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On $\mathcal{I}$-Lacunary Double Statistical Convergence of Weight g

Research Article

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Abstract. In this paper, our aim is to introduce new notions, namely, $\mathcal{I}$-statistical double convergence of weight g and $\mathcal{I}$-lacunary double statistical convergence of weight g. We mainly investigate their relationship and also make some observations about these classes.

Keywords. Ideal; Filter; $\mathcal{I}$-double statistical convergence of weight g; $\mathcal{I}$-lacunary double statistical convergence of weight g; Closed subspace

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1. Introduction

In this section we recall some notations and fundamental definitions.

The idea of convergence of a real sequence was extended to statistical convergence by Fast [8] (see also, Schoenberg [31]) as follows: If $\mathbb{N}$ denotes the set of natural numbers and $K \subset \mathbb{N}$ then $K(m,n)$ denotes the cardinality of the set $K \cap [m,n]$. The upper and lower natural density of the subset K is defined by

$$\overline{d}(K) = \limsup_{n\to\infty} \frac{K(1,n)}{n} \quad \text{and} \quad \underline{d}(K) = \liminf_{n\to\infty} \frac{K(1,n)}{n}.$$

If $\overline{d}(K) = \underline{d}(K)$ then we say that the natural density of K exists and it is denoted simply by $d(K)$. Clearly, $d(K) = \lim_{n\to\infty} \frac{K(1,n)}{n}$.

A sequence (x_n) of real numbers is said to be statistically convergent to L if for arbitrary $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$ has natural density zero.

Statistical convergence turned out to be one of the most active areas of research in summability theory after the works of Fridy [9] and Šalát [20]. For some very interesting investigations concerning statistical convergence one may consult the papers of Savas [29], and Mursaleen and Alotaibi [15] where more references on this important summability method can be found.

The idea of statistical convergence was further extended to $\mathcal{I}$-convergence in [12] using the notion of ideals of $\mathbb{N}$ with many interesting consequences. More investigations in this direction and more applications of ideals can be found in [5, 6, 11, 13, 14, 19, 24–26, 30] where many important references can be found.

In another direction, a new type of convergence called lacunary statistical convergence was introduced in [10] as follows. A lacunary sequence is an increasing integer sequence $\theta = (k_r) = (k_0 < k_1 < \dots)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$, as $r \to \infty$. Let $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$.

A sequence (x_n) of real numbers is said to be lacunary statistically convergent to L (or, S_θ-convergent to L) if for any $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0,$$

where $|A|$ denotes the cardinality of $A \subset \mathbb{N}$. In [10] the relation between lacunary statistical convergence and statistical convergence was established among other things. More results on this convergence can be seen from [3, 7, 16, 17, 21–23, 27, 28].

Recently in [6, 24] we used ideals to introduce the concepts of $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence which naturally extend the notions of the above mentioned convergence.

On the other hand in [2, 4] a different direction was given to the study of statistical convergence where the notion of statistical convergence of order α, $0 < \alpha < 1$ was introduced by using the notion of natural density of order α (where n is replaced by n^α in the denominator in the definition of natural density).

Very recently it has been shown in [1] that one can further extend the concept of natural or asymptotic density (as well as natural density of order α) by considering natural density of weight g where $g : \mathbb{N} \to [0, \infty)$ is a function with $\lim_{n\to\infty} g(n) = \infty$ and $\frac{n}{g(n)} \nrightarrow 0$, as $n \to \infty$.

In a natural way, in this paper we introduce new and more general summability methods, namely, $\mathcal{I}$-double statistical convergence of weight g and $\mathcal{I}$-lacunary double statistical convergence of weight g. In this context it should be mentioned that the concept of double lacunary statistical convergence of weight g (which happens to be a special case of $\mathcal{I}$-lacunary double statistical convergence of weight g) has also not been studied till now. We mainly investigate their relationship and also make some observations about these classes.

2. Basic Definitions and Preliminaries

The following definitions and notions will be needed in the sequel.

Definition 2.1. A non-empty family $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to be an ideal of $\mathbb{N}$ if the following conditions hold:

(a) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$,

(b) $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$.

Definition 2.2. A non-empty family $\mathcal{F} \subset 2^{\mathbb{N}}$ is said to be a filter of $\mathbb{N}$ if the following conditions hold:

(a) $\phi \notin \mathcal{F}$,

(b) $A, B \in \mathcal{F}$ imply $A \cap B \in \mathcal{F}$,

(c) $A \in \mathcal{F}$, $A \subset B$ imply $B \in \mathcal{F}$.

