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Research Article

Generalized Turan-Type Inequalities for the (q,k)-Polygamma Functions

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Abstract. This study provides generalizations of the results of Merovci concerning some Turan-type inequalities involving the (q,k)-Polygamma functions.

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1. Introduction

In this paper, we provide generalizations of the results of Merovci concerning some Turan-type inequalities involving the (q,k)-Polygamma functions. In the meantime, we present the following definitions and preliminary materials.

The (q,k)-analogue of the Gamma function, $\Gamma_{q,k}(x)$ is defined for $x>0$, $q\in(0,1)$ and $k>0$ by any of the following equivalent definitions (see [3], [4] and the references therein).

$$\begin{aligned}\Gamma_{q,k}(x) &= \int_0^{\left(\frac{[k]_q}{1-q^k}\right)^{\frac{1}{k}}} t^{x-1} E_{q,k}^{-\frac{q^k t^k}{[k]_q}} \, d_q t \\ &= \frac{(1-q^k)_{q,k}^{\frac{x}{k}-1}}{(1-q)^{\frac{x}{k}-1}}\end{aligned}$$

$$= \frac{1}{(1-q)^{\frac{x}{k}-1}} \prod_{n=0}^{\infty} \frac{1-q^{(n+1)k}}{1-q^{nk+x}}$$

where $(1+x)_{q,k}^{t} = \frac{(1+x)_{q,k}^{\infty}}{(1+q^t x)_{q,k}^{\infty}}$, $(1+x)_{q,k}^{\infty} = \prod_{i=0}^{\infty}(1+q^{ik}x)$ for $x,t \in \mathbb{R}$ and $E_{q,k}^{x} = \sum_{n=0}^{\infty} \frac{q^{kn(n-1)/2}x^n}{[n]_{q^k}!} = \left(1+(1-q^k)x\right)_{q,k}^{\infty}$ is the (q,k)-analogue of the classical exponential function. The function $\Gamma_{q,k}(x)$ satisfies the basic identities [3]

$$\Gamma_{q,k}(x+k) = [x]_q \Gamma_{q,k}(x), \qquad \Gamma_{q,k}(k) = 1,$$

where $[x]_q = \frac{1-q^x}{1-q}$. The (q,k)-Digamma function, $\psi_{q,k}(x)$ and the (q,k)-Polygamma functions, $\psi_{q,k}^{(m)}(x)$ are defined as follows (see [2], [6]).

$$\begin{aligned}
\psi_{q,k}(x) = \frac{d}{dx}\ln\Gamma_{q,k}(x) &= -\frac{1}{k}\ln(1-q) + (\ln q)\sum_{n=0}^{\infty}\frac{q^{nk+x}}{1-q^{nk+x}} \\
&= -\frac{1}{k}\ln(1-q) + (\ln q)\sum_{n=0}^{\infty}\frac{q^{(n+1)x}}{1-q^{(n+1)k}} \\
&= -\frac{1}{k}\ln(1-q) + (\ln q)\sum_{n=1}^{\infty}\frac{q^{nkx}}{1-q^{nk}}, \\
\psi_{q,k}^{(m)}(x) = \frac{d^m}{dx^m}\psi_{q,k}(x) & \\
&= (\ln q)^{m+1}\sum_{n=1}^{\infty}\frac{(nk)^m q^{nkx}}{1-q^{nk}},
\end{aligned} \tag{1.1}$$

where $m \in \mathbb{N}$ and $\psi_{q,k}^{(0)}(x) \equiv \psi_{q,k}(x)$.

In 2013, Merovci [6] established the following Turan-type inequalities involving the function $\psi_{q,k}^{(m)}(x)$.

Theorem 1.1. *For $m,n = 1,2,3,\dots$,*

$$\psi_{q,k}^{\left(\frac{m}{a}+\frac{n}{b}\right)}\left(\frac{x}{a}+\frac{y}{b}\right) \le \left(\psi_{q,k}^{(m)}(x)\right)^{\frac{1}{a}}\left(\psi_{q,k}^{(n)}(y)\right)^{\frac{1}{b}}, \tag{1.2}$$

where $\frac{m+n}{2}$ is an integer, $a > 1$, $\frac{1}{a}+\frac{1}{b} = 1$.

