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Research Article

# On Asymptotically $f$-statistical Equivalent Set Sequences in the Sense of Wijsman

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**Abstract.** The aim of this paper is to introduce a generalization of statistical convergence of asymptotically equivalent set sequences and examine some inclusion relations related to a new concept of Wijsman asymptotically equivalent statistical convergence of sequences of sets with respect to a modulus function $f$.

**Keywords.** Statistical convergence; Sequence space; Modulus function; Asymptotically equivalent set sequences; Wijsman convergence

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## 1. Introduction

The concept of statistical convergence was defined by Steinhaus [23], Fast [9] and later reintroduced by Schoenberg [21], independently. Although statistical convergence was introduced over nearly the last sixty years, it has become an active area of research in recent years.

The notion of a modulus function was introduced by Nakano [13]. Ruckle [20] and Maddox [11] introduced and discussed some properties of sequence spaces defined by using a modulus function. In the year 2014, Aizpuru et al. [1] defined a new concept of density with the help of an unbounded modulus function and, as a consequence, they obtained a new concept of non-matrix convergence, namely, $f$-statistical convergence, which is intermediate between the ordinary convergence and the statistical convergence and agrees with the statistical convergence when the modulus function is the identity mapping. Quite recently, Bhardwaj and Dhawan [3], and Bhardwaj *et al.* [4], have introduced and studied the concepts of $f$-statistical convergence of order $\alpha$ and $f$-statistical boundedness, respectively, by using the approach of Aizpuru *et al.* [1] (see also [5–8]).

In 1993, Marouf [12] presented definitions for asymptotically equivalent and asymptotic regular matrices. In 2003, Patterson [18] extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity for nonnegative summability matrices. In 2006, Patterson and Savaş extended the definitions presented in [19] to lacunary sequences.

The concept of Wijsman statistical convergence is implementation of the concept of statistical convergence to sequences of sets presented by Nuray and Rhoades [16] in 2012. Similar to this concept, the concept of Wijsman lacunary statistical convergence was presented by Ulusu and Nuray [25] in 2012. For further results one may refer to [6, 10, 15–17, 24–29]. The notion of Wijsman statistical convergence has been extended by Bhardwaj *et al.* [7] to that of $f$-Wijsman statistical convergence, where $f$ is an unbounded modulus.

With the aid of modulus functions, we have defined a generalization of statistical convergence of asymptotically equivalent set sequences and obtained some inclusion relations related to this concept.

## 2. Definitions and Preliminaries

In this section, we present some definitions and notations needed throughout the paper. By $\mathbb{N}$ and $\mathbb{R}$, we mean the set of all natural and real numbers, respectively. For brevity, we also mean $\lim_{k\to\infty} x_k$ by the notation $\lim_k x_k$.

**Definition 2.1** ([14])**.** A number sequence $x=(x_k)$ is said to be statistically convergent to the number $\ell$ if for each $\varepsilon>0$ the set $\{k\in\mathbb{N}:|x_k-\ell|\geq\varepsilon\}$ has natural density zero, where the natural density of a subset $K\subset\mathbb{N}$ is defined by $d(K)=\lim_{n\to\infty}|\{k\leq n:k\in K\}|$, where $|\{k\leq n:k\in K\}|$ denotes the number of elements of $K$ not exceeding $n$. Obviously, we have $d(K)=0$ provided that $K$ is a finite set of positive integers. If a sequence is statistically convergent to $\ell$, then we write it as $S$-$\lim_k x_k=\ell$ or $x_k\to\ell(S)$. The set of all statistically convergent sequences is denoted by $S$.

**Definition 2.2** ([12])**.** Two nonnegative sequences $x=(x_k)$ and $y=(y_k)$ are said to be asymptotically equivalent if $\lim_k \frac{x_k}{y_k}=1$ (denoted by $x\sim y$). If the limit is $\ell$, then it will be denoted by $x\overset{\ell}{\sim}y$.

**Definition 2.3** ([18])**.** Two nonnegative sequences $x=(x_k)$ and $y=(y_k)$ are said to be asymptotically statistical equivalent of multiple $\ell$ provided that for every $\varepsilon>0$ $\lim_{n\to\infty}\frac{1}{n}\left|\left\{k\leq n:\left|\frac{x_k}{y_k}-\ell\right|\geq\varepsilon\right\}\right|=0$ (denoted by $x\overset{S_\ell}{\sim}y$).