If $\mathcal{I}$ is a proper nontrivial ideal of $\mathbb{N}$ (i.e. $\mathbb{N} \notin \mathcal{I}$), then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$ is a filter of $\mathbb{N}$. It is called the filter associated with the ideal $\mathcal{I}$. A proper ideal $\mathcal{I}$ is said to be admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Definition 2.3 ([12]). (i) A sequence (x_n) of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in \mathcal{I}$.

(ii) A sequence (x_n) of elements of $\mathbb{R}$ is said to be $\mathcal{I}^*$-convergent to $L \in \mathbb{R}$ if there exists $M \in \mathcal{F}(\mathcal{I})$ such that $(x_n)_{n \in M}$ converges to L.

We now present the basis of our main discussions. Let $g : \mathbb{N} \to [0, \infty)$ be a function with $\lim_{n\to\infty} g(n) = \infty$. The upper density of weight g was defined in [1] by the formula

$$\overline{d}_g(A) = \limsup_{n\to\infty} \frac{A(1,n)}{g(n)}$$

for $A \subset \mathbb{N}$ where as before $A(1,n)$ denotes the cardinality of the set $A \cap [1,n]$. Then the family

$$\mathcal{I}_g = \{A \subset \mathbb{N} : \overline{d}_g(A) = 0\}$$

forms an ideal. It has been observed in [1] that $\mathbb{N} \in \mathcal{I}_g$ if and only if $\frac{n}{g(n)} \to 0$, as $n \to \infty$. So we additionally assume that $n/g(n) \nrightarrow 0$, as $n \to \infty$ so that $\mathbb{N} \notin \mathcal{I}_g$ and $\mathcal{I}_g$ is a proper admissible ideal of $\mathbb{N}$. The collection of all such weight functions g satisfying the above properties will be denoted by G. As a natural consequence we can introduce the following definition.

Quite recently, $\mathcal{I}$-double statistical convergence has been established as a better tool than double statistical convergence. It is found very interesting that some results on sequences, series and summability can be proved by replacing the double statistical convergence by $\mathcal{I}$-double statistical convergence. Statistical convergence of double sequences $x=(x_{kl})$ has been defined and studied by Mursaleen and Edely [18]. Now, it would be helpful to give some definitions.

Let $K\subseteq\mathbb{N}\times\mathbb{N}$ be a two-dimensional set of positive integers and let $K(m,n)$ be the numbers of (k,l) in K such that $k\le m$ and $l\le n$. Then the two-dimensional analogue of natural density can be defined as follows [17].

A double sequence $x=(x_{kl})$ of real numbers is said to be convergent in the Pringsheim's sense or P-convergent if for each $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that $|x_{kl}-L|<\varepsilon$ whenever $k,l\ge N$ and L is called Pringsheim limit (denoted by $P-\lim x=L$).

The *lower asymptotic density* of the set $K\subseteq\mathbb{N}\times\mathbb{N}$ is defined as

$$\underline{\delta_2}(K)=\liminf_{m,n}\frac{K(m,n)}{mn}.$$

In case the sequence $(K(m,n)/mn)$ has a limit then we say that K has a double natural density and is defined as

$$\lim_{m,n}\frac{K(m,n)}{mn}=\delta_2(K).$$

Definition 2.4 ([17])**.** A real double sequence $x=(x_{k,l})$ is said to be *statistically convergent* to the number L if for each $\varepsilon>0$, the set

$$\{(k,l),k\le m \text{ and } l\le n:|x_{k,l}-L|\ge\varepsilon\}$$

has double natural density zero. In this case we write $st_2\text{-}\lim_{k,l}x_{k,l}=L$.

Let $K\subseteq\mathbb{N}\times\mathbb{N}$. As before if $m,n\in\mathbb{N}\times\mathbb{N}$, by $K_{m,n}$ here we denote the cardinality of the set $\{(k,l)\in K:1\le k\le m \text{ and } 1\le l\le n\}$. Let $g\in G$ and define (k,l) in K such that $k\le n$ and $l\le m$. Define

$$\underline{\delta_2^g}(K)=\text{P-}\liminf_{m,n}\frac{K_{m,n}}{g(mn)},\quad \overline{\delta_2^g}(K)=\text{P-}\limsup_{m,n}\frac{K_{m,n}}{g(mn)}.$$

These are called the lower and upper double density of weight g of the set K, respectively. If the limit $\text{P-}\lim_{m,n}\frac{K_{m,n}}{g(m,n)}$ exists in Pringsheim's sense then we say that the double density of weight g of the set K exists and we denote it by $\delta_2^g(K)$.