Theorem 1.2. *For $m,n = 1,2,3,\dots$,*

$$\left(\psi_{q,k}^{(m)}(x) + \psi_{q,k}^{(n)}(y)\right)^{\frac{1}{u}} \le \left(\psi_{q,k}^{(m)}(x)\right)^{\frac{1}{u}} + \left(\psi_{q,k}^{(n)}(y)\right)^{\frac{1}{u}}, \tag{1.3}$$

where $\frac{m+n}{2}$ is an integer, $u \ge 1$.

Theorem 1.3. *For every $x > 0$ and integers $m \ge 1$, we have:*

If m is odd, then $\quad \left(\exp\psi_{q,k}^{(m)}(x)\right)^2 \ge \exp\psi_{q,k}^{(m+1)}(x)\cdot\exp\psi_{q,k}^{(m-1)}(x);$ (1.4)

If m is even, then $\quad \left(\exp\psi_{q,k}^{(m)}(x)\right)^2 \le \exp\psi_{q,k}^{(m+1)}(x)\cdot\exp\psi_{q,k}^{(m-1)}(x).$ (1.5)

Also, in a recent paper [9], the author gave improvements of Theorems 1.1 and 1.2.

In the present work, the main objective is to provide generalizations of the above inequalities. In order to achieve this, the following lemmas shall be employed.

Lemma 1.4 (Generalized Hölder's Inequality). *Let $i = 1,2,3,\dots,N$ and $n = 1,2,3,\dots,T$ such that the sums exist. Then the inequality*

$$\sum_{n=1}^{T}\left|\prod_{i=1}^{N} Q_{i,n}\right| \le \prod_{i=1}^{N}\left(\sum_{n=1}^{T}|Q_{i,n}|^{\alpha_i}\right)^{\frac{1}{\alpha_i}} \tag{1.6}$$

is valid for $\alpha_i > 1$ such that $\sum_{i=1}^{N}\frac{1}{\alpha_i} = 1$.

Lemma 1.5 (Generalized Minkowski's Inequality). *Let $i = 1,2,3,\dots,N$ and $n = 1,2,3,\dots,T$ such that the sums exist. Then the inequality*

$$\left(\sum_{n=1}^{T}\left|\sum_{i=1}^{N} Q_{i,n}\right|^{u}\right)^{\frac{1}{u}} \le \sum_{i=1}^{N}\left(\sum_{n=1}^{T}|Q_{i,n}|^{u}\right)^{\frac{1}{u}} \tag{1.7}$$

is valid for $u \ge 1$.

For proofs of Lemmas 1.4 and 1.5, see [8] and the references therein.

Lemma 1.6 (Weighted AM-GM Inequality). *For $i = 1,2,\dots,n$, let $Q_i \ge 0$ and $\lambda_i \ge 0$ such that $\sum_{i=1}^{n}\lambda_i = 1$. Then the inequality*

$$\sum_{i=1}^{n}\lambda_i Q_i \ge \prod_{i=1}^{n} Q_i^{\lambda_i} \tag{1.8}$$

is valid .

Lemma 1.6 is well-known in the literature. See for instance [5] and the related references.

2. Main Results

In this section, we present generalizations of the results of Merovci as shown in Theorem 1.1, 1.2 and 1.3.

Theorem 2.1. *For $i = 1,2,\dots,N$, let $\alpha_i > 1$, $\sum_{i=1}^{N}\frac{1}{\alpha_i} = 1$ and $m_i \in \mathbb{N}$ such that $\sum_{i=1}^{N}\frac{m_i}{\alpha_i} \in \mathbb{N}$. Then the inequality*

$$\left|\psi_{q,k}^{\left(\sum_{i=1}^{N}\frac{m_i}{\alpha_i}\right)}\left(\sum_{i=1}^{N}\frac{x_i}{\alpha_i}\right)\right| \le \prod_{i=1}^{N}\left|\psi_{q,k}^{(m_i)}(x_i)\right|^{\frac{1}{\alpha_i}} \tag{2.1}$$

is valid for $x_i > 0$.