**Definition 2.4.** Recall that a modulus function $f$ is a function from $[0,\infty)$ to $[0,\infty)$ such that

(1) $f(x)=0 \Leftrightarrow x=0$,

(2) $f(x+y)\le f(x)+f(y)$ for $x\ge 0$, $y\ge 0$,

(3) $f$ is increasing,

(4) $f$ is continuous from the right at 0.

A modulus may be unbounded or bounded. For example, $f(x)=x^p$ where $0<p\le 1$, is unbounded, but $f(x)=\frac{x}{1+x}$ is bounded. It is said that a modulus $f:[0,\infty)\to[0,\infty)$ is slowly varying if the limit relation $\lim\limits_{x\to\infty}\frac{f(ax)}{x}=1$ holds for every $a>0$. All bounded modulus are slowly varying. The function $f(x)=\log(x+1)$ is an example of unbounded, slowly varying modulus (see [22, Chapter 1]).

**Lemma 2.1** ([1, Lemma 3.4])**.** *Let $K$ be an infinite subset of $\mathbb{N}$. Then there is an unbounded, concave and slowly varying modulus $f:[0,\infty)\to[0,\infty)$ such that $d^f(K)=1$.*

**Lemma 2.2** ([11])**.** *Let $f:[0,\infty)\to[0,\infty)$ be a modulus. Then there is a finite $\lim\limits_{t\to\infty}\frac{f(t)}{t}$ and equality*

$$\lim_{t\to\infty}\frac{f(t)}{t}=\inf\left\{\frac{f(t)}{t}:t\in(0,\infty)\right\}$$

*holds.*

**Definition 2.5** ([1])**.** Let $f$ be an unbounded modulus function. The $f$-density of a set $A\subset\mathbb{N}$ is defined by

$$d^f(A)=\lim_{n\to\infty}\frac{f(|\{k\le n:k\in A\}|)}{f(n)}$$

in case this limit exists. Clearly, finite sets have zero $f$-density and $d^f(\mathbb{N}-A)=1-d^f(A)$ does not hold, in general. But if $d^f(A)=0$ then $d^f(\mathbb{N}-A)=1$.

For example, if we take $f(x)=\log(x+1)$ and $A=\{2n:n\in\mathbb{N}\}$, then $d^f(A)=d^f(\mathbb{N}-A)=1$. For any unbounded modulus $f$ and $A\subset\mathbb{N}$, $d^f(A)=0$ implies that $d(A)=0$. But converse need not be true in the sense that a set having zero natural density may have non-zero $f$-density with respect to some unbounded modulus $f$. For example, if we take $f(x)=\log(x+1)$ and $A=\{1,4,9,\dots\}$, then $d(A)=0$ but $d^f(A)=1/2$. However, $d(A)=0$ implies $d^f(A)=0$ is always true in case of any finite set $A\subset\mathbb{N}$, irrespective of the choice of unbounded modulus $f$ (see [3]).

**Definition 2.6** ([1])**.** Let $f$ be an unbounded modulus function. A number sequence $x=(x_k)$ is said to be $f$-statistically convergent to $\ell$ or $S^f$-convergent to $\ell$, if for each $\varepsilon>0$

$$d^f(\{k\in\mathbb{N}:|x_k-\ell|\ge\varepsilon\})=0,$$

that is,

$$\lim_{n\to\infty}\frac{f(|\{k\le n:|x_k-\ell|\ge\varepsilon\}|)}{f(n)}=0,$$

and one writes it as $S^f\text{-}\lim\limits_k x_k=l$ or $x_k\to l(S^f)$. The set of all $f$-statistically convergent sequences is denoted by $S^f$.

Let $(X,\rho)$ be a metric space. For any point $x \in X$ and any nonempty subset $A$ of $X$, we define the distance from $x$ to $A$ by

$$d(x,A) = \inf_{a\in A} \rho(x,a).$$

**Definition 2.7** ([2]). Let $(X,\rho)$ be a metric space. For any nonempty closed subsets $A, A_k \subseteq X$, it is said that the sequence $\{A_k\}$ is Wijsman convergent to $A$ if

$$\lim_{k\to\infty} d(x,A_k) = d(x,A)$$

for each $x \in X$ and it is denoted by $W\text{-}\lim A_k = A$.

