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Research Article

On Generalized Absolute Matrix Summability of Infinite Series

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Abstract. In this paper, we have generalized a known theorem on $|\bar{N}, p_n|_k$ summability factors of infinite series with a new summability method by using almost increasing sequences. This new theorem also includes several new and known results.

Keywords. Summability factors; Absolute matrix summability; Almost increasing sequence; Infinite series; Hölder inequality; Minkowski inequality

MSC. 26D15; 40D15; 40F05; 40G99

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1. Introduction

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Obviously every increasing sequence is almost increasing sequence but the converse need not be true as can be seen from the example $b_n = ne^{(-1)^n}$. Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{1.1}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.2}$$

defines the sequence (σ_n) of the Riesz mean or simply the $(\bar{N},p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [8]). The series $\sum a_n$ is said to be summable $|\bar{N},p_n|_k$, $k\geq 1$, if (see [6])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|\sigma_n-\sigma_{n-1}|^k<\infty, \tag{1.3}$$

and it is said to be summable $|\bar{N},p_n,\beta;\delta|_k$, $k\geq 1$, $\delta\geq 0$ and β is a real number, if (see [7])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}|\sigma_n-\sigma_{n-1}|^k<\infty. \tag{1.4}$$

If we take $\beta=1$, then $|\bar{N},p_n,\beta;\delta|_k$ summability reduces to $|\bar{N},p_n;\delta|_k$ summability (see [5]). Also, if we take $\beta=1$ and $\delta=0$, then $|\bar{N},p_n,\beta;\delta|_k$ summability reduces to $|\bar{N},p_n|_k$ summability.

Let $A=(a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s=(s_n)$ to $As=(A_n(s))$, where

$$A_n(s)=\sum_{v=0}^{n}a_{nv}s_v, \quad n=0,1,\dots \tag{1.5}$$

The series $\sum a_n$ is said to be summable $|A,p_n|_k, k\geq 1$, if (see [12])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|\bar{\Delta}A_n(s)|^k<\infty. \tag{1.6}$$

We say that the series $\sum a_n$ is summable $|A,p_n,\beta;\delta|_k$, $k\geq 1$, $\delta\geq 0$ and β is a real number, if

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}|\bar{\Delta}A_n(s)|^k<\infty, \tag{1.7}$$

where

$$\bar{\Delta}A_n(s)=A_n(s)-A_{n-1}(s). \tag{1.8}$$

If we take $\beta=1$, then $|A,p_n,\beta;\delta|_k$ summability reduces to $|A,p_n;\delta|_k$ summability (see [10]). Also, if we take $\beta=1$ and $\delta=0$, then $|A,p_n,\beta;\delta|_k$ summability reduces to $|A,p_n|_k$ summability.

Before stating the main theorem we must first introduce some further notations.

Given a normal matrix $A=(a_{nv})$, we associate two lower semimatrices $\bar{A}=(\bar{a}_{nv})$ and $\hat{A}=(\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv}=\sum_{i=v}^{n}a_{ni}, \quad n,v=0,1,\dots \tag{1.9}$$

and

$$\hat{a}_{00}=\bar{a}_{00}=a_{00}, \;\; \hat{a}_{nv}=\bar{a}_{nv}-\bar{a}_{n-1,v}, \quad n=1,2,\dots. \tag{1.10}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s)=\sum_{v=0}^{n}a_{nv}s_v=\sum_{v=0}^{n}\bar{a}_{nv}a_v \tag{1.11}$$

and

$$\bar{\Delta}A_n(s)=\sum_{v=0}^{n}\hat{a}_{nv}a_v. \tag{1.12}$$

2. Known Result

In [3], Bor has proved the following theorem for $|\bar{N}, p_n|_k$ summability factors of infinite series.

Theorem 2.1. *Let (X_n) be an almost increasing sequence and let there be sequences (β_n) and (λ_n) such that*

$$|\Delta\lambda_n| \le \beta_n, \tag{2.1}$$

$$\beta_n \to 0 \ \ as \ \ n \to \infty, \tag{2.2}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty, \tag{2.3}$$

$$|\lambda_n|X_n = O(1) \tag{2.4}$$

and

$$\sum_{v=1}^{n} \frac{|t_v|^k}{v} = O(X_n) \ \ as \ \ n \to \infty, \tag{2.5}$$

where (t_n) is the n-th $(C,1)$ mean of the sequence (na_n). Suppose further, the sequence (p_n) is such that

$$P_n = O(np_n), \tag{2.6}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{2.7}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ is summable $|\bar{N}, p_n|_k$, $k \ge 1$.