Proof. From (1.1) and (1.6) we obtain

$$\begin{aligned}
\left|\psi_{q,k}^{\left(\sum_{i=1}^{N}\frac{m_i}{\alpha_i}\right)}\left(\sum_{i=1}^{N}\frac{x_i}{\alpha_i}\right)\right| &= |\ln q|^{\sum_{i=1}^{N}\frac{m_i}{\alpha_i}+1}\sum_{n=1}^{\infty}\frac{(nk)^{\sum_{i=1}^{N}\frac{m_i}{\alpha_i}}q^{nk\sum_{i=1}^{N}\frac{x_i}{\alpha_i}}}{1-q^{nk}} \\
&= \sum_{n=1}^{\infty}|\ln q|^{\sum_{i=1}^{N}\frac{m_i+1}{\alpha_i}}\frac{(nk)^{\sum_{i=1}^{N}\frac{m_i}{\alpha_i}}q^{nk\sum_{i=1}^{N}\frac{x_i}{\alpha_i}}}{\left(1-q^{nk}\right)^{\sum_{i=1}^{N}\frac{1}{\alpha_i}}}
\end{aligned}$$

$$= \sum_{n=1}^{\infty}\prod_{i=1}^{N}|\ln q|^{\frac{m_i+1}{\alpha_i}}\frac{(nk)^{\frac{m_i}{\alpha_i}}q^{nk\frac{x_i}{\alpha_i}}}{\left(1-q^{nk}\right)^{\frac{1}{\alpha_i}}}$$
$$\leq \prod_{i=1}^{N}\left(\sum_{n=1}^{\infty}\left[|\ln q|^{\frac{m_i+1}{\alpha_i}}\frac{(nk)^{\frac{m_i}{\alpha_i}}q^{nk\frac{x_i}{\alpha_i}}}{\left(1-q^{nk}\right)^{\frac{1}{\alpha_i}}}\right]^{\alpha_i}\right)^{\frac{1}{\alpha_i}}$$
$$= \prod_{i=1}^{N}\left(|\ln q|^{m_i+1}\sum_{n=1}^{\infty}\frac{(nk)^{m_i}q^{nkx_i}}{1-q^{nk}}\right)^{\frac{1}{\alpha_i}}$$
$$= \prod_{i=1}^{N}\left|\psi_{q,k}^{(m_i)}(x_i)\right|^{\frac{1}{\alpha_i}}$$

which yields the result (2.1). □

Remark 2.2. Let $N=2$, $\alpha_1=a$, $\alpha_2=b$, $m_1=m$, $m_2=n$, $x_1=x$ and $x_2=y$ in Theorem 2.1. Then, we get

$$\left|\psi_{q,k}^{(\frac{m}{a}+\frac{n}{b})}\left(\frac{x}{a}+\frac{y}{b}\right)\right| \leq \left|\psi_{q,k}^{(m)}(x)\right|^{\frac{1}{a}}\left|\psi_{q,k}^{(n)}(y)\right|^{\frac{1}{b}}$$

as obtained in [9]. Particularly, if m and n are odd, then we obtain the result (1.2).

Remark 2.3. By letting $q\to 1$ as $k\to 1$ in Theorem 2.1, we recover Theorem 2.1 of [1].

Theorem 2.4. *For $i=1,2,\dots,N$, let $m_i\in\mathbb{N}$. Then the inequality*

$$\left(\sum_{i=1}^{N}\left|\psi_{q,k}^{(m_i)}(x_i)\right|\right)^{\frac{1}{u}} \leq \sum_{i=1}^{N}\left|\psi_{q,k}^{(m_i)}(x_i)\right|^{\frac{1}{u}} \tag{2.2}$$

is valid for $x_i>0$ and $u\geq 1$.

