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Research Article

A New Approach for the Solution of Space-Time Fractional Order Heat-Like Partial Differential Equations by Residual Power Series Method

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Abstract. The main concern of this article has been to apply the *Residual Power Series Method* (RPSM) effectively to find the exact solutions of fractional-order space-time dependent nonhomogeneous partial differential equations in the Caputo sense. Our first step is to reduce fractional-order space-time dependent non-homogeneous partial differential equations to fractional-order space-time dependent homogeneous partial differential equations before applying the proposed method. Obtaining fractional power series solutions of the problem and reproducing the exact solution is the main step. The illustrative examples reveal that RPSM is a very significant and powerful method for obtaining the solution of any-order time-space fractional non-homogeneous partial differential equations in the form of fractional power series.

Keywords. Residual power series method; Space-time fractional partial differential equations; Caputo derivative

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1. Introduction

Fractional calculus, including integrals and derivatives of arbitrary order, is a generalization of classical integer-order differentiation and integration [19]. By using fractional calculus, some models in different branches of science and engineering such as fluid mechanics, electric network, signal processing, control theory of dynamical system, image processing, optics, visco-elasticity [10, 11, 17, 20–22] can be described more reasonably and applicably. To find an approximate or analytical solutions of nonlinear fractional partial differential equations various methods are available in our literature like Adomian's decomposition methods [18], Laplace decomposition method [26], homotopy perturbation method [13], homotopy analysis method [1, 25], homotopy analysis transform method [12, 14] and Differential transform method [23]. Among these, *Residual Power Series Method* (RPSM) is a new algorithm. RPSM (proposed by the Jordan mathematician Arqub [5]) was developed as an efficient method for determining values of coefficients of the power series solution for fuzzy differential equations. This method is based on constructing power series expansion solution for different nonlinear equations without linearization, perturbation, or discretization [2–4, 6–9, 15, 16, 24]. With the help of residual error concepts, this method computes the coefficient of the power series by a chain of algebraic equations of one or more variables and finally we get a series solution, in practice a truncated series solution. The main advantage of this method over the other method is it can be applied directly to the given problem by choosing an appropriate initial guess approximation.

2. Preliminaries

We first give the main definitions and various features of the fractional calculus theory in this section.

Definition 1. The Riemann-Liouville fractional integral operator of order α ($\alpha \ge 0$) is defined as

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0,\ x > 0,$$
$$J^0 f(x) = f(x).$$

Definition 2. The Caputo fractional derivatives of order α are defined as

$$D^{\alpha} f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \quad m-1 < \alpha \le m,\ x > 0,$$

where D^m is the classical differential operator of order m.

For the Caputo derivative, we have

$$D^{\alpha} x^{\beta} = 0, \quad \beta < \alpha,$$
$$D^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, \quad \beta \ge \alpha.$$

Definition 3. Let n be the smallest integer greater than α, the Caputo time fractional derivative operator of order α of $u(x,t)$ is defined as [2, 3, 7–9, 15, 16, 24]

$$D_t^{\alpha} u(x,t) = \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x,\tau)}{\partial t^n} d\tau, & n-1 < \alpha \le n \\ \frac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n \in N \end{cases}$$

and the space fractional derivative of order β of $u(x,t)$ is defined as

$$D_x^{\beta} u(x,t) = \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} = \begin{cases} \frac{1}{\Gamma(n-\beta)} \int_0^x (x-\tau)^{n-\beta-1} \frac{\partial^n u(x,\tau)}{\partial x^n} d\tau, & n-1 < \beta \le n \\ \frac{\partial^n u(x,t)}{\partial x^n}, & \beta = n \in N. \end{cases}$$

Definition 4. The power series expansions about $t = t_0$ and $x = x_0$

$$\sum_{k=0}^{\infty}\sum_{l=0}^{m-1} f_{kl}(x)(t-t_0)^{k\alpha+l}, \quad 0 \le m-1 < \alpha \le m,\ t \ge t_0$$

and

$$\sum_{k=0}^{\infty}\sum_{l=0}^{n-1} g_{kl}(t)(x-x_0)^{k\alpha+l}, \quad 0 \le n-1 < \alpha \le n,\ x \ge x_0$$

are called multiple fractional power series, where $f_{kl}(x)$ and $g_{kl}(t)$ are called the coefficients of the series.