**Definition 2.8** ([16]). Let $(X,\rho)$ be a metric space and let $A, A_k$ be any nonempty closed subsets of $X$. It is said that the sequence $\{A_k\}$ is Wijsman statistically convergent to $A$ if $d(x,A_k)$ is statistically convergent to $d(x,A)$, that is, for $\varepsilon > 0$ and for each $x \in X$

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |d(x,A_k) - d(x,A)| \ge \varepsilon\}| = 0.$$

In this case, it is written by $[WS]$-lim $A_k = A$ or $A_k \to A(WS)$.

**Definition 2.9** ([16]). Let $(X,\rho)$ be a metric space. For any nonempty closed subsets $A_k$ of $X$, it is said that the sequence $\{A_k\}$ is bounded if $\sup_k d(x,A_k) < \infty$ for each $x \in X$. In this case, it is written as $\{A_k\} \in L_\infty$, where $L_\infty$ denotes the set of all bounded sequences of sets.

**Definition 2.10** ([27]). Let $(X,\rho)$ be a metric space. For any nonempty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$. It is said that the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically equivalent, in the sense of Wijsman, if for each $x \in X$,

$$\lim_k \frac{d(x,A_k)}{d(x,B_k)} = 1$$

(denoted by $A_k \overset{W}{\sim} B_k$).

As an example, consider the following sequences of circles in the $(x,y)$-plane:

$$A_k = \{(x,y) : x^2 + y^2 + 2kx = 0\}, \qquad B_k = \{(x,y) : x^2 + y^2 - 2kx = 0\}.$$

Since

$$\lim_k \frac{d(x,A_k)}{d(x,B_k)} = 1,$$

the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically equivalent, in the sense of Wijsman, that is, $A_k \overset{W}{\sim} B_k$.

**Definition 2.11** ([27, Definition 11]). Let $(X,\rho)$ be a metric space. For any nonempty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$. It is said that the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically statistical equivalent (Wijsman sense) of multiple $L$ if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \le n : \left| \frac{d(x,A_k)}{d(x,B_k)} - L \right| \ge \varepsilon \right\} \right| = 0$$

(denoted by $A_k \overset{WS_L}{\sim} B_k$) and simply asymptotically statistical equivalent, in the sense of Wijsman, if $L = 1$.

Now, we present the following definition which has been defined by Bhardwaj *et al.* [7].

**Definition 2.12** ([7]). Let $(X,\rho)$ be a metric space and $f:[0,\infty)\to[0,\infty)$ be an unbounded modulus. For any nonempty closed subsets $A, A_k$ of $X$, the sequence $\{A_k\}$ is said to be $f$-Wijsman statistically convergent to $A$ if the sequence $(d(x,A_k))$ is $f$-statistically convergent to $(d(x,A))$ for each $x\in X$ and for every $\varepsilon>0$, that is

$$\lim_{n\to\infty}\frac{1}{f(n)}f(|\{k\le n:|d(x,A_k)-d(x,A)|\ge\varepsilon\}|)=0$$

(denoted by $[WS^f]$-lim $A_k = A$) if $(A_k)$ is $f$-Wijsman statistically convergent to $A$.

**Definition 2.13** ([16]). Let $(X,\rho)$ be a metric space. For any nonempty closed subsets $A, A_k$ of $X$, the sequence $\{A_k\}$ is said to be Wijsman strongly Cesaro summable to $A$ if $(d(x,A_k))$ is strongly summable to $d(x,A)$; i.e., for each $x\in X$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|d(x,A_k)-d(x,A)|=0.$$

## 3. Main Results

Following the previous results, we introduce the new notions called Wijsman $f$-asymptotically statistical equivalence, strongly Cesaro $f$-asymtotically equivalence in the sense of Wijsman and examine some relations related to these concepts.