3. Main Results

In this section, we give the main definitions and also main results of this paper.

Definition 3.1. A double sequence $x = (x_{k,l})$ is said to be $\mathcal{I}-$double statistically convergent of weight g to L or $S(\mathcal{I})_2^g$-convergent to L if for each $\varepsilon > 0$ and $\delta > 0$

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{g(mn)}|\{k \le m \text{ and } l \le n : |x_{k,l} - L| \ge \varepsilon\}| \ge \delta\right\} \in \mathcal{I}.$$

In this case we write $x_{k,l} \to L(S(\mathcal{I})_2^g)$. The class of all sequences that are $\mathcal{I}$-double statistically convergent of weight g will be denoted simply by $S(\mathcal{I})_2^g$.

Remark 3.1. For $\mathcal{I}_2 = \mathcal{I}_{fin} = \{A \subset \mathbb{N}\times\mathbb{N}, A \text{ is a finite}\}$, $S(\mathcal{I})_2^g$-convergence coincidences with double statistical convergence of weight g. Further taking $g(mn) = (mn)^\alpha$, it reduces to $\mathcal{I}$-double statistical convergence of order α.

Definition 3.2. The double sequence x is bounded if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l,

$$\|x\|_{(\infty,2)} = \sup_{k,l}|x_{k,l}| < \infty.$$

We will denote the set of all bounded double sequences by l''_∞.

Note that in contrast to the case for single sequence, a P-convergent double sequence need to be bounded but every P-convergent real (or complex) double sequences is Cauchy and convergent.

Definition 3.3. The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing of integers such that

$$k_0 = 0,\ h_r = k_r - k_{k-1} \to \infty \ \text{ as } \ r \to \infty$$

and

$$l_0 = 0,\ \bar{h}_s = l_s - l_{s-1} \to \infty \ \text{ as } \ s \to \infty.$$

Notations. $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \bar{h}_s$, $\theta_{r,s}$ is determine by $I_{r,s} = \{(k,l) : k_{r-1} < k \le k_r \& l_{s-1} < l \le l_s\}$, $q_r = \frac{k_r}{k_{r-1}}$, $\bar{q}_s = \frac{l_s}{l_{s-1}}$, and $q_{r,s} = q_r \bar{q}_s$. We will denote the set of all double lacunary sequences by $\mathbf{N}_{\theta_{r,s}}$.

Definition 3.4. Let $\theta_{r,s}$ be a double lacunary sequence. A sequence $x = (x_{k,l})$ is said to be $\mathcal{I}$-double lacunary statistically convergent of weight g to L or $S_{\theta_{r,s}}(\mathcal{I})_2^g$-convergent to L if for each $\varepsilon > 0$ and $\delta > 0$

$$\left\{(r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{g(h_{r,s})}|\{(k,l) \in I_{r,s} : |x_{k,l} - L| \ge \varepsilon\}| \ge \delta\right\} \in \mathcal{I}.$$

In this case we write $x_{k,l} \to L(S_{\theta_{r,s}}(\mathcal{I})_2^g)$. The class of all $\mathcal{I}$-double lacunary sequences of weight g will be denoted simply by $S_{\theta_{r,s}}(\mathcal{I})_2^g$.

Theorem 3.1. *Let $g_1, g_2 \in G$ be such that there exist $M > 0$ and $(m_0, n_0) \in \mathbb{N} \times \mathbb{N}$ such that $\frac{g_1(mn)}{g_2(mn)} \le M$ for all $m \ge m_0$ and $n \ge n_0$. Then $S(\mathcal{I})_2^{g_1} \subset S(\mathcal{I})_2^{g_2}$.*