Remark 2.2. It should be noted that, from the hypotheses of Theorem 2.1, (λ_n) is bounded and $\Delta\lambda_n = O(1/n)$ (see [2]).

3. Main Result

The aim of this paper is to generalize Theorem 2.1 for absolute matrix summability.

Now, we shall prove the following theorem:

Theorem 3.1. *Let $A = (a_{nv})$ be a positive normal matrix such that*

$$\overline{a}_{n0} = 1, \quad n = 0, 1, \dots, \tag{3.1}$$

$$a_{n-1,v} \ge a_{nv}, \ \ for \ n \ge v+1, \tag{3.2}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{3.3}$$

$$|\hat{a}_{n,v+1}| = O(v|\Delta_v(\hat{a}_{nv})|). \tag{3.4}$$

Let (X_n) be an almost increasing sequence. If the conditions (2.1)-(2.4) and (2.6)-(2.7) of Theorem 2.1 and

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k} |t_n|^k = O(X_m) \ \ as \ \ m \to \infty, \tag{3.5}$$

$$\sum_{n=v+1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k+1} |\Delta_v(\hat{a}_{nv})| = O\left(\left(\frac{P_v}{p_v}\right)^{\beta(\delta k+k-1)-k}\right), \tag{3.6}$$

are satisfied, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ *is summable* $|A, p_n, \beta; \delta|_k$, $k \geq 1$, $\delta \geq 0$ *and* $-\beta(\delta k + k - 1) + k > 0$.

We need the following lemmas for the proof of Theorem 3.1.

Lemma 3.2 ([11]). *If* (X_n) *is an almost increasing sequence, then under the conditions* (2.2)-(2.3)*, we have that*

$$nX_n\beta_n = O(1), \tag{3.7}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{3.8}$$

Lemma 3.3 ([4]). *If the conditions* (2.6) *and* (2.7) *are satisfied, then* $\Delta(P_n/p_n n^2) = O(1/n^2)$.

4. Proof of Theorem 3.1

Let (I_n) denotes A-transform of the series $\sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{n p_n}$. Then, by (1.11) and (1.12), we have

$$\bar{\Delta} I_n = \sum_{v=1}^{n} \hat{a}_{nv} \frac{a_v P_v \lambda_v}{v p_v}.$$

Applying Abel's transformation to this sum, we get that

$$\begin{aligned}
\bar{\Delta} I_n &= \sum_{v=1}^{n} \hat{a}_{nv} \frac{v a_v P_v \lambda_v}{v^2 p_v} \\
&= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} P_v \lambda_v}{v^2 p_v} \right) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn} P_n \lambda_n}{n^2 p_n} \sum_{r=1}^{n} r a_r \\
&= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} P_v \lambda_v}{v^2 p_v} \right) (v+1) t_v + \frac{a_{nn} P_n \lambda_n}{n^2 p_n} (n+1) t_n \\
&= \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv}) \frac{(v+1)}{v^2} \frac{P_v \lambda_v}{p_v} t_v + \sum_{v=1}^{n-1} \frac{\hat{a}_{n,v+1} P_v}{p_v} \Delta \lambda_v t_v \frac{(v+1)}{v^2} \\
&\quad + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \Delta \left(\frac{P_v}{v^2 p_v} \right) t_v (v+1) + \frac{a_{nn} P_n \lambda_n}{n^2 p_n} (n+1) t_n \\
&= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.
\end{aligned}$$

To complete the proof of Theorem 3.1, by Minkowski's inequality, it is enough to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\beta(\delta k + k - 1)} |I_{n,r}|^k < \infty, \quad \text{for } r = 1, 2, 3, 4. \tag{4.1}$$

First, using the fact that $P_v = O(v p_v)$ by (2.6), we have that

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{\beta(\delta k + k - 1)} |I_{n,1}|^k = O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{\beta(\delta k + k - 1)} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_v| |t_v| \right)^k.$$

Now, applying Hölder's inequality with indices k and k', where $k>1$ and $\frac{1}{k}+\frac{1}{k'}=1$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}|I_{n,1}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k\left|t_v\right|^k\right)\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)} a_{nn}^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k\left|t_v\right|^k\right).
\end{aligned}$$

Now, using the fact that $a_{nn}=O(\frac{p_n}{P_n})$ by (3.3), we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}|I_{n,1}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k+1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k\left|t_v\right|^k\right)\\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^k|t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k+1}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\beta(\delta k+k-1)-k}|\lambda_v|^{k-1}|\lambda_v||t_v|^k\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\beta(\delta k+k-1)-k}|\lambda_v||t_v|^k\\
&= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\beta(\delta k+k-1)-k}|t_r|^k\\
&\quad + O(1)|\lambda_m|\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\beta(\delta k+k-1)-k}|t_v|^k\\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v+O(1)|\lambda_m|X_m\\
&= O(1)\sum_{v=1}^{m-1}\beta_v X_v+O(1)|\lambda_m|X_m\\
&= O(1) \quad \text{as } m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.2.