Proof. From (1.1) we obtain

$$\left(\sum_{i=1}^{N}\left|\psi_{q,k}^{(m_i)}(x_i)\right|\right)^{\frac{1}{u}} = \left(\sum_{i=1}^{N}\sum_{n=1}^{\infty}\frac{|\ln q|^{m_i+1}(nk)^{m_i}q^{nkx_i}}{1-q^{nk}}\right)^{\frac{1}{u}}$$
$$= \left(\sum_{n=1}^{\infty}\sum_{i=1}^{N}\left[\frac{|\ln q|^{\frac{m_i+1}{u}}(nk)^{\frac{m_i}{u}}q^{\frac{nkx_i}{u}}}{(1-q^{nk})^{\frac{1}{u}}}\right]^{u}\right)^{\frac{1}{u}}$$
$$\leq \left(\sum_{n=1}^{\infty}\left[\sum_{i=1}^{N}\frac{|\ln q|^{\frac{m_i+1}{u}}(nk)^{\frac{m_i}{u}}q^{\frac{nkx_i}{u}}}{(1-q^{nk})^{\frac{1}{u}}}\right]^{u}\right)^{\frac{1}{u}}$$
$$\leq \sum_{i=1}^{N}\left(\sum_{n=1}^{\infty}\left[\frac{|\ln q|^{\frac{m_i+1}{u}}(nk)^{\frac{m_i}{u}}q^{\frac{nkx_i}{u}}}{(1-q^{nk})^{\frac{1}{u}}}\right]^{u}\right)^{\frac{1}{u}}$$
$$= \sum_{i=1}^{N}\left(\sum_{n=1}^{\infty}\frac{|\ln q|^{m_i+1}(nk)^{m_i}q^{nkx_i}}{1-q^{nk}}\right)^{\frac{1}{u}}$$
$$= \sum_{i=1}^{N}\left|\psi_{q,k}^{(m_i)}(x_i)\right|^{\frac{1}{u}}$$

which yields the result (2.2). (**Note.** The first inequality follows from the fact that $\sum_{i=1}^{n} a_i^u \leq \left(\sum_{i=1}^{n} a_i\right)^u$, for $a_i \geq 0$, $u \geq 1$, $n \in \mathbb{N}$ whiles the second inequality is as a result of the generalized Minkowski's inequality (1.7)). □

Remark 2.5. By letting $N = 2$, $m_1 = m$, $m_2 = n$, $x_1 = x$ and $x_2 = y$ in Theorem 2.4, we get

$$\left(\left|\psi_{q,k}^{(m)}(x)\right| + \left|\psi_{q,k}^{(n)}(y)\right|\right)^{\frac{1}{u}} \leq \left|\psi_{q,k}^{(m)}(x)\right|^{\frac{1}{u}} + \left|\psi_{q,k}^{(n)}(y)\right|^{\frac{1}{u}}$$

as obtained in [9]. Particularly, if m and n are odd, then we obtain the result (1.3).

Theorem 2.6. *For $i = 1, 2, \ldots, N$, let $\alpha_i > 1$, $\sum_{i=1}^{N} \frac{1}{\alpha_i} = 1$ and $m_i \in \mathbb{N}$ such that $\sum_{i=1}^{N} \frac{m_i}{\alpha_i} \in \mathbb{N}$. Then, the following inequalities are valid for $x_i > 0$.*

$$\exp \psi_{q,k}^{\left(\sum_{i=1}^{N} \frac{m_i}{\alpha_i}\right)}\left(\sum_{i=1}^{N} \frac{x_i}{\alpha_i}\right) \leq \prod_{i=1}^{N} \left(\exp \psi_{q,k}^{(m_i)}(x_i)\right)^{\frac{1}{\alpha_i}} \quad \text{if } m_i, \sum_{i=1}^{N} \frac{m_i}{\alpha_i} \text{ are odd} \tag{2.3}$$

and

$$\exp \psi_{q,k}^{\left(\sum_{i=1}^{N} \frac{m_i}{\alpha_i}\right)}\left(\sum_{i=1}^{N} \frac{x_i}{\alpha_i}\right) \geq \prod_{i=1}^{N} \left(\exp \psi_{q,k}^{(m_i)}(x_i)\right)^{\frac{1}{\alpha_i}} \quad \text{if } m_i, \sum_{i=1}^{N} \frac{m_i}{\alpha_i} \text{ are even}. \tag{2.4}$$

