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Research Article

Stability of *n*-bi-Jordan Homomorphisms on Commutative Algebras

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Abstract. In this paper, we prove that every n-bi-Jordan homomorphism between commutative algebras is an n-bi-ring homomorphism, and then we employ this result to show that to each approximate n-bi-Jordan homomorphism φ between commutative Banach algebras there corresponds a unique n-bi-ring homomorphism near to φ.

Keywords. bi-additive; n-bi-homomorphism; n-bi-Jordan homomorphism

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1. Introduction

Let $\mathscr{A}$ and $\mathscr{B}$ be complex Banach algebras and $\varphi : \mathscr{A} \to \mathscr{B}$ be a linear map. Then φ is called n-homomorphism if for all $a_1, a_2, \dots, a_n \in \mathscr{A}$,

$$\varphi(a_1 a_2 \dots a_n) = \varphi(a_1)\varphi(a_2)\dots\varphi(a_n).$$

The concept of n-homomorphism was studied for complex algebras by Hejazian *et al*. in [5]. A 2-homomorphism is then just a homomorphism, in the usual sense. One may refer to [1], for certain properties of 3-homomorphisms.

In [6], Herstein introduced the concept of an n-Jordan homomorphism. A linear map φ between Banach algebras $\mathscr{A}$ and $\mathscr{B}$ is called an n-Jordan homomorphism if

$$\varphi(a^n) = \varphi(a)^n, \quad a \in \mathscr{A}.$$

A 2-Jordan homomorphism is called simply a Jordan homomorphism. For characterization of Jordan and 3-Jordan homomorphism the reader is referred to [12], [13] and [14] and the references therein.

From the above definitions it follows that every n-homomorphism is an n-Jordan homomorphism, but in general the converse is false. The converse statement may be true under certain conditions. For example, Herstein in [6] proved the following theorem.

Theorem 1.1. *If φ is a Jordan homomorphism of a ring R onto a prime ring R' of characteristic deferent from 2 and 3, then either φ is a homomorphism or an anti-homomorphism.*

The next theorem is due to Zelazko [12]. Also, see [13] for another approach to the same result.

Theorem 1.2. *Suppose that $\mathscr{A}$ is a Banach algebra, which need not be commutative, and suppose that $\mathscr{B}$ is a semisimple commutative Banach algebra. Then each Jordan homomorphism $\varphi : \mathscr{A} \to \mathscr{B}$ is a homomorphism.*

Also it is shown in [2] that every n-Jordan homomorphism between two commutative Banach algebras is an n-homomorphism for $n \in \{2,3,4\}$ and this result is extended to the case $n = 5$ in [3]. Lee in [8] generalized this result and proved it for all $n \in \mathbb{N}$. See also [4] for another proof of Lee's Theorem.

A classical question in the theory of functional equations is that "When is it true that a mapping which approximately satisfies a functional equation $\mathscr{E}$ must be somehow close to an exact solution of $\mathscr{E}$?" Such a problem was formulated by Ulam [11] in 1940 and solved in the next year for the Cauchy functional equation by Hyers [7]. It gave rise to the stability theory for functional equations.

Th. M. Rassias [10] considered a generalized version of the Hyers's result which permitted the Cauchy difference to become unbounded.

In [9], Miura *et al*. investigated the Hyers-Ulam-Rassias stability of Jordan homomorphisms, and it is extended to n-Jordan homomorphisms in [3] and [8].

Let $\mathscr{A}$ and $\mathscr{B}$ be a two normed (Banach) algebra and set $\mathscr{U} = \mathscr{A} \times \mathscr{B}$. Then $\mathscr{U}$ is a normed (Banach) algebra for the multiplication

$$(a,b)(x,y) = (ax,by), \quad (a,b),(x,y) \in \mathscr{U},$$

and with norm

$$\|(a,b)\| = \|a\| + \|b\|.$$

Let $\mathscr{D}$ be a normed (Banach) algebra and let $\varphi : \mathscr{U} \to \mathscr{D}$ be a map. Then we say that φ is bi-additive, if

$$\varphi(a+x,b+y) = \varphi(a,b) + \varphi(x,y), \quad (a,b),(x,y) \in \mathscr{U},$$