3. Construction of RPSM for Space-time FPDE

Case A. Consider the following space-time FPDE

$$D_t^{\alpha} u = D_x^{\beta} u + f(x), \quad m-1 < \alpha \le m,\ n-1 < \beta \le n \tag{1}$$

subject to the initial condition

$$u(x,0) = \varphi(x). \tag{2}$$

Applying the transformation

$$u(x,t) = D_x^{2-\beta} v(x,t) - I_x^{\beta} f(x) \tag{3}$$

then we have

$$D_t^{\alpha}(D_x^{2-\beta} v) = D_{xx} v\,, \tag{4}$$

$$v(x,0) = D_x^{\beta-2}\varphi(x) + I_x^2 f(x). \tag{5}$$

Our purpose is to construct a power series solution for eqs. (4) and (5) by its power series expansion among its truncated residual function. The main steps of this procedure are shown as follows:

Step 1. Suppose that the solution of eqs. (4) and (5) is expressed in the form of fractional power series expansion about the initial point $t = 0$ as follows:

$$v(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{m-1} f_{ij}(x)\frac{t^{i\alpha+j}}{\Gamma(1+i\alpha+j)}, \quad m-1 < \alpha \le m,\ x \in R,\ t \ge 0.$$

The RPSM guarantees that the analytical approximate solution for eqs. (4) and (5) are in the form of an infinite multiple fractional power series. To get the numerical values from this series, let $v_{kl}(x,t)$ denotes the (k,l)-truncated series of $v(x,t)$. That is,

$$v_{kl}(x,t) = \sum_{i=0}^{k}\sum_{j=0}^{l} f_{ij}(x)\frac{t^{i\alpha+j}}{\Gamma(1+i\alpha+j)}, \quad m-1 < \alpha \le m,\ x \in R,\ t \ge 0.$$

Take the initial condition, the residual power series approximate solution of $v(x,t)$ can be written in the following form:

$$v_{00}(x,t) = f_{00}(x) = v(x,0) = D_x^{\beta-2}\varphi(x) + I_x^2 f(x).$$

As a result, we can reformulate the expansion form as

$$v_{kl}(x,t) = D_x^{\beta-2}\varphi(x) + I_x^2 f(x) + \sum_{i=1}^{k}\sum_{j=0}^{l} f_{ij}(x)\frac{t^{i\alpha+j}}{\Gamma(1+i\alpha+j)}, \quad m-1 < \alpha \le m,\ x \in R,\ t \ge 0$$

where $k = 1,2,3,\dots$ and $l = 0,1,2,\dots,m-1$.

Step 2. Define the residual function for eqs. (4) and (5) as follows:

$$Res(x,t) = D_t^{\alpha}(D_x^{2-\beta} v(x,t)) - D_{xx} v(x,t)$$

and the (k,l)-truncated residual function can be expressed as

$$Res_{kl}(x,t) = D_t^{\alpha}(D_x^{2-\beta} v_{kl}(x,t)) - D_{xx} v_{kl}(x,t).$$

As in [2, 3, 7–9, 15, 16, 24], some useful results in the residual power series solution are stated as follows:

(1) $Res(x,t) = 0$

(2) $\lim_{k\to\infty} Res_k(x,t) = Res(x,t)$

(3) $D_t^{(i-1)\alpha} D_t^j Res(x,0) = D_t^{(i-1)\alpha} D_t^j Res_{ij}(x,0) = 0,\ i = 1,2,3,\dots,k,\ j = 0,1,2,\dots,l$

Step 3. Substitute the (k,l)-truncated series of $v(x,t)$ into $Res_{kl}(x,t)$ and calculate the fractional derivative at $t = 0$, the following algebraic system is obtained:

$$D_t^{(k-1)\alpha} D_t^l Res_{kl}(x,0) = 0,\ m-1 < \alpha \le m,\ k = 1,2,3,\dots,\ l = 0,1,2,\dots,m-1$$

Step 4. After solving the above system, the values of the coefficients $f_{ij}(x)$, $i = 1,2,3,\dots,k$, $j = 0,1,2,\dots,l$ are obtained. Thus, the (k,l) residual power series approximate solution is derived.

In the next discussion, the residual power series approximate solutions are determined by the following the above steps for $m = 1$.