**Definition 3.1.** Let $(X,\rho)$ be a metric space. For any nonempty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k)>0$ and $d(x,B_k)>0$ for each $x\in X$ and an unbounded modulus function $f:[0,\infty)\to[0,\infty)$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are $f$-asymptotically statistical equivalent, in the sense of Wijsman, of multiple $L$ if for every $\varepsilon>0$ and for each $x\in X$,

$$\lim_{n\to\infty}\frac{1}{f(n)}f\left(\left|\left\{k\le n:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\ge\varepsilon\right\}\right|\right)=0$$

(denoted by $\{A_k\}\overset{WS_L^f}{\sim}\{B_k\}$) and simply $f$-asymptotically statistical equivalent, in the sense of Wijsman, if $L=1$.

For $f(x)=x$, Definition 3.1 reduces to the Definition 2.11. Then one may consider the following sequences which are asymptotically statistical equivalent (Wijsman sense), for $f(x)=x$, that is, $\{A_k\}\overset{WS}{\sim}\{B_k\}$

$$A_k=\begin{Bmatrix}(x,y):x^2+y^2-2ky=0, & \text{if } k \text{ is a square integer}\\ \{(1,1)\}, & \text{otherwise}\end{Bmatrix},$$

$$B_k=\begin{Bmatrix}(x,y):x^2+y^2+2ky=0, & \text{if } k \text{ is a square integer}\\ \{(1,1)\}, & \text{otherwise}\end{Bmatrix}.$$

**Theorem 3.1.** *Let $f:[0,\infty)\to[0,\infty)$ be an unbounded modulus function, $(X,\rho)$ be a metric space and let $A_k, B_k$ be non-empty closed subsets of $X$. If $\{A_k\}\overset{WS_L^f}{\sim}\{B_k\}$ then $\{A_k\}\overset{WS_L}{\sim}\{B_k\}$.*

*Proof.* The proof can be done in a similar manner as in unpublished paper [7].
For all $x\in X$, $\varepsilon>0$ and $n\in\mathbb{N}$ let us write

$$K_{x,\varepsilon}(n):=\left\{k\le n:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\ge\varepsilon\right\}. \tag{3.1}$$

If $\{A_k\}$ and $\{B_k\}$ are not asymptotically statistical equivalent (Wijsman sense) of multiple $L$, then there are $x\in X$ and $\varepsilon>0$ such that

$$\limsup_{n\to\infty}\frac{|K_{x,\varepsilon}(n)|}{n}>0.$$

Hence there exist $p\in\mathbb{N}$ and a sequence $(n_m)\subset\mathbb{N}$ such that

$$\lim_{m\to\infty}n_m=\infty \tag{3.2}$$

and $\frac{|K_{x,\varepsilon}(n_m)|}{n_m}\ge\frac{1}{p}$ for every $m\in\mathbb{N}$. From the last inequality we have

$$n_m\le p|K_{x,\varepsilon}(n_m)|. \tag{3.3}$$

Using the subadditivity of $f$ and (3.3), we obtain

$$f(n_m)\le pf(|K_{x,\varepsilon}(n_m)|).$$

Consequently the inequality

$$\frac{f(|K_{x,\varepsilon}(n_m)|)}{f(n_m)}\ge\frac{1}{p} \tag{3.4}$$

holds for every $m\in\mathbb{N}$. Equality (3.2) and inequality (3.4) imply

$$\limsup_{n\to\infty}\frac{f(|K_{x,\varepsilon}(n_m)|)}{f(n_m)}\ge\frac{1}{p},$$

contrary to $\{A_k\}\overset{WS_L^f}{\sim}\{B_k\}$. □

**Theorem 3.2.** *Let $(X,\rho)$ be a metric space and $f,g$ be unbounded modulus functions. Then for all $A_k,B_k$ non-empty closed subsets of $X$,*

$$\{A_k\}\overset{WS_{L_1}^f}{\sim}\{B_k\}\text{ and }\{A_k\}\overset{WS_{L_2}^g}{\sim}\{B_k\} \tag{3.5}$$

*imply $L_1=L_2$.*

*Proof.* Let $(X,\rho)$ be a metric space, $A_k,B_k$ be non-empty closed subsets of $X$ and let (3.5) hold. From Theorem 3.1, $\{A_k\}$ and $\{B_k\}$ are asymptotically statistical equivalent both of multiple $L_1$ and $L_2$, that is,

$$\{A_k\}\overset{WS_{L_1}^f}{\sim}\{B_k\}\Rightarrow\{A_k\}\overset{WS_{L_1}}{\sim}\{B_k\}$$

and

$$\{A_k\}\overset{WS_{L_2}^g}{\sim}\{B_k\}\Rightarrow\{A_k\}\overset{WS_{L_2}}{\sim}\{B_k\}.$$