Proof. For any $\varepsilon > 0$,

$$\frac{|\{k \le m \text{ and } l \le n : |x_{k,l} - L| \ge \varepsilon\}|}{g_2(mn)} = \frac{g_1(mn)}{g_2(mn)} \cdot \frac{|\{k \le m \text{ and } l \le n : |x_{k,l} - L| \ge \varepsilon\}|}{g_1(mn)}$$
$$\le M \cdot \frac{|\{k \le m \text{ and } l \le n : |x_{k,l} - L| \ge \varepsilon\}|}{g_1(mn)}.$$

for $m \ge m_0$ and $n \ge n_0$. Hence for any $\delta > 0$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{|\{k \le m \text{ and } l \le n : |x_{k,l} - L| \ge \varepsilon\}|}{g_2(mn)} \ge \delta\right\}$$
$$\subset \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{|\{k \le m \text{ and } l \le n : |x_{k,l} - L| \ge \varepsilon\}|}{g_1(mn)} \ge \frac{\delta}{M} \cup A\right\},$$

where A is the union of the first m_0 rows and the first n_0 columns of the double sequence of $x = (x_{k,l})$.

If $x = (x_{k,l}) \in \mathrm{S}(\mathcal{I})_2^{g_1}$ then the set on the right hand side belongs to the ideal $\mathcal{I}$ and so the set on the left hand side also belongs to $\mathcal{I}$. This shows that $S(\mathcal{I})_2^{g_1} \subset S(\mathcal{I})_2^{g_2}$. □

Similar inclusion relations hold for $\mathrm{S}_{\theta_{r,s}}(\mathcal{I})_2^g$ also.

It is easy to check that both $\mathrm{S}(\mathcal{I})_2^g$ and $\mathrm{S}_{\theta_{r,s}}(\mathcal{I})_2^g$ are linear subspaces of the space of all real sequences. Next, we present a topological characterization of these spaces. As both the proofs are similar, we give the detailed proof for the class $\mathrm{S}_{\theta_{r,s}}(\mathcal{I})_2^g$ only.

Theorem 3.2. $\mathrm{S}_{\theta_{r,s}}(\mathcal{I})^g \cap \ell''_\infty$ *is a closed subset of ℓ''_∞ where ℓ''_∞ is the Banach space of all double bounded real sequences endowed with the sup norm where $g \in G$ is such that $\frac{g(mn)}{mn} \le M$ for all $m \ge m_0$ and $n \ge n_0$ for some $M > 0$ and $(m_0, n_0) \in \mathbb{N} \times \mathbb{N}$.*

Proof. Suppose that $(x^{m,n}) \subseteq S_{\theta_{r,s}}(\mathcal{I})^g \cap \ell''_\infty$, $0 < \alpha \le 1$, is a convergent sequence and it converges to $x \in l''_\infty$. We need to prove that $x \in S_{\theta_{r,s}}(\mathcal{I})^g \cap l''_\infty$. Assume that $(x^{m,n}) \to L_{m,n}(S_{\theta_{r,s}}(\mathcal{I})_2^g)$, $\forall\ (m,n) \in \mathbb{N} \times \mathbb{N}$. Take a positive strictly decreasing sequence $(\varepsilon_{m,n})$ converging to 0. Choose a positive integer (m,n) such that $\|x - x^{m,n}\|_\infty < \frac{\varepsilon_{m,n}}{4}$. Let $0 < \delta < 1$. Then

$$A = \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{g(h_{r,s})} \left|\left\{(k,l) \in I_{rs} : |x_{k,l}^{m,n} - L_{m,n}| \ge \frac{\varepsilon_{m,n}}{4}\right\}\right| < \frac{\delta}{3}\right\} \in F(\mathcal{I})$$

and

$$B = \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{g(h_{r,s})} \left|\left\{(k,l) \in I_{rs} : |x_{k,l}^{m+1,n+1} - L_{m+1,n+1}| \ge \frac{\varepsilon_{m+1,n+1}}{4}\right\}\right| < \frac{\delta}{3}\right\} \in F(\mathcal{I}).$$