To determine form of the first unknown coefficient $f_{10}(x)$, we should substitute the $(1,0)$-truncated series

$$v_{10}(x,t) = D_x^{\beta-2}\varphi(x) + I_x^2 f(x) + f_{10}(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}$$

into the $(1,0)$-truncated residual function

$$Res_{10}(x,t) = D_t^{\alpha}(D_x^{2-\beta} v_{10}(x,t)) - D_{xx} v_{10}(x,t).$$

Then, we have

$$Res_{10}(x,t) = D_x^{2-\beta} f_{10}(x) - D_x^{\beta}\varphi(x) - f''_{10}(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}.$$

From *Step* 3 of *Case* A, the substituting of $t = 0$ back into $Res_{10}(x,t)$ will yields

$$f_{10}(x) = D_x^{2\beta-2}\varphi(x) + I_x^{2-\beta} f(x).$$

Again, to find out form of the second unknown coefficient $f_{20}(x)$, we substitute the $(2,0)$-truncated series solution

$$v_{20}(x,t) = D_x^{\beta-2}\varphi(x) + I_x^2 f(x) + \left(D_x^{2\beta-2}\varphi(x) + I_x^{2-\beta} f(x)\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + f_{20}(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

into the $(2,0)$-truncated residual function

$$Res_{20}(x,t) = D_t^{\alpha}(D_x^{2-\beta} v_{20}(x,t)) - D_{xx} v_{20}(x,t)$$

Then, we obtain

$$Res_{20}(x,t) = D_x^{\beta}\varphi(x) + f(x) + D_x^{2-\beta} f_{20}(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)} - D_x^{\beta}\varphi(x) - f(x)$$

$$-\left(D_x^{2\beta}\varphi(x)+D_x^{\beta}f(x)\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)}-f_{20}''(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}.$$

From *Step* 3 of *Case* A, we have $D_t^{\alpha}Res_{20}(x,0)=0$. Solving the resulting equation for $f_{20}(x)$, we have

$$f_{20}(x)=D_x^{3\beta-2}\varphi(x)+I_x^{2-2\beta}f(x).$$

Therefore, collecting the previous results, the RPS solution of eqs. (4) and (5) can be constructed in the form of fractional power series as follows:

$$v(x,t)=D_x^{\beta-2}\varphi(x)+I_x^{2}f(x)+\left(D_x^{2\beta-2}\varphi(x)+I_x^{2-\beta}f(x)\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)}$$
$$+\left(D_x^{3\beta-2}\varphi(x)+I_x^{2-2\beta}f(x)\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+\left(D_x^{4\beta-2}\varphi(x)+I_x^{2-3\beta}f(x)\right)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\dots.$$

Again, applying the transformation (3), we obtain

$$u(x,t)=\varphi(x)+\left(D_x^{\beta}\varphi(x)+f(x)\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)}+\left(D_x^{2\beta}\varphi(x)+D_x^{\beta}f(x)\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$
$$+\left(D_x^{3\beta}\varphi(x)+D_x^{2\beta}f(x)\right)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\dots.$$

Case B. Consider the following space-time FPDE

$$D_t^{\alpha}u=D_x^{\beta}u+g(t),\quad m-1<\alpha\le m,\ n-1<\beta\le n \tag{6}$$

subject to the nonhomogeneous initial condition

$$u(0,t)=\mu_1(t),\quad u_x(0,t)=\mu_2(t). \tag{7}$$

Applying the transformation

$$u(x,t)=D_t^{1-\alpha}v(x,t)+I_t^{\alpha}g(t) \tag{8}$$

then, we have

$$D_t v=D_x^{\beta}(D_t^{1-\alpha}v), \tag{9}$$

$$v(0,t)=D_t^{\alpha-1}\mu_1(t)-I_t g(t),\quad v_x(0,t)=D_t^{\alpha-1}\mu_2(t). \tag{10}$$

The main steps to construct the residual power series solution for eqs. (9) and (10) are shown as follows:

Step 1. Suppose that the solution of eqs. (9) and (10) is expressed in the form of fractional power series expansion about the initial point $x=0$ as follows:

$$v(x,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{n-1}g_{ij}(t)\frac{x^{i\beta+j}}{\Gamma(1+i\beta+j)},\quad n-1<\beta\le n,\ t\in R,\ x\ge 0.$$

The RPSM guarantees that the analytical approximate solution for eqs. (9) and (10) are in the form of an infinite multiple fractional power series. To get the numerical values from this series, let $v_{kl}(x,t)$ denotes the (k,l)-truncated series of $v(x,t)$. That is,

$$v_{kl}(x,t)=\sum_{i=0}^{k}\sum_{j=0}^{l}g_{ij}(t)\frac{x^{i\beta+j}}{\Gamma(1+i\beta+j)},\quad n-1<\beta\le n,\ t\in R,\ x\ge 0.$$

Take the nonhomogeneous initial condition, the residual power series approximate solution of $v(x,t)$ can be written in the following form:

$$v_{01}(x,t) = g_{00}(t) + g_{01}(t)x = v(0,t) + v_x(0,t)x = D_t^{\alpha-1}\mu_1(t) - I_t g(t) + D_t^{\alpha-1}\mu_2(t)x$$

As a result, we can reformulate the expansion form as

$$v_{kl}(x,t) = D_t^{\alpha-1}\mu_1(t) - I_t g(t) + D_t^{\alpha-1}\mu_2(t)x + \sum_{i=1}^{k}\sum_{j=0}^{l} g_{ij}(t)\frac{x^{i\beta+j}}{\Gamma(1+i\beta+j)},\ n-1<\beta\le n,\ t\in R,\ x\ge 0$$

where $k = 1,2,3,\dots$ and $l = 0,1,2,\dots,n-1$.