Using the uniqueness of statistical limits of numerical sequences we obtain that $L_1=L_2$. □

**Corollary 3.3.** *Let $(X,\rho)$ be a metric space and $A_k, B_k$ be non-empty closed subsets of $X$. Then for every unbounded modulus $f:[0,\infty)\to[0,\infty)$, the set sequences $\{A_k\}$ and $\{B_k\}$ are $f$-asymptotically statistical equivalent of an unique multiple $L$ if it exists.*

**Theorem 3.4.** *Let $(X,\rho)$ be a metric space. Then for all $A_k, B_k$ be non-empty closed subsets of $X$. Then the following statements are equivalent*

(1) $\{A_k\}\overset{W}{\sim}\{B_k\}$,

(2) $\{A_k\}\overset{WS^f}{\sim}\{B_k\}$ *holds for every unbounded modulus $f$,*

(3) $\{A_k\}\overset{WS^f}{\sim}\{B_k\}$ *holds for every $f\in MUCS$ (The set of all unbounded, concave and slowly varying moduli).*

*Proof.* $(1)\Rightarrow(2)$: Let (1) hold. Since $\{A_k\}\overset{W}{\sim}\{B_k\}$, the set

$$K_{x,\varepsilon}:=\left\{k\in\mathbb{N}:\left|\frac{d(x,A_k)}{d(x,B_k)}-1\right|\geq\varepsilon\right\}$$

is finite for all $x\in X$ and $\varepsilon>0$. Let $f:[0,\infty)\to[0,\infty)$ be an unbounded modulus. The equality

$$\lim_{n\to\infty}\frac{f(|K_{x,\varepsilon}|)}{f(n)}=0,$$

holds because $f$ is unbounded and increasing. Thus $\{A_k\}\overset{WS^f}{\sim}\{B_k\}$ holds.

$(2)\Rightarrow(3)$: It is trivial.

$(3)\Rightarrow(1)$: Let (3) holds. Suppose $\{A_k\}$ and $\{B_k\}$ are not asymptotically equivalent, in the sense of Wijsman, for all $x\in X$. Then the set $K_{x,\varepsilon}$ is infinite for some $x\in X$ and $\varepsilon>0$. Now by Lemma 2.1 there exists $f\in MUCS$ such that $d^f(K_{x,\varepsilon})=1$ which contradicts $\{A_k\}\overset{WS^f}{\sim}\{B_k\}$. □

**Definition 3.2.** Let $(X,\rho)$ be a metric space and $f$ be an unbounded modulus. For any nonempty closed subsets $A_k, B_k\subseteq X$ such that $d(x,A_k)>0$ and $d(x,B_k)>0$ for each $x\in X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are strongly Cesaro $f$-asymptotically equivalent (in the sense of Wijsman) of multiple $L$ if for each $x\in X$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f\left(\left|\frac{d(x,A_k)}{d(x,A_k)}-L\right|\right)=0 \tag{3.6}$$

(denoted by $\{A_k\}\overset{[Ces_L^W(f)]}{\sim}\{B_k\}$) and simply strongly Cesaro $f$-asymptotically equivalent, in the sense of Wijsman, if $L=1$.

If $f(x)=x$ in (3.6), then the limit reduces to

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|\frac{d(x,A_k)}{d(x,A_k)}-L\right|=0,$$

for each $x\in X$, (denoted by $\{A_k\}\overset{[Ces_L^W]}{\sim}\{B_k\}$) and we say that the sequences $\{A_k\}$ and $\{B_k\}$ are strongly Cesaro asymptotically equivalent (in the sense of Wijsman) of multiple $L$.