Since $A \cap B \in F(\mathcal{I})$ and $\phi \notin F(\mathcal{I})$, so we can choose $r,s \in A \cap B$. Then

$$\frac{1}{g(h_{r,s})} \left|\left\{(k,l) \in I_{rs} : |x_{k,l}^{m,n} - L_{m,n}| \ge \frac{\varepsilon_{m,n}}{4}\right\}\right| < \frac{\delta}{3}$$

and

$$\frac{1}{g(h_{r,s})}\left|\left\{(k,l)\in I_{rs}:|x_{k,l}^{m+1,n+1}-L_{m+1,n+1}|\geq\frac{\varepsilon_{m+1,n+1}}{4}\right\}\right|<\frac{\delta}{3}$$

and so

$$\frac{1}{g(h_{r,s})}\left|\left\{(k,l)\in I_{rs}:|x_{k,l}^{m,n}-L_{m,n}|\geq\frac{\varepsilon_{m,n}}{4}\vee|x_{k,l}^{m+1,n+1}-L_{m+1,n+1}|\geq\frac{\varepsilon_{m+1,n+1}}{4}\right\}\right|<\delta<1.$$

Hence there exists a $(k,l)\in I_{rs}$ for which

$$|x_{k,l}^{m,n}-L_{m,n}|<\frac{\varepsilon_{m,n}}{4}$$

and

$$|x_{k,l}^{m+1,n+1}-L_{m+1,n+1}|<\frac{\varepsilon_{m+1,n+1}}{4}.$$

Then, we can write

$$\begin{aligned}|L_{m,n}-L_{m+1,n+1}|&\leq|L_{m,n}-x_{k,l}^{m,n}|+|x_{k,l}^{m,n}-x_{k,l}^{m+1,n+1}|+|x_{k,l}^{m+1,n+1}-L_{m+1,n+1}|\\&\leq|x_{k,l}^{m,n}-L_{m,n}|+|x_{k,l}^{m+1,n+1}-L_{m+1,n+1}|+\|x-x^{m,n}\|_\infty+\|x-x_{k,l}^{m+1,n+1}\|_\infty\\&\leq\frac{\varepsilon_{m,n}}{4}+\frac{\varepsilon_{m+1,n+1}}{4}+\frac{\varepsilon_{m,n}}{4}+\frac{\varepsilon_{m+1,n+1}}{4}\leq\varepsilon_{m,n}\ .\end{aligned}$$

This implies that $(L_{m,n})$ is a Cauchy sequence in $\mathbb{R}$, and so there is a real number L such that $L_{m,n}\to L$, as $m,n\to\infty$. We need to prove that $x\to L(S_{\theta_{r,s}}(\mathfrak{I})^g)$. For any $\varepsilon>0$, choose $(m,n)\in\mathbb{N}\times\mathbb{N}$ such that $\varepsilon_{m,n}<\frac{\varepsilon}{4}$, $\|x-x^{m,n}\|_\infty<\frac{\varepsilon}{4}$, $|L_{m,n}-L|<\frac{\varepsilon}{4}$. Then

$$\begin{aligned}\frac{1}{g(h_{r,s})}|\{(k,l)\in I_{rs}:|x_{k,l}-L|\geq\varepsilon\}|&\leq\frac{1}{g(h_{r,s})}|\{k\in I_r:|x_{k,l}^{m,n}-L_{m,n}|+\|x_{k,l}-x_{k,l}^{m,n}\|_\infty+|L_{m,n}-L|\geq\varepsilon\}|\\&\leq\frac{1}{g(h_{r,s})}\left|\left\{(k,l)\in I_{rs}:|x_{k,l}^{m,n}-L_{m,n}|+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}\geq\varepsilon\right\}\right|\\&\leq\frac{1}{g(h_{r,s})}\left|\left\{(k,l)\in I_{rs}:|x_{k,l}^{m,n}-L_{m,n}|\geq\frac{\varepsilon}{2}\right\}\right|.\end{aligned}$$

This implies

$$\begin{aligned}&\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{g(h_{r,s})}|\{(k,l)\in I_{rs}:\ |x_{k,l}-L|\geq\varepsilon\}|<\delta\right\}\\&\quad\supseteq\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{g(h_{r,s})}\left|\left\{(k,l)\in I_{rs}:|x_{k,l}^{m,n}-L_{m,n}|\geq\frac{\varepsilon}{2}\right\}\right|<\delta\right\}\in F(\mathfrak{I}).\end{aligned}$$

So

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{g(h_{r,s})}|\{(k,l)\in I_{rs}:|x_{k,l}-L|\geq\varepsilon\}|<\delta\right\}\in F(\mathfrak{I})$$

and so

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{g(h_{r,s})}|\{(k,l)\in I_{rs}:|x_{k,l}-L|\geq\varepsilon\}|\geq\delta\right\}\in\mathfrak{I}.$$

This gives that $x\to L(S_{\theta_{r,s}}(\mathfrak{I})_2^g)$ and completes the proof of the theorem. □

Theorem 3.3. *$S(I)^g \cap \ell''_\infty$ is a closed subset of ℓ''_∞ where $g \in G$ is such that $\frac{g(mn)}{mn} \le M$ for all $m \ge m_0$ and $n \ge n_0$, for some $M > 0$ and $(m_0, n_0) \in \mathbb{N} \times \mathbb{N}$.*

Remark 3.2. Theorem 2.13 and Theorem 2.14 of [7] are single cases of Theorem 3.2 and Theorem 3.3, respectively.

