Communications in Mathematics and Applications
Vol. 8, No. 1, pp. 1–6, 2017
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

Research Article

Double Total Domination in Harary Graphs

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Abstract. Let G be a graph with minimum degree at least 2. A set $D \subseteq V$ is a double total dominating set of G if each vertex is adjacent to at least two vertices in D. The double total domination number $\gamma_{\times 2,t}(G)$ of G is the minimum cardinality of a double total dominating set of G. In this paper, we will find double total domination number of Harary graphs.

Keywords. Double total domination number; Harary graph

MSC. 05C69

Received: November 7, 2013 **Accepted:** December 31, 2013

1. Introduction

Let G be a simple graph with the *vertex set* $V = V(G)$ and the *edge set* $E = E(G)$. The *order* $|V|$ of G is denoted by $n = n(G)$. The *open neighborhood* and the *closed neighborhoods* of a vertex $v \in V$ are $N_G(v) = \{u \in V \mid uv \in E\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. The *degree* of a vertex $v \in V$ is $deg(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively.

The research of domination in graphs has been an evergreen of the graph theory. Its basic concept is the dominating set and the domination number. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi and Slater [1, 2]. And many variants of the dominating set were introduced and the corresponding numerical invariants were defined for them. For example, the k-tuple total domination number is defined in [3] by Henning and Kazemi, which is an extension of the total domination number (for more information see [4, 5]).

Here, we find double domination number of Harary graphs $H_{m,n}$. The next known results are useful for our investigations.

Definition 1.1 ([3]). Let $k \geq 1$ be an integer and let G be a graph with $\delta(G) \geq k$. A subset $S \subseteq V(G)$ is called a *k-tuple total dominating set,* briefly kTDS, in G, if for each $x \in V(G)$, $|N(x) \cap S| \geq k$. The minimum number of vertices of a k-tuple total dominating set in a graph G is called the *k-tuple total domination number* of G and denoted by $\gamma_{\times k,t}(G)$.

When $k = 2$, k-tuple total dominating set and k-tuple total domination number are known as *double total dominating set* and *double total domination number*, respectively.

Harary Graph ([7]). Given $m < n$, place n vertices $1, 2, \ldots, n$ around a circle, equally spaced. If m is even, form $H_{m,n}$ by making each vertex adjacent to the nearest $\frac{m}{2}$ vertices in each direction around the circle. If m is odd and n is even, form $H_{m,n}$ by making each vertex adjacent to the nearest $\frac{m-1}{2}$ vertices in each direction and to the diametrically opposite vertex. In each case, $H_{m,n}$ is m-regular. When m and n are both odd, index the vertices by the integers modulo n. Construct $H_{m,n}$ from $H_{m-1,n}$ by adding the edges $i \leftrightarrow i + \frac{n-1}{2}$ for $0 \leq i \leq \frac{n-1}{2}$.

Here, we find double domination number of Harary graphs $H_{m,n}$. The next known results are useful for our investigations.

Proposition 1.2 (Henning and Kazemi [3]). *For any graph G of order n with $\delta(G) \geq k$,*

(i) $\max\{\gamma_{\times k}(G), k+1\} \leq \gamma_{\times k,t}(G) \leq n$,

(ii) *if G is a spanning subgraph of a graph H, then $\gamma_{\times k,t}(H) \leq \gamma_{\times k,t}(G)$,*

(iii) *if v is a degree-k vertex in G, then $N_G(v)$ is a subset of every kTDS of G.*

Proposition 1.3 (Henning and Kazemi [4]). *Let G be a graph of order with minimum degree at least k. Then $\gamma_{\times k,t}(G) \geq \left\lceil \frac{kn}{\Delta(G)} \right\rceil$.*

2. Harary Graphs

As a first result of Proposition 1.3, we obtain the next result.

Proposition 2.1. (i) $\gamma_{\times k,t}(H_{2m,n}) \geq \left\lceil \frac{kn}{2m} \right\rceil$,

(ii) $\gamma_{\times k,t}(H_{2m+1,2n}) \geq \left\lceil \frac{2kn}{2m+1} \right\rceil$,

(iii) $\gamma_{\times k,t}(H_{2m+1,2n+1}) \geq \left\lceil \frac{k(2n+1)}{2m+2} \right\rceil$.

In this section, we calculate double total domination number of Harary graphs.