Now, using Hölder's inequality, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}|I_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}\left(\sum_{v=1}^{n-1}v\,|\Delta_v(\hat{a}_{nv})|\,\beta_v|t_v|\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}\left(\sum_{v=1}^{n-1}v\beta_v\,|\Delta_v(\hat{a}_{nv})|\,|t_v|^k\right)\left(\sum_{v=1}^{n-1}v\beta_v\,|\Delta_v(\hat{a}_{nv})|\right)^{k-1}.
\end{aligned}$$

Since

$$\Delta_v(\hat{a}_{nv})=\hat{a}_{nv}-\hat{a}_{n,v+1}=\bar{a}_{nv}-\bar{a}_{n-1,v}-\bar{a}_{n,v+1}+\bar{a}_{n-1,v+1}=a_{nv}-a_{n-1,v}$$

we get that

$$\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| = \sum_{v=1}^{n-1}(a_{n-1,v} - a_{nv}) \le a_{nn}.$$

Thus, we obtain

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)} |I_{n,2}|^k = O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} v\beta_v |\Delta_v(\hat{a}_{nv})| |t_v|^k\right).$$

Now, using (3.3), we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)} |I_{n,2}|^k &= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k+1} \left(\sum_{v=1}^{n-1} v\beta_v |\Delta_v(\hat{a}_{nv})| |t_v|^k\right) \\
&= O(1) \sum_{v=1}^{m} v\beta_v |t_v|^k \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k+1} |\Delta_v(\hat{a}_{nv})| \\
&= O(1) \sum_{v=1}^{m} v\beta_v |t_v|^k \left(\frac{P_v}{p_v}\right)^{\beta(\delta k+k-1)-k} \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} \left(\frac{P_r}{p_r}\right)^{\beta(\delta k+k-1)-k} |t_r|^k \\
&\quad + O(1) m\beta_m \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\beta(\delta k+k-1)-k} |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} |\Delta(v\beta_v)| X_v + O(1) m\beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} v|\Delta\beta_v| X_v + O(1) \sum_{v=1}^{m-1} \beta_v X_v + O(1) m\beta_m X_m \\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.2.

Since $\Delta\left(\frac{P_v}{v^2 p_v}\right) = O\left(\frac{1}{v^2}\right)$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)} |I_{n,3}|^k &= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\lambda_{v+1}| \frac{|t_v|}{v}\right)^k \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_{v+1}| |t_v|\right)^k \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_{v+1}|^k |t_v|^k\right) \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})|\right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_{v+1}|^k |t_v|^k\right).
\end{aligned}$$

By using (3.3), as in $I_{n,1}$, we have that

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)} |I_{n,3}|^k = O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k+1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_{v+1}| |t_v|^k\right)$$

$$\begin{aligned}
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}||t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k+1}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\beta(\delta k+k-1)-k}|\lambda_{v+1}||t_v|^k\\
&= O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}$$

by virtue of hypotheses of Theorem 3.1, Lemma 3.2 and Lemma 3.3.

Finally, by using Abel's transformation, as in $I_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}|I_{n,4}|^k &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}a_{nn}^k|\lambda_n|^k|t_n|^k\\
&= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k}|\lambda_n|^{k-1}|\lambda_n||t_n|^k\\
&= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k}|\lambda_n||t_n|^k\\
&= O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.2.

This completes the proof of Theorem 3.1.

5. Corollaries

Corollary 1. *If we take $\beta = 1$ and $\delta = 0$, then we get a theorem dealing with $|A, p_n|_k$ summability (see* [9]*).*

Corollary 2. *If we take $\beta = 1$, $\delta = 0$ and $a_{nv} = \frac{p_v}{P_n}$, then we get Theorem* 2.1.

6. Conclusion

We prove a general theorem for absolute matrix summability of infinite series by virtue of almost increasing sequence. This general theorem enrich the literature of summability theory and create basis for future researches.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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