Step 2. Define the residual function for eqs. (9) and (10) as follows:

$$Res(x,t) = D_t v(x,t) - D_x^{\beta}(D_t^{1-\alpha} v(x,t))$$

and the (k,l)-truncated residual function can be expressed as

$$Res_{kl}(x,t) = D_t v_{kl}(x,t) - D_x^{\beta}(D_t^{1-\alpha} v_{kl}(x,t))$$

As in [2, 3, 7–9, 15, 16, 24], some useful results in the residual power series solution are stated as follows:

(1) $Res(x,t) = 0$

(2) $\lim_{k\to\infty} Res_k(x,t) = Res(x,t)$

(3) $D_x^{(i-1)\beta} D_x^{j} Res(0,t) = D_x^{(i-1)\beta} D_x^{j} Res_{ij}(0,t) = 0,\ i = 1,2,3,\dots,k,\ j = 0,1,2,\dots,l$

Step 3. Substitute the (k,l)-truncated series of $v(x,t)$ into $Res_{kl}(x,t)$ and calculate the fractional derivative at $x = 0$, the following algebraic system is obtained:

$$D_x^{(k-1)\alpha} D_x^{l} Res_{kl}(0,t) = 0, \quad n-1<\beta\le n,\ k = 1,2,3,\dots,\ l = 0,1,2,\dots,n-1.$$

Step 4. After solving the above system, the values of the coefficients $g_{ij}(t)$, $i = 1,2,3,\dots,k$, $j = 0,1,2,\dots,l$ are obtained. Thus, the (k,l) residual power series approximate solution is derived.

In the next discussion, the residual power series approximate solutions are determined by the following the above steps for $n = 2$.

To determine form of the first unknown coefficient $g_{10}(t)$, we should substitute the $(1,0)$-truncated series

$$v_{10}(x,t) = D_t^{\alpha-1}\mu_1(t) - I_t^2 g(t) + D_t^{\alpha-1}\mu_2(t)x + g_{10}(t)\frac{x^{\beta}}{\Gamma(1+\beta)}$$

into the $(1,0)$-truncated residual function

$$Res_{10}(x,t) = D_t v_{10}(x,t) - D_x^{\beta}(D_t^{1-\alpha} v_{10}(x,t)).$$

Then, we have

$$Res_{10}(x,t) = D_t^{\alpha}\mu_1(t) - I_t^2 g(t) + D_t^{\alpha}\mu_2(t)x + g'_{10}(t)\frac{x^{\beta}}{\Gamma(1+\beta)} - D_t^{1-\alpha} g_{10}(t).$$

From *Step* 3 of *Case* B, the substituting of $x = 0$ back into $Res_{10}(x,t)$ will yields

$$g_{10}(t) = D_t^{2\alpha-1}\mu_1(t) - I_t^{3-\alpha} g(t).$$

Again, to find out form of the second unknown coefficient $g_{11}(t)$, we substitute the $(1,1)$-truncated series solution

$$v_{11}(x,t) = D_t^{\alpha-1}\mu_1(t) - I_t g(t) + D_t^{\alpha-1}\mu_2(t)x + \left(D_t^{2\alpha-1}\mu_1(t) - I_t^{3-\alpha}g(t)\right)\frac{x^\beta}{\Gamma(1+\beta)} + g_{11}(t)\frac{x^{1+\beta}}{\Gamma(2+\beta)}$$

into the $(1,1)$-truncated residual function

$$Res_{11}(x,t) = D_t v_{11}(x,t) - D_x^\beta(D_t^{1-\alpha}v_{11}(x,t)).$$

Then, we obtain

$$Res_{11}(x,t) = D_t^\alpha\mu_1(t) - I_t^2 g(t) + D_t^{\alpha-1}\mu_2(t)x\left(D_t^{2\alpha}\mu_1(t) - I_t^{4-\alpha}g(t)\right)\frac{x^\beta}{\Gamma(1+\beta)} + g_{11}'(t)\frac{x^{1+\beta}}{\Gamma(2+\beta)}$$
$$- D_t^\alpha\mu_1(t) + I_t^2 g(t) - D_t^{1-\alpha}g_{11}(t)x.$$

From *Step* 3 of *Case* B, we have $D_x Res_{11}(0,t) = 0$. Solving the resulting equation for $g_{11}(t)$, we have $g_{11}(t) = D_t^{2\alpha-1}\mu_2(t)$.