and it is called n-bi-multiplicative, if

$$\varphi(x_1x_2\dots x_n, y_1y_2\dots y_n) = \varphi(x_1,y_1)\varphi(x_2,y_2)\dots\varphi(x_n,y_n),$$

for all $(x_i,y_i) \in \mathscr{U}$. If φ is bi-additive and n-bi-multiplicative, then it is called n-bi-ring homomorphism. We say that a bi-additive mapping $\varphi : \mathscr{U} \to \mathscr{D}$ is an n-bi-Jordan homomorphisms if φ satisfies

$$\varphi(x^n, y^n) = \varphi(x, y)^n, \quad (x, y) \in \mathscr{U}.$$

We remark that in case $n = 2$ we speak about bi-ring homomorphism and bi-Jordan homomorphism, respectively. It is obvious that each n-bi-ring homomorphism is an n-bi-Jordan homomorphism, but in general the converse is false.

For bi-Jordan homomorphism the next result obtained by the author in [15].

Theorem 1.3. *Suppose that $\mathscr{U}$ is a Banach algebra, which need not be commutative, and suppose $\mathscr{D}$ is a commutative semisimple Banach algebra. Then each bi-Jordan homomorphism $\varphi : \mathscr{U} \to \mathscr{D}$ is a bi-ring homomorphism.*

In this paper, we first prove that each n-bi-Jordan homomorphism $\varphi : \mathscr{U} \to \mathscr{D}$, between commutative algebras, is an n-bi-ring homomorphism and then we applying this fact to prove that to each approximate n-bi-Jordan homomorphism φ there corresponds a unique n-bi-ring homomorphism near to φ.

In the next section we present basic concepts and some needed results to construct Hyers-Ulam-Rassias stability of n-bi-Jordan homomorphism between commutative algebras. The conclusion will be presented at the end.

2. Main Results

Let G, H be two abelian groups, X be a complex linear space and $f : G \times H \to X$ a function. For all $(a, b) \in G \times H$, we define the *difference operator* $\Delta_{(a,b)}$ on f by

$$\Delta_{(a,b)} f(x, y) = f(a + x, b + y) - f(x, y),$$

whenever $(x, y) \in G \times H$. Further for all positive integer n and for $(a_i, b_i) \in G \times H$, with $1 \le i \le n$, let

$$\Delta_{(a_1,b_1),\dots,(a_n,b_n)} f = \Delta_{(a_1,b_1)} \dots \Delta_{(a_n,b_n)} f.$$

The function $F : (G \times H)^n \to X$ is called n-bi-additive if F is bi-additive in each of its variables.

For the sake of brevity we use the notation $(G \times H)^0 = G \times H$ and we call constant functions from $G \times H$ to X, 0-bi-additive.

Suppose that $F : (G \times H)^n \to X$ is an arbitrary function. By the trace of F we understand the function $\Phi : G \times H \to X$ arising from F by putting all the variables from $G \times H$ equal, that is,

$$\Phi(x, y) = F[(x, y), \dots, (x, y)], \quad (x, y) \in G \times H.$$

The function $f : G \times H \to X$ is called *bi-polynomial function* of degree at most n, if for all (x, y), $(a_i, b_i) \in G \times H$, with $1 \le i \le n + 1$, the equation

$$\Delta_{(a_1,b_1),\dots,(a_{n+1},b_{n+1})} f(x, y) = 0,$$

is satisfied. For example, the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $f(x, y) = x + y$ is a *bi-polynomial function* of degree at most one.

Lemma 2.1. *Let $F:(G\times H)^n \to X$ be a symmetric and n-biadditive function. Then*

$$\Delta_{(a_1,b_1),\dots,(a_k,b_k)}\Phi(x,y)=\begin{cases} n!F[(a_1,b_1),\dots,(a_n,b_n)] & \text{for } k=n,\\ 0 & \text{for } k>n,\end{cases}$$

whenever $(x,y),(a_1,b_1),\dots,(a_n,b_n)\in G\times H$ and $\Phi:G\times H\to X$ denotes the trace of F.

Proof. The proof is straightforward. □

Now we give a characterization of n-bi-Jordan homomorphism.