Next, we establish a result for $\mathcal{I}$-lacunary statistical convergence of weight g. For this we introduce the following definition.

Definition 3.5. Let $\theta_{r,s}$ be a lacunary sequence. A sequence $x = (x_{k,l})$ is said to be $N_{\theta_{r,s}}(\mathcal{I})_2^g$-convergent to L if for any $\varepsilon > 0$,

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{g(h_{r,s})} \sum_{(k,l) \in I_{rs}} |x_{k,l} - L| \ge \varepsilon \right\} \in \mathcal{I}.$$

It is denoted by $x_{k,l} \to L(N_{\theta_{r,s}}(\mathcal{I})_2{}^g)$ and the class of all such sequences is denoted simply by $N_{\theta_{r,s}}(\mathcal{I})_2{}^g$.

Theorem 3.4. *Let $\theta = (k_{rs})$ be a double lacunary sequence. Then*

(a) $x_{k,l} \to L(N_{\theta_{r,s}}(\mathcal{I})_2{}^g) \Rightarrow x_{k,l} \to L(S_{\theta_{r,s}}(\mathcal{I})_2^g)$,

(b) $N_{\theta_{r,s}}(\mathcal{I})_2^g$ *is a proper subset of* $S_{\theta_{r,s}}(\mathcal{I})_2^g$.

Proof. (a) If $\varepsilon > 0$ and $x_{k,l} \to L(N_{\theta_{r,s}}(\mathcal{I})_2^g)$, we can write

$$\sum_{(k,l) \in I_{r,s}} |x_{k,l} - L| \ge \varepsilon |\{(k,l) \in I_{r,s} :\ |x_{k,l} - L| \ge \varepsilon\}|$$

and so

$$\frac{1}{\varepsilon\, g(h_{r,s})} \sum_{(k,l) \in I_{r,s}} |x_{k,l} - L| \ge \frac{1}{g(h_{r,s})} |\{(k,l) \in I_{r,s} :\ |x_{k,l} - L| \ge \varepsilon\}|.$$

Then for any $\delta > 0$

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{g(h_{r,s})} |\{(k,l) \in I_{r,s} : |x_{k,l} - L| \ge \varepsilon\}| \ge \delta \right\}$$
$$\subseteq \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{g(h_{r,s})} \sum_{(k,l) \in I_{r,s}} |x_{k,l} - L| \ge \varepsilon\delta \right\} \in \mathcal{I}.$$

This proves the result.

(b) Let x be defined as follows:

$$x_{k,l} := \begin{pmatrix} 1 & 2 & 3 & \cdots & [\sqrt[3]{h_{r,s}}] & 0 & \cdots \\ 2 & 2 & 3 & \cdots & [\sqrt[3]{h_{r,s}}] & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & [\sqrt[3]{h_{r,s}}] & \cdots & \ldots & [\sqrt[3]{h_{r,s}}] & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now for any $\varepsilon > 0$,

$$\frac{1}{g(h_{r,s})}|\{(k,l)\in I_{r,s} : |x_{k,l} - L| \geq \varepsilon\}| \leq \frac{[\sqrt[3]{h_{r,s}}]}{g(h_{r,s})}$$

and consequently for any $\delta > 0$, we get

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{g(h_{r,s})}|(k,l)\in I_{r,s} : |x_{k,l} - 0| \geq \varepsilon| \geq \delta\right\} \subseteq \left\{(r,s)\in\mathbb{N}\times\mathbb{N} : \frac{[\sqrt[3]{h_{r,s}}]}{g(h_{r,s})} \geq \delta\right\}.$$