Therefore, collecting the previous results, the RPS solution of eqs. (9) and (10) can be constructed in the form of fractional power series as follows:

$$v(x,t) = D_t^{\alpha-1}\mu_1(t) - I_t g(t) + D_t^{\alpha-1}\mu_2(t)x + \left(D_t^{2\alpha-1}\mu_1(t) - I_t^{3-\alpha}g(t)\right)\frac{x^\beta}{\Gamma(1+\beta)}$$
$$+ D_t^{2\alpha-1}\mu_2(t)\frac{x^{1+\beta}}{\Gamma(2+\beta)} + \left(D_t^{3\alpha-1}\mu_1(t) - I_t^{5-2\alpha}g(t)\right)\frac{x^{2\beta}}{\Gamma(1+2\beta)} + \dots.$$

Again, applying the transformation (8), we obtain

$$u(x,t) = \mu_1(t) + \mu_2(t)x + \left(D_t^\alpha\mu_1(t) - I_t^2 g(t)\right)\frac{x^\beta}{\Gamma(1+\beta)}$$
$$+ D_t^\alpha\mu_2(t)\frac{x^{1+\beta}}{\Gamma(2+\beta)} + \left(D_t^{2\alpha}\mu_1(t) - I_t^{4-\alpha}g(t)\right)\frac{x^{2\beta}}{\Gamma(1+2\beta)} + \dots.$$

4. Numerical Results

Example 1. Consider the following space-time FPDE

$$D_t^\alpha u = D_x^\beta u + 1 - 6x, \quad 0 < \alpha \le 1,\ 1 < \beta \le 2, \tag{11}$$

$$u(x,0) = e^x + x^3. \tag{12}$$

Applying the transformation (3), we have

$$D_t^\alpha(D_x^{2-\beta}v) = D_{xx}v, \tag{13}$$

$$v(x,0) = \sum_{k=0}^{\infty}\frac{x^{k-\beta+2}}{\Gamma(k-\beta+3)} + \Gamma(4)\frac{x^{5-\beta}}{\Gamma(6-\beta)} + \frac{x^2}{2} - x^3. \tag{14}$$

To determine form of the first unknown coefficient $f_{10}(x)$, we substitute the $(1,0)$-truncated series of

$$v_{10}(x,t) = \sum_{k=0}^{\infty}\frac{x^{k-\beta+2}}{\Gamma(k-\beta+3)} + \Gamma(4)\frac{x^{5-\beta}}{\Gamma(6-\beta)} + \frac{x^2}{2} - x^3 + f_{10}(x)\frac{t^\alpha}{\Gamma(1+\alpha)}$$

into the $(1,0)$-truncated residual function to get the following result:

$$Res_{10}(x,t) = D_x^{2-\beta}f_{10}(x) - \sum_{k=0}^{\infty}\frac{x^{k-\beta}}{\Gamma(k-\beta+1)} - \Gamma(4)\frac{x^{3-\beta}}{\Gamma(4-\beta)} - 1 + 6x - f_{10}''(x)\frac{t^\alpha}{\Gamma(1+\alpha)}.$$

Depending on the result of *Step* 3 of *Case* A, we have

$$f_{10}(x)=\sum_{k=0}^{\infty}\frac{x^{k-2\beta+2}}{\Gamma(k-2\beta+3)}+\Gamma(4)\frac{x^{5-2\beta}}{\Gamma(6-2\beta)}+\frac{x^{2-\beta}}{\Gamma(3-\beta)}-6\frac{x^{3-\beta}}{\Gamma(4-\beta)}.$$

Similarly, to find out form of the second unknown coefficient $f_{20}(x)$, we substitute the $(2,0)$-truncated series solution of eqs. (13) and (14) into the $(2,0)$-truncated residual function to obtain

$$\begin{aligned}Res_{20}(x,t)=&\sum_{k=0}^{\infty}\frac{x^{k-\beta}}{\Gamma(k-\beta+1)}+\Gamma(4)\frac{x^{3-\beta}}{\Gamma(4-\beta)}+1-6x+D_x^{2-\beta}f_{20}(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}\\&-\left(\sum_{k=0}^{\infty}\frac{x^{k-2\beta}}{\Gamma(k-2\beta+1)}+\Gamma(4)\frac{x^{3-2\beta}}{\Gamma(4-2\beta)}+\frac{x^{-\beta}}{\Gamma(1-\beta)}-6\frac{x^{1-\beta}}{\Gamma(2-\beta)}\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)}\\&-f_{20}''(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}.\end{aligned}$$