Theorem 2.2. *Suppose that $\mathscr{U}$ and $\mathscr{D}$ are two commutative algebra. Then each n-bi-Jordan homomorphism $\varphi:\mathscr{U}\to\mathscr{D}$ is a n-bi-ring homomorphism.*

Proof. Define $F:\mathscr{U}^2\to\mathscr{D}$ by

$$F[(a,b),(x,y)]=\varphi(ax,by)-\varphi(a,b)\varphi(x,y),$$

and let Φ be a trace of F. Since φ is bi-additive, the function F is bi-additive and symmetric, therefore by Lemma 2.1,

$$\Delta_{(a,b),(x,y)}\Phi(u,v)=2F[(a,b),(x,y)],$$

for all $(a,b),(x,y),(u,v)\in\mathscr{U}$.
Now suppose that φ is bi-Jordan homomorphism. Then $\Phi(u,v)=0$, and so

$$2F[(a,b),(x,y)]=\Delta_{(a,b),(x,y)}\Phi(u,v)=0,$$

which proves that $F[(a,b),(x,y)]=0$ for all $(a,b),(x,y)\in\mathscr{U}$. Hence

$$\varphi(ax,by)=\varphi(a,b)\varphi(x,y),$$

for all $(a,b),(x,y)\in\mathscr{U}$. Thus, the result is valid for $n=2$. A similar discussion reveals that the result will be established for $n>2$. □

The following result is Theorem 2.5 and Theorem 2.6 of [15].

Theorem 2.3. *Let $\mathscr{U}$ be a normed algebra, let $\mathscr{D}$ be a Banach algebra, let δ and ε be nonnegative real numbers, and let p,q be a real numbers such that $(p-1)(q-1)>0$, $q\ge 0$, or $(p-1)(q-1)>0$, $q<0$ and $\varphi(0,0)=0$. Assume that $\varphi:\mathscr{U}\to\mathscr{D}$ satisfies*

$$\|\varphi(a+x,b+y)-\varphi(a,b)-\varphi(x,y)\|\le\varepsilon\big(\|(a,b)\|^p+\|(x,y)\|^p\big), \tag{2.1}$$

$$\|\varphi(x^n,y^n)-\varphi(x,y)^n\|\le\delta\|(x,y)\|^{nq}, \tag{2.2}$$

for all $(a,b),(x,y)\in\mathscr{U}$. Then, there exists a unique n-bi-Jordan homomorphism $F:\mathscr{U}\to\mathscr{D}$ such that

$$\|F(x,y)-\varphi(x,y)\|\le\frac{2\varepsilon}{|2-2^p|}\|(x,y)\|^p, \tag{2.3}$$

for all $(x,y)\in\mathscr{U}$.

As a consequence of Theorem 2.2 and 2.3, we have the following.

Corollary 2.4. *By hypotheses of above Theorem, if $\mathscr{U}$ and $\mathscr{D}$ are commutative, then there exists a unique n-bi-ring homomorphism $F:\mathscr{U}\to\mathscr{D}$ such that satisfies* (2.3).

By a same method of [4, Theorem 1.4], we get the following result.

Theorem 2.5. *The function $P: G\times H \to X$ is a bi-polynomial of degree at most n if and only if there exist symmetric, k-bi-additive functions $F_k:(G\times H)^k \to X$, $k=0,1,\dots,n$ such that*

$$P(x,y)=\sum_{k=0}^{n}\Phi_k(x,y),$$

where $\Phi_k: G\times H\to X$ denotes the trace of the function F_k.

Theorem 2.6. *Let G and H be two abelian groups and let X be a locally convex topological linear space. If a bi-polynomial $P: G\times H\to X$ is bounded on $G\times H$, then it is constant.*

Proof. By Theorem 2.5,

$$P(x,y)=\sum_{k=0}^{n}\Phi_k(x,y),$$

where $\Phi_k: G\times H\to X$ denotes the trace of symmetric, k-bi-additive function $F_k:(G\times H)^k\to X$. That is, for $k=0,1,\dots,n$,

$$\Phi_k(x,y)=F_k[(x,y),\dots,(x,y)].$$

Obviously, it is enough to prove that $\Phi_k(x,y)=0$, for all $0\le k\le n$. It follows from Lemma 2.1 that

$$F_n[(x_1,y_1),\dots,(x_n,y_n)]=\frac{1}{n!}\Delta_{(x_1,y_1),\dots,(x_n,y_n)}P(x,y). \tag{2.4}$$

Since the right hand of the equality (2.4) is of the form

$$\sum(-1)^{n-k}P(x+x_{i_1}+\dots+x_{j_k},y+y_{j_1}+\dots+y_{j_k}),$$

where

$$0\le i_1<\dots<i_k<n \quad \text{and} \quad 0\le j_1<\dots<j_k<n,$$

so F_n is bounded.