Proof. Let m_i and $\sum_{i=1}^{N} \frac{m_i}{\alpha_i}$ both be odd. Then,

$$\begin{aligned}
&\psi_{q,k}^{\left(\sum_{i=1}^{N} \frac{m_i}{\alpha_i}\right)}\left(\sum_{i=1}^{N} \frac{x_i}{\alpha_i}\right) - \sum_{i=1}^{N} \frac{\psi_{q,k}^{(m_i)}(x_i)}{\alpha_i} \\
&= \sum_{n=1}^{\infty} \frac{(\ln q)^{\sum_{i=1}^{N} \frac{m_i}{\alpha_i}+1} (nk)^{\sum_{i=1}^{N} \frac{m_i}{\alpha_i}} q^{nk \sum_{i=1}^{N} \frac{x_i}{\alpha_i}}}{1-q^{nk}} - \sum_{i=1}^{N} \frac{1}{\alpha_i} \sum_{n=1}^{\infty} \frac{(\ln q)^{m_i+1} (nk)^{m_i} q^{nkx_i}}{1-q^{nk}} \\
&= \sum_{n=1}^{\infty} \frac{1}{1-q^{nk}} \left[\prod_{i=1}^{N} (\ln q)^{\frac{m_i+1}{\alpha_i}} (nk)^{\frac{m_i}{\alpha_i}} q^{\frac{nkx_i}{\alpha_i}} - \sum_{i=1}^{N} \frac{1}{\alpha_i} (\ln q)^{m_i+1} (nk)^{m_i} q^{nkx_i}\right] \\
&\leq 0
\end{aligned}$$

which follows from the weighted AM-GM inequality (1.8). Hence

$$\psi_{q,k}^{\left(\sum_{i=1}^{N} \frac{m_i}{\alpha_i}\right)}\left(\sum_{i=1}^{N} \frac{x_i}{\alpha_i}\right) \leq \sum_{i=1}^{N} \frac{\psi_{q,k}^{(m_i)}(x_i)}{\alpha_i}. \tag{2.5}$$

Similarly, if m_i and $\sum_{i=1}^{n} \frac{m_i}{\alpha_i}$ are both even, then we obtain

$$\psi_{q,k}^{\left(\sum_{i=1}^{N} \frac{m_i}{\alpha_i}\right)}\left(\sum_{i=1}^{N} \frac{x_i}{\alpha_i}\right) \geq \sum_{i=1}^{N} \frac{\psi_{q,k}^{(m_i)}(x_i)}{\alpha_i}. \tag{2.6}$$

Finally, by exponentiating (2.5) and (2.6), we obtain (2.3) and (2.4), respectively. □

3. Concluding Remarks

By using the generalized Hölder's inequality, the generalized Minkowski's inequality and the weighted AM-GM inequality, we provide generalizations of the results of Merovci [6]. Interested readers can also refer to the work [7] for similar results involving the m-th derivative of the (q,k)-Gamma function.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

References

[1] T. Batbold, Some remarks on results of Mortici, *Kragujevac Journal of Mathematics* **36**(1) (2012), 73 – 76.

[2] K. Brahim and Y. Sidomou, Some inequalities for the q,k-Gamma and Beta functions, *Malaya Journal of Matematik* **2**(1) (2014), 61 – 71.

[3] R. Díaz and C. Teruel, q,k-generalized gamma and beta functions, *Journal of Nonlinear Mathematical Physics* **12**(1) (2005), 118 – 134.

[4] C.G. Kokologiannaki, Some properties of $\Gamma_{q,k}(t)$ and related functions, *International Journal of Contemporary Mathematical Sciences* **11**(1) (2016), 1 – 8.

[5] Y. Li and X.-M. Gu, The weighted AM-GM inequality is equivalent to the Hölder inequality, *arXiv:1504.02718v3* [math.FA], 5 pages, available online `https://arxiv.org/abs/1504.02718v3`.

[6] F. Merovci, Turan type inequalities for some (q,k)-special functions, *Acta Universitatis Apulensis* **34** (2013), 69 – 76.

[7] K. Nantomah and S. Nasiru, Inequalities for the m-th derivative of the (q