Hence, the application of the operator D_t^{α} on both sides of above equation will gives the first Caputo derivative of $Res_{20}(x,t)$ with respect to t as

$$D_t^{\alpha}Res_{20}(x,t)=D_x^{2-\beta}f_{20}(x)-\sum_{k=0}^{\infty}\frac{x^{k-2\beta}}{\Gamma(k-2\beta+1)}-\Gamma(4)\frac{x^{3-2\beta}}{\Gamma(4-2\beta)}-\frac{x^{-\beta}}{\Gamma(1-\beta)}+6\frac{x^{1-\beta}}{\Gamma(2-\beta)}-f_{20}''(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}.$$

From $D_t^{\alpha}Res_{20}(x,0)=0$, we obtain

$$f_{20}(x)=\sum_{k=0}^{\infty}\frac{x^{k-3\beta+2}}{\Gamma(k-3\beta+3)}.$$

Therefore, the RPS solution of eqs. (13) and (14) can be constructed as follows:

$$\begin{aligned}v(x,t)=&\sum_{k=0}^{\infty}\frac{x^{k-\beta+2}}{\Gamma(k-\beta+3)}+\Gamma(4)\frac{x^{5-\beta}}{\Gamma(6-\beta)}+\frac{x^2}{2}-x^3\\&+\left(\sum_{k=0}^{\infty}\frac{x^{k-\beta+2}}{\Gamma(k-\beta+3)}+\Gamma(4)\frac{x^{5-2\beta}}{\Gamma(6-2\beta)}+\frac{x^{2-\beta}}{\Gamma(3-\beta)}-6\frac{x^{3-\beta}}{\Gamma(4-\beta)}\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)}\\&+\sum_{k=0}^{\infty}\frac{x^{k-3\beta+2}}{\Gamma(k-3\beta+3)}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+\dots.\end{aligned}$$

Applying the transformation (3), we obtain

$$\begin{aligned}u(x,t)=&\sum_{k=0}^{\infty}\frac{x^{k}}{\Gamma(k+1)}+x^3+\left(\sum_{k=0}^{\infty}\frac{x^{k-\beta}}{\Gamma(k-\beta+1)}+\Gamma(4)\frac{x^{3-\beta}}{\Gamma(4-\beta)}+1-6x\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)}\\&+\sum_{k=0}^{\infty}\frac{x^{k-2\beta}}{\Gamma(k-2\beta+1)}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+\dots.\end{aligned}$$

As a special case when $\alpha=1$ and $\beta=2$, the RPS solution of eqs. (11) and (12) has the general pattern form which is coinciding with the exact solution in terms of fractional power series $u(x,t)=e^{x+t}+x^3+t$. To show this accuracy for Example 1, we report the consecutive error which is defined by $Res(x,t)$, where $x,t\geq 0$ and u_{ij} is the (i,j)-truncated series of $u(x,t)$ obtained from the RPS method. In Table 1, the numerical values of residual errors have been calculated when $(\alpha=1,\beta=2)$, $(\alpha=0.75,\beta=1.75)$, $(\alpha=0.5,\beta=1.5)$, $(\alpha=0.25,\beta=1.25)$ and for various x and t. The computational results of tables provide a numerical estimate for the convergence of the RPS method.

Table 1. The residual errors for $u_{30}(x,t)$ of Example 1

x	t	$\alpha = 1$ $\beta = 2$	$\alpha = 0.75$ $\beta = 1.75$	$\alpha = 0.5$ $\beta = 1.5$	$\alpha = 0.25$ $\beta = 1.25$
0.2	−0.10	0.00020	−0.29438	−1.87786	−5.75598
	−0.05	0.00003	−0.23420	−1.93893	−5.57721
	0.05	−0.00003	−0.12819	−1.66489	−5.41935
	0.10	−0.00020	−0.71793	−2.97769	−6.87618
0.4	−0.10	0.00023	−0.42845	−1.85083	−5.48856
	−0.05	0.00003	−0.34520	−1.92541	−5.07227
	0.05	−0.00003	−0.12263	−1.84835	−6.09028
	0.10	−0.00023	−0.74493	−3.06425	−7.31811
0.6	−0.10	0.00027	−0.49516	−1.81787	−5.58063
	−0.05	0.00003	−0.34235	−1.90894	−5.12981
	0.05	−0.00003	0.15034	−1.38859	−5.89703
	0.10	−0.00027	−0.50263	−2.67907	−7.30404
0.8	−0.10	0.00030	−0.44924	−1.77780	−6.06194
	−0.05	0.00004	−0.20592	−1.88892	−5.77021
	0.05	−0.00004	0.61653	−0.38050	−4.83443
	0.10	−0.00030	−0.11484	−1.93814	−6.83626