On the other hand, for $k>0$ the k-bi-additivity of F_k implies that

$$\Phi_k(mx,my)=m^k\Phi_k(x,y),$$

for all $(x,y)\in G\times H$, and for all $m\in\mathbb{N}$. Now assume that $\Phi_k(x_0,y_0)\neq 0$ for some $(x_0,y_0)\in G\times H$. Choose a balanced and absorbing neighborhood $U\subset X$ of the zero such that $\Phi_k(x_0,y_0)\notin U$. As Φ_k is bounded, there is a real λ for which

$$m^k\Phi_k(x_0,y_0)=\Phi_k(mx_0,my_0)\in\lambda U,$$

for all positive integers m. Then $\lambda m^{-k}<1$ for some m, and we have

$$\Phi_k(x_0,y_0)=m^{-k}\Phi_k(mx_0,my_0)\in\lambda m^{-k}U\subset U,$$

which is a contradiction. Thus, $\Phi_k(x,y)=0$ for all $(x,y)\in G\times H$ and $0\le k\le n$. □

Theorem 2.7. *Let $\varphi:\mathscr{U}\to\mathscr{D}$ be a bi-additive function between normed algebras $\mathscr{U}$ and $\mathscr{D}$. Suppose that*

$$\|\varphi(x^n,y^n)-\varphi(x,y)^n\|\le\delta\|(x,y)\|,$$

for some $\delta>0$ and for all $(x,y)\in\mathscr{U}$. Then φ is an n-bi-Jordan homomorphism.

Proof. With the help of the function φ we define the mapping F on $\mathscr{U}^n$ by

$$F[(x_1,y_1),\dots,(x_n,y_n)] = \sum_{\sigma\in S_n} \varphi[(x_1,y_1)_{\sigma(1)}\dots(x_n,y_n)_{\sigma(n)}] - \varphi[(x_1,y_1)_{\sigma(1)}]\dots\varphi[(x_n,y_n)_{\sigma(n)}],$$

where S_n denotes the symmetric group of $\{1,2,\dots,n\}$. Clearly, the function F is symmetric under all permutations of its variables. Due to the bi-additivity of the function φ, the function F is n-bi-additive. So its trace

$$\Phi(x,y) = F[(x,y),\dots,(x,y)] = n![\varphi(x^n,y^n) - \varphi(x,y)^n], \quad (x,y)\in\mathscr{U}.$$

is a bi-polynomial function of degree at most n. On the other hand, from the assumption of the theorem, the function Φ is bounded on $\mathscr{U}$, therefore by Theorem 2.6 we get $\Phi(x,y) = c$, where c is the constant element. Since φ is bi-additive we have $\varphi(0,0) = 0$, hence

$$c = \Phi(0,0) = n![\varphi(0,0) - \varphi(0,0)^n] = 0.$$

Therefore, $\Phi(x,y) = 0$ for all $(x,y)\in\mathscr{U}$. That is,

$$\varphi(x^n,y^n) = \varphi(x,y)^n,$$

holds for all $(x,y)\in\mathscr{U}$. This complete the proof. □

As a consequence of Theorems 2.2 and 2.7 we deduce the next result.

Corollary 2.8. *By hypotheses of Theorem* 2.7*, if $\mathscr{U}$ and $\mathscr{D}$ are commutative, then φ is a n-bi-ring homomorphism.*

3. Conclusion

This paper characterize of n-bi-Jordan homomorphism, and then generalize some well-known results in the area of Hyers-Ulam-Rassias stability of n-bi-Jordan homomorphism between commutative algebras. On the other word, the paper prove that to each approximate n-bi-Jordan homomorphism $\varphi:\mathscr{U}\to\mathscr{D}$ there corresponds a unique n-bi-ring homomorphism near to φ. Concluding remarks, the superstability of n-bi-Jordan homomorphism is also obtained.

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Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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