Note that the set on the right hand side is a finite set and so is a member of $\mathcal{I}$. Thus $x_{kl} \to 0(S_\theta(\mathcal{I})_2^g)$. Again observe that

$$\frac{1}{g(h_{r,s})}\sum_{(k,l)\in I_{r,s}} |x_{k,l} - 0| = \frac{1}{g(h_{r,s})}\frac{[\sqrt[3]{h_{r,s}}]([\sqrt[3]{h_{r,s}}]([\sqrt[3]{h_{r,s}}]+1))}{2}.$$

Hence

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{g(h_{r,s})}\sum_{(k,l)\in I_{r,s}} |x_{k,l} - 0| \geq \frac{1}{4}\right\}$$
$$= \left\{(r,s)\in\mathbb{N}\times\mathbb{N} : \frac{[\sqrt[3]{h_{r,s}}]([\sqrt[3]{h_{r,s}}]([\sqrt[3]{h_{r,s}}]+1))}{2g(h_{r,s})} \geq \frac{1}{2}\right\}$$

which evidently belongs to $\mathcal{F}(\mathcal{I})$ as $\mathcal{I}$ is admissible. Therefore $x_{k,l} \nrightarrow 0\left(N_\theta(\mathcal{I})^g\right)$. □

Remark 3.3. *It is not clear whether the following result hold for any $g \in G$ and we leave it as an open problem.*

(c) $x \in \ell''_\infty$ *and* $x_{k,l} \to L(S_\theta(\mathcal{I})) \Rightarrow x_{k,l} \to L(N_\theta(\mathcal{I}))$,

(d) $S_\theta(\mathcal{I}) \cap \ell''_\infty = N_\theta(\mathcal{I}) \cap \ell''_\infty$

In the remaining results we intend to investigate the relationship between the two new convergence methods introduced above, namely, $\mathcal{I}$-double statistical and $\mathcal{I}$-double lacunary statistical convergence of weight g.

Theorem 3.5. *For any lacunary sequence $\theta_{r,s}$, $\mathcal{I}$-double statistical convergence of weight g implies $\mathcal{I}$-double lacunary statistical convergence of weight g if* $\liminf_r \frac{g(h_{r,s})}{g(k_{r,s})} > 1$.

Proof. Since $\liminf_{r,s} \frac{g(h_{r,s})}{g(k_{r,s})} > 1$, so we can find a $H > 1$ such that for sufficiently large r, s, we have $\frac{g(h_{r,s})}{g(k_{r,s})} \geq H$.

Since $x_{k,l} \to L(S(\mathcal{I})^g)$, hence for every $\varepsilon > 0$ and sufficiently large r, s, we have

$$\frac{1}{g(k_{r,s})}|\{k \leq k_r \text{ and } l \leq l_s : |x_{k,l} - L| \geq \varepsilon\}| \geq \frac{1}{g(k_{r,s})}|\{(k,l)\in I_{r,s} : |x_{k,l} - L| \geq \epsilon\}|$$
$$\geq H\frac{1}{g(h_{r,s})}|\{(k,l)\in I_{r,s} : |x_{k,l} - L| \geq \varepsilon\}|.$$

Then for any $\delta > 0$, we get

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N}: \frac{1}{g(h_{r,s})}|\{(k,l)\in I_{r,s}: |x_{k,l}-L|\geq\varepsilon\}|\geq\delta\right\}$$
$$\subseteq\left\{(r,s)\in\mathbb{N}\times\mathbb{N}: \frac{1}{g(k_{r,s})}|\{k\leq k_r \text{ and } l\leq l_s: |x_{k,l}-L|\geq\varepsilon\}|\geq H\delta\right\}\in\mathcal{I}.$$

This shows that $x_{k,l}\to L(S_{\theta_{r,s}}(\mathcal{I})_2{}^g)$. □

It is known that double lacunary statistical convergence implies double statistical convergence if and only if $\limsup_{rs} q_{rs} < \infty$ (see [27]). However for arbitrary admissible ideal $\mathcal{I}$, this is not clear and we leave it as an open problem.

Problem 1. *When $\mathcal{I}$-double lacunary statistical convergence of weight g implies $\mathcal{I}$-double statistical convergence of weight g?*

4. Conclusions

Double lacunary sequence was studied by Savas and Patterson. It is natural question that whether this concept will be work for lacunary double statistical convergence of weighted g. In this paper, we gave some answers of this question and also we prove that $\mathcal{I}$-double statistical convergence a better tool than double statistical convergence.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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