**Theorem 3.5.** *Let $(X,\rho)$ be a metric space and $A_k, B_k$ be non-empty closed subsets of $X$.*

*If $f:[0,\infty)\to[0,\infty)$ is a modulus such that $\beta=\lim_{t\to\infty}\frac{f(t)}{t}>0$ and $\{A_k\}\overset{[Ces_L^W(f)]}{\sim}\{B_k\}$ then $\{A_k\}\overset{[Ces_L^W]}{\sim}\{B_k\}$.*

*Proof.* Let a modulus $f$ satisfy the condition $\beta=\lim_{t\to\infty}\frac{f(t)}{t}>0$ and let $\{A_k\}\overset{[Ces_L^W(f)]}{\sim}\{B_k\}$. By Lemma 2.2 we have $\beta=\inf\left\{\frac{f(t)}{t}:t>0\right\}$. Consequently

$$f(t)\ge\beta t \tag{3.7}$$

holds for every $t>0$. For every $x\in X$ it follows from (3.7) that

$$\frac{1}{n}\sum_{k=1}^{n}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\le\beta^{-1}\frac{1}{n}\sum_{k=1}^{n}f\left(\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\right).$$

Since $\{A_k\}\overset{[Ces_L^W(f)]}{\sim}\{B_k\}$, then we have $\{A_k\}\overset{[Ces_L^W]}{\sim}\{B_k\}$ from the inequality given above. □

**Theorem 3.6.** *Let $(X,\rho)$ be a metric space and $A_k,B_k$ be non-empty closed subsets of $X$. Assume that $f:[0,\infty)\to[0,\infty)$ is an unbounded modulus satisfying the inequalities*

$$\lim_{t\to\infty}\frac{f(t)}{t}>0\quad\text{and}\quad f(xy)\ge cf(x)f(y) \tag{3.8}$$

*with some $c\in(0,\infty)$ for all $x,y\in[0,\infty)$. Then the following statements hold*

(1) *If $\{A_k\}\overset{[Ces_L^W(f)]}{\sim}\{B_k\}$ then $\{A_k\}\overset{WS_L^f}{\sim}\{B_k\}$.*

(2) *If $\{A_k\}\in L_\infty$ and $\{A_k\}\overset{WS_L^f}{\sim}\{B_k\}$ then $\{A_k\}\overset{[Ces_L^W(f)]}{\sim}\{B_k\}$.*

*Proof.* Let $K_{x,\varepsilon}(n)$ be defined as in (3.1) for all $x\in X$, $\varepsilon\in(0,\infty)$ and $n\in\mathbb{N}$.

(1) Let $\{A_k\}\overset{[Ces_L^W(f)]}{\sim}\{B_k\}$. By subadditivity of moduli we have

$$\sum_{k=1}^{n}f\left(\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\right)\ge f\left(\sum_{k=1}^{n}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\right)\ge f\left(\sum_{\substack{k=1\\k\in K_{x,\varepsilon}(n)}}^{n}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\right).$$

Using the second inequality from (3.8) we obtain

$$f\left(\sum_{\substack{k=1\\k\in K_{x,\varepsilon}(n)}}^{n}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\right)\ge f(|K_{x,\varepsilon}(n)|\varepsilon)\ge cf(|K_{x,\varepsilon}(n)|)f(\varepsilon).$$

Hence

$$\frac{1}{n}\sum_{k=1}^{n}f\left(\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\right)\ge c\,\frac{f(|K_{x,\varepsilon}(n)|)}{f(n)}\frac{f(n)}{n}f(\varepsilon).$$

Hence the result is obtained from the inequality above.

(2) Let $\{A_k\}$ be Wijsman bounded and let $\{A_k\}\overset{WS_L^f}{\sim}\{B_k\}$. Since $\{A_k\}$ is Wijsman bounded we may assume that there exists a $M>0$ such that

$$\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\le M \tag{3.9}$$

for each $x \in X$ and all $k$. Given $\varepsilon > 0$, we obtain

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) = \frac{1}{n}\sum_{\substack{k=1\\ k\in K_{x,\varepsilon}(n)}}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) + \frac{1}{n}\sum_{\substack{k=1\\ k\notin K_{x,\varepsilon}(n)}}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)$$

$$\le \frac{|K_{x,\varepsilon}(n)|}{n} f(M) + \frac{1}{n} n f(\varepsilon).$$

Letting $n \to \infty$, we get

$$0 \le \lim_{n} \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \le f(\varepsilon)$$

in view of Theorem 3.1 and (3.9). Hence the result is obtained. □

We present the following corollaries as a result of Theorem 3.6, Theorem 3.5 and Theorem 3.1.