Theorem 2.2. *Let $G = H_{2m,n}$ be a Harary graph. Then $\gamma_{\times 2,t}(G) = \left\lceil \frac{n}{m} \right\rceil$.*

Proof. Let $V(G) = \{i \mid 1 \le i \le n\}$. Since $S = \left\{im+1 \middle| 0 \le i \le \left\lceil \frac{n}{m} \right\rceil - 1\right\}$ is a 2TDS of G, Proposition 2.1(i) implies $\gamma_{\times 2,t}(G) = \left\lceil \frac{n}{m} \right\rceil$. □

Theorem 2.3. *Let $G = H_{2m+1,2n}$ be a Harary graph. Let $2n = (2m+1)\ell + r$, where $0 \le r \le 2m$, $\ell \ge 1$. Also let $\ell + r = 2m\ell' + r'$, where $0 \le r' < 2m$ and $\ell' \ge 0$. Then*

$$\left\lceil \frac{4n}{2m+1} \right\rceil \le \gamma_{\times 2,t}(G) \le \left\lceil \frac{4n}{2m+1} \right\rceil + 1,$$

if $1 \le r \le m$ and $(r', \ell') \notin \{1,2,\dots,m\} \times \{0\}$, and $\gamma_{\times 2,t}(G) = \left\lceil \frac{4n}{2m+1} \right\rceil$ otherwise.

Proof. Let $V(G) = \{i \mid 1 \le i \le 2n\}$. Proposition 2.1(ii) implies that

$$\gamma_{\times 2,t}(G) \ge \left\lceil \frac{4n}{2m+1} \right\rceil = \begin{cases} 2\ell & \text{if } r = 0, \\ 2\ell + 1 & \text{if } 1 \le r \le m, \\ 2\ell + 2 & \text{if } m+1 \le r \le 2m, \end{cases}$$

where r and ℓ are both even or both odd. If $(\ell, r, m) = (1,1,1)$, then $G = K_4$ and $\gamma_{\times 2,t}(G) = 3 = \left\lceil \frac{8}{3} \right\rceil$. Now let $(\ell, r, m) \ne (1,1,1)$. If $r = 0$, set

$$S = \{(2m+1)i + 1, (2m+1)i + (m+1) | 0 \le i \le \ell - 1\},$$

which is a 2TDS of cardinality $\left\lceil \frac{4n}{2m+1} \right\rceil$. Now let $r \ne 0$. If $\ell' = 0$ and $1 \le r' < m$, set

$$S = \{2mi + 1, 2mi + (m+1) | 0 \le i \le \ell - 1\} \cup \{2n - m + 1\},$$

which is a 2TDS of G of cardinality $2\ell + 1 = \left\lceil \frac{4n}{2m+1} \right\rceil$. Otherwise, for odd r, set

$$\begin{aligned} S = &\left\{(2m+1)i + (n+m+1), (2m+1)i + (n+2m+2) \middle| 0 \le i \le \left\lfloor \frac{n}{2m+1} \right\rfloor - 1\right\} \\ &\cup \left\{(2m+1)i + 1, (2m+1)i + (m+1) \middle| 0 \le i \le \left\lceil \frac{n}{2m+1} \right\rceil - 1\right\} \\ &\cup \left\{n+1, (2m+1)\left(\left\lceil \frac{n}{m+1} \right\rceil - 1\right) + (n+m+1)\right\}, \end{aligned}$$

and for even r, set

$$\begin{aligned} S = &\left\{(2m+1)i + (n+1), (2m+1)i + (n+m+1) \middle| 0 \le i \le \frac{\ell}{2} - 1\right\} \\ &\cup \left\{(2m+1)i + 1, (2m+1)i + (m+1) \middle| 0 \le i \le \frac{\ell}{2} - 1\right\} \\ &\cup \left\{n + 1 - \frac{r}{2}, 2n + 1 - \frac{r}{2}\right\}. \end{aligned}$$

In each case, S is a 2TDS of G of cardinality $2\ell + 2$, and this completes our proof. □

Theorem 2.4. *Let $G = H_{2m+1,2n+1}$ be a Harary graph. Let $2n+1 = (2m+1)\ell + r$, where $0 \le r \le 2m$ and $\ell \ge 1$. Also let $\ell + r = 2m\ell' + r'$, where $0 \le r' < 2m$, and $\ell' \ge 0$. Then*

$$\gamma_{\times 2,t}(G) = \left\lceil \frac{4n+1}{2m+1} \right\rceil$$

if $(r,r',\ell')\in\{i\mid 2\le i\le m\}\times\{j\mid 1\le j\le m\}\times\{0\}$ *or* $r\in\{1\}\cup\{i\mid m+2\le i\le 2m\}$, *and*

$$\left\lceil\frac{4n+1}{2m+1}\right\rceil\le\gamma_{\times 2,t}(G)\le\left\lceil\frac{4n+1}{2m+1}\right\rceil+1,$$

otherwise.