Example 2. Consider the following space-time FPDE

$$D_t^\alpha u = D_x^\beta u + g(t), \quad 0 < \alpha \le 1,\ 1 < \beta \le 2 \tag{15}$$

subject to the nonhomogeneous initial condition

$$u(0,t) = e^t + t^2, \quad u_x(0,t) = e^t. \tag{16}$$

Applying the transformation (8), then we have

$$D_t v = D_x^\beta (D_t^{1-\alpha} v), \tag{17}$$

$$v(0,t) = \sum_{k=0}^{\infty} \frac{t^{k-\alpha+1}}{\Gamma(k-\alpha+2)} + \Gamma(3)\frac{t^{3-\alpha}}{\Gamma(4-\alpha)} - t^2 + 2t, \quad v_x(0,t) = \sum_{k=0}^{\infty} \frac{t^{k-\alpha+1}}{\Gamma(k-\alpha+2)}. \tag{18}$$

To determine form of the first unknown coefficient $g_{10}(t)$, we substitute the $(1,0)$-truncated series of

$$v_{10}(x,t) = \sum_{k=0}^{\infty} \frac{t^{k-\alpha+1}}{\Gamma(k-\alpha+2)} + \Gamma(3)\frac{t^{3-\alpha}}{\Gamma(4-\alpha)} - t^2 + 2t + \sum_{k=0}^{\infty} \frac{t^{k-\alpha+1}}{\Gamma(k-\alpha+2)} x + g_{10}(t)\frac{x^\beta}{\Gamma(1+\beta)}$$

into the $(1,0)$-truncated residual function to get the following result:

$$Res_{10}(x,t) = \sum_{k=0}^{\infty} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} + \Gamma(3)\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} - 2t + 2 + \sum_{k=0}^{\infty} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} x + g'_{10}(t)\frac{x^\beta}{\Gamma(1+\beta)} - D_t^{1-\alpha} g_{10}(t).$$

Depending on the result of *Step* 3 of *Case* A, we have

$$g_{10}(t) = \sum_{k=0}^{\infty} \frac{t^{k-2\alpha+1}}{\Gamma(k-2\alpha+2)} + \Gamma(3)\frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)} - 2\frac{x^{2-\alpha}}{\Gamma(3-\alpha)} + 2\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}.$$

Similarly, to find out form of the second unknown coefficient $g_{11}(t)$, we substitute the $(1,1)$-truncated series solution of eqs. (17) and (18) into the $(1,1)$-truncated residual function to obtain

$$\begin{aligned} Res_{11}(x,t) &= \sum_{k=0}^{\infty}\frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}+\Gamma(3)\frac{t^{2-\alpha}}{\Gamma(3-\alpha)}-2t+2+\sum_{k=0}^{\infty}\frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}x \\ &+\left(\sum_{k=0}^{\infty}\frac{t^{k-2\alpha}}{\Gamma(k-2\alpha+1)}+\Gamma(3)\frac{t^{2-2\alpha}}{\Gamma(3-2\alpha)}-2\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}+2\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right)\frac{x^{\beta}}{\Gamma(1+\beta)} \\ &+g'_{11}(t)\frac{x^{1+\beta}}{\Gamma(2+\beta)}-\sum_{k=0}^{\infty}\frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}-\Gamma(3)\frac{t^{2-\alpha}}{\Gamma(3-\alpha)}+2t-2-D_t^{1-\alpha}g_{11}(t)x. \end{aligned}$$

Hence, the application of the operator D_x on both sides of above equation will gives the first partial derivative of $Res_{11}(x,t)$ with respect to x as

$$\begin{aligned} D_xRes_{11}(x,t) &= \sum_{k=0}^{\infty}\frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}+\left(\sum_{k=0}^{\infty}\frac{t^{k-2\alpha}}{\Gamma(k-2\alpha+1)}+\Gamma(3)\frac{t^{2-2\alpha}}{\Gamma(3-2\alpha)}-2\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}+2\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right)\frac{x^{\beta-1}}{\Gamma(\beta)} \\ &+g'_{11}(t)\frac{x^{\beta}}{\Gamma(1+\beta)}-D_t^{1-\alpha}g_{11}(t). \end{aligned}$$

From $D_xRes_{11}(0,t)=0$, we obtain $g_{11}(t)=\sum_{k=0}^{\infty}\frac{x^{k-2\alpha+1}}{\Gamma(k-2\alpha+2)}$.