**Corollary 3.7.** *Let $(X,\rho)$ be a metric space and $A_k, B_k$ be non-empty closed subsets of $X$. Assume that $f:[0,\infty)\to[0,\infty)$ is an unbounded modulus satisfying the inequalities*

$$\lim_{t\to\infty} \frac{f(t)}{t} > 0 \quad \textit{and} \quad f(xy) \ge c f(x) f(y) \tag{3.10}$$

*with some $c \in (0,\infty)$ for all $x, y \in [0,\infty)$. If $\{A_k\} \overset{[Ces_L^W(f)]}{\sim} \{B_k\}$ then $\{A_k\} \overset{WS_L}{\sim} \{B_k\}$.*

**Corollary 3.8.** *Let $(X,\rho)$ be a metric space and $A_k, B_k$ be non-empty closed subsets of $X$. Assume that $f:[0,\infty)\to[0,\infty)$ is an unbounded modulus. If $\{A_k\} \in L_\infty$ and $\{A_k\} \overset{WS_L(f)}{\sim} \{B_k\}$ then $\{A_k\} \overset{[Ces_L^W]}{\sim} \{B_k\}$.*

**Theorem 3.9.** *Let $(X,\rho)$ be a metric space and $A_k, B_k$ be non-empty closed subsets of $X$ and let $f:[0,\infty)\to[0,\infty)$ be an unbounded modulus. If $\{A_k\} \overset{[Ces_L^W]}{\sim} \{B_k\}$ then $\{A_k\} \overset{[Ces_L^W(f)]}{\sim} \{B_k\}$.*

*Proof.* Suppose that

$$\lim_{n} \frac{1}{n}\sum_{k=1}^{n} \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| = 0 \tag{3.11}$$

holds for each $x \in X$. Let $\varepsilon > 0$ and choose $\delta \in (0,1)$ such that $f(t) < \varepsilon$ for all $t \in [0,\delta]$. Consider

$$\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) = \sum_1 + \sum_2,$$

where the first summation is over the set $\left\{k \le n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \le \delta\right\}$ and the second is the over $\left\{k \le n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| > \delta\right\}$. It is clear that $\sum_1 \le n\varepsilon$. To estimate $\sum_2$ we use the inequalities

$$\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| < \frac{\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|}{\delta} \le \left[\left|\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\delta^{-1}\right|\right]$$

where $[|.|]$ is the ceiling function (which gives the smallest integer equal or greater than its value). The modulus functions are increasing and subadditive. Hence

$$f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \le f(1)\left[\left|\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\delta^{-1}\right|\right] \le 2f(1)\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\delta^{-1}$$

holds whenever $\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|>\delta$. Thus we have

$$\sum_2 \le 2f(1)\delta^{-1}\sum_{k=1}^{n}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|,$$

which together with $\sum_1 \le n\varepsilon$ yields

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\right)\le \varepsilon+2f(1)\delta^{-1}\frac{1}{n}\sum_{k=1}^{n}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|.$$

From (3.11), we obtain

$$0\le \lim_n \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\right)\le \varepsilon$$

and hence the result. □

In view of Theorem 3.5 and Theorem 3.9 we have the following corollary.

**Corollary 3.10.** *Let $(X,\rho)$ be a metric space and $A_k, B_k$ be non-empty closed subsets of $X$ and let $f:[0,\infty)\to[0,\infty)$ be an unbounded modulus such that $\lim_{t\to\infty}\frac{f(t)}{t}>0$. Then $\{A_k\}\overset{[Ces_L^W(f)]}{\sim}\{B_k\}\Leftrightarrow \{A_k\}\overset{[Ces_L^W]}{\sim}\{B_k\}$.*

## 4. Concluding Remarks

In this work, we introduce a generalization of statistical convergence of asymptotically equivalent set sequences called Wijsman $f$-asymptotically statistical equivalence, strongly Cesaro $f$-asymptotically equivalence in the sense of Wijsman and examine some connections related to these concepts under some conditions depending $f$.

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### Competing Interests

The authors declare that they have no competing interests.

### Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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