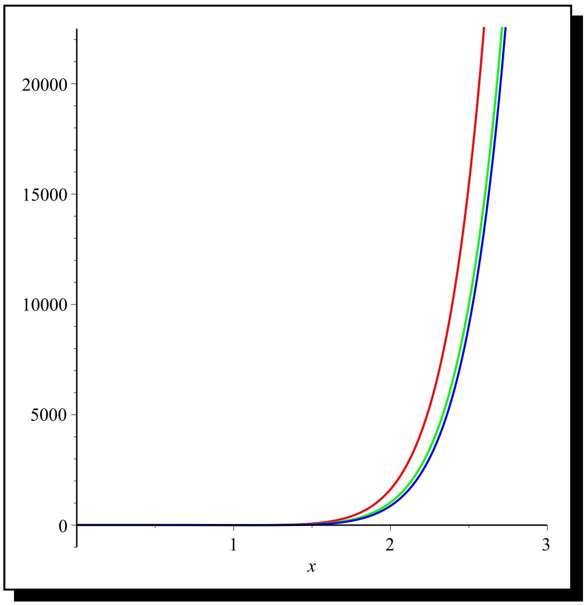

Figure 1. The convergence of $\mathscr{K}_{100}^{(100)}(f;x)$ and $K_{100}(f;x)$ to $f(x)$

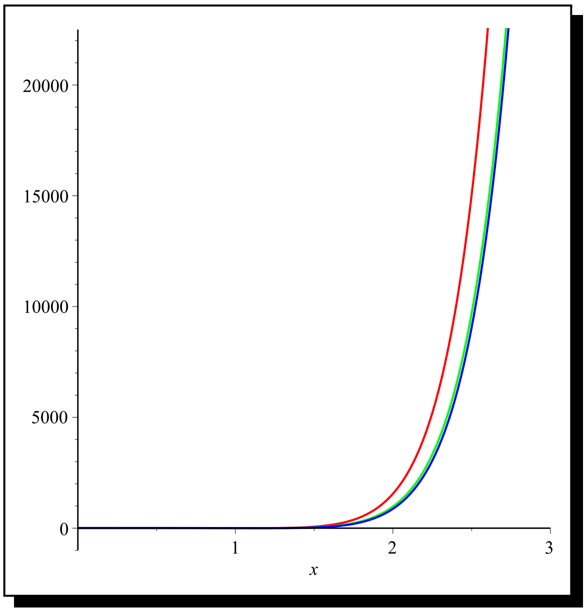

Figure 2. The convergence of $\mathscr{K}_{150}^{(150)}(f;x)$ and $K_{150}(f;x)$ to $f(x)$

5. Conclusion

In this paper, we have proven the basic convergence theorem for the operators (1.3) by using Bohman-Korovkin criterion. We have also found estimates of the rate of convergence involving modulus of continuity and Lipschitz function. In addition, we have studied weighted approximation and pointwise convergence of the operators in terms of quantitative Voronovskaja type theorem. Finally, we have shown the rate of convergence of these operators to certain functions with the help of the illustrations using Maple algorithms.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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