Therefore, the RPS solution of eqs. (17) and (18) can be constructed as follows:

$$\begin{aligned} v(x,t) &= \sum_{k=0}^{\infty}\frac{t^{k-\alpha+1}}{\Gamma(k-\alpha+2)}+\Gamma(3)\frac{t^{3-\alpha}}{\Gamma(4-\alpha)}-t^2+2t+\sum_{k=0}^{\infty}\frac{t^{k-\alpha+1}}{\Gamma(k-\alpha+2)}x \\ &+\left(\sum_{k=0}^{\infty}\frac{t^{k-\alpha+1}}{\Gamma(k-\alpha+2)}+\Gamma(3)\frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)}-2\frac{x^{2-\alpha}}{\Gamma(3-\alpha)}+2\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right)\frac{x^{\beta}}{\Gamma(1+\beta)} \\ &+\sum_{k=0}^{\infty}\frac{t^{k-2\alpha+1}}{\Gamma(k-2\alpha+2)}\frac{x^{1+\beta}}{\Gamma(2+\beta)}+\dots. \end{aligned}$$

Applying the transformation (8), we obtain

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty}\frac{t^{k}}{\Gamma(k+1)}+t^2+\sum_{k=0}^{\infty}\frac{t^{k}}{\Gamma(k+1)}x+\left(\sum_{k=0}^{\infty}\frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}+\Gamma(3)\frac{t^{2-\alpha}}{\Gamma(3-\alpha)}-2t+2\right)\frac{x^{\beta}}{\Gamma(1+\beta)} \\ &+\sum_{k=0}^{\infty}\frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}\frac{x^{1+\beta}}{\Gamma(2+\beta)}+\dots \end{aligned}$$

As a special case when $\alpha=1$ and $\beta=2$, the RPS solution of eqs. (15) and (16) has the general pattern form which is coinciding with the exact solution in terms of fractional power series $u(x,t)=e^{x+t}+x^2+t^2$. To show this accuracy for Example 2, we report the consecutive error which is defined by $Res(x,t)$, where $x,t\geq 0$ and u_{ij} is the (i,j)-truncated series of $u(x,t)$ obtained from the RPS method. In Table 2, the numerical values of residual errors have been calculated when $(\alpha=1,\beta=2)$, $(\alpha=0.75,\beta=1.75)$, $(\alpha=0.5,\beta=1.5)$, $(\alpha=0.25,\beta=1.25)$ and for various x and t. The computational results of tables provide a numerical estimate for the convergence of the RPS method.

Table 2. The residual errors for $u_{11}(x,t)$ of Example 2

x	t	$\alpha = 1$ $\beta = 2$	$\alpha = 0.75$ $\beta = 1.75$	$\alpha = 0.5$ $\beta = 1.5$	$\alpha = 0.25$ $\beta = 1.25$
0.2	−0.10	0.01930	−0.20867	0.17579	1.10393
	−0.05	0.02029	−0.03937	0.35762	1.22312
	0.05	0.02243	0.14285	0.48423	1.33477
	0.10	0.02358	0.02985	0.39222	1.32800
0.4	−0.10	0.08204	0.20439	0.92362	1.89431
	−0.05	0.08624	0.27804	1.07778	2.06049
	0.05	0.09532	0.95680	1.67295	2.37035
	0.10	0.10020	0.76413	1.51965	2.35059
0.6	−0.10	0.19544	0.29706	1.14610	2.19894
	−0.05	0.20547	0.22797	1.29317	2.44497
	0.05	0.22707	1.70841	2.55196	3.01160
	0.10	0.23872	1.38091	2.30026	2.97507
0.8	−0.10	0.36676	0.22055	1.16445	2.37687
	−0.05	0.38556	−0.04683	1.31103	2.72308
	0.05	0.42612	2.55772	3.42849	3.59591
	0.10	0.44796	2.03976	3.04443	3.53650

5. Conclusion

The fundamental goal of this work has been to demonstrate the feasibility of the RPSM for solving time-space non-homogeneous partial differential equations in the Caputo sense. The above results and all of the discussed examples reveal that the goal has been achieved successfully. As a result RPSM can be used as an significant method to obtain analytical solutions of non-homogeneous partial differential equations arising in different branches of science.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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