Proof. Let $V(G)=\{i\mid 1\le i\le 2n+1\}$. First we show that $\gamma_{\times 2,t}(G)\ge\left\lceil\frac{4n+1}{2m+1}\right\rceil$. Let S be a 2TDS such that $n+1\not\in S$. Let

$$t:=\min\{n+1-j\mid j\in S,\text{ and } n+1-j \text{ is positive}\}.$$

Then one can verify that $S'=\{j+t|j\in S\}$ is also a 2TDS such that $n+1\in S'$, and $|S'|=|S|$. Therefore

$$\gamma_{\times 2,t}(G)=\min\{|S|\mid S \text{ is a 2TDS of } G,\text{ and } n+1\in S\}.$$

Now let S be an arbitrary 2TDS of G, and $n+1\in S$. Since every vertex of $V(G)$ is counted at least two times in the union of the neighborhoods of the vertices of S, we have $\sum_{j\in S}\deg(j)\ge 2(2n+1)$. So $|S|(2m+1)+1\ge 2(2n+1)$, and this implies $|S|\ge\left\lceil\frac{4n+1}{2m+1}\right\rceil$. Hence $\gamma_{\times 2,t}(G)\ge\left\lceil\frac{4n+1}{2m+1}\right\rceil$.

Now let $2n+1=2m\ell+(\ell+r)$, and let $\ell+r=2m\ell'+r'$, where $0\le r'<2m$ and $\ell'\ge 0$. Notice that ℓ is odd if and only if r is even. Recall that

$$\left\lceil\frac{4n+1}{2m+1}\right\rceil=\begin{cases}2\ell & \text{if } r=0,\\ 2\ell+1 & \text{if } 1\le r\le m,\\ 2\ell+2 & \text{if } m+1\le r\le 2m.\end{cases}$$

Let $\alpha_i=(2m+1)i$ be an arbitrary vertex of G. Continue the proof in the following cases.

Case 1: $0\le r\le 1$.
For $r=0$, set

$$S_0=\left\{\alpha_i+1,\alpha_i+(m+1),\alpha_i+(n+1),\alpha_i+(n+m+1)\middle|0\le i\le\left\lfloor\frac{\ell}{2}\right\rfloor-1\right\}\cup\{n-m+1,2n-m+1,2n+1\},$$

and for $r=1$, set

$$S_1=(S_0-\{n-m+1,2n-m+1,2n+1\})\cup\{2n+1\}.$$

Then S_0 and S_1 are two double total dominating sets of G of cardinality $2\ell+1$.

Case 2: $2\le r\le 2m$.
Let $1\le r'=\ell+r\le m$. Then $S=\{2mi+1,2mi+(m+1)|0\le i\le\ell-1\}\cup\{2n+2-m\}$ is a 2TDS of G of cardinality $2\ell+1$. Otherwise, if r is odd, set

$$S_o=\left\{\alpha_i+1,\alpha_i+(m+1),\alpha_i+(n+1),\alpha_i+(n+m+1)\middle|0\le i\le\left\lfloor\frac{\ell}{2}\right\rfloor-1\right\}\cup\left\{n+1-\left(\frac{r-1}{2}\right),2n+1-\left(\frac{r-1}{2}\right)\right\},$$

and if r is even, set

$$S_e = \left\{\alpha_i + 1, \alpha_i + (m+1), \alpha_i + (n+1), \alpha_i + (n+m+1) \middle| 0 \le i \le \left\lfloor \frac{\ell}{2} \right\rfloor - 1 \right\}$$
$$\cup \left\{ n - \left(\frac{r-2}{2}\right), n - m - \left(\frac{r-2}{2}\right), 2n+1 - \left(\frac{r-2}{2}\right), 2n+1-m-\left(\frac{r-2}{2}\right) \right\}.$$

In each case, the given set is a 2TDS of G of cardinality $\left\lceil \frac{4n+1}{2m+1} \right\rceil$ or $\left\lceil \frac{4n+1}{2m+1} \right\rceil + 1$, and this completes our proof. □

3. Some Problems

We finish our paper with the following problems.

Problem 3.1. *Characterize Harary graphs in which their double total domination number achieves the lower or upper bounds given in Theorem* 2.3 *or Theorem* 2.4.

Problem 3.2. *Find general formula for* $\gamma_{\times k,t}(G)$, *when* $k \ge 2$ *and* G *is a graph from some of the known family of graphs, such as Supergeneralized Petersen graphs, which was proved the special case by A.P. Kazemi and B. Pahlavsay in* [6].

4. Conclusion

The double total domination number in Harary graphs was proved as an algorithm because of the importance of this graphs. A natural way to extend this study is finding a general formula for $\gamma_{\times k,t}(H_{m,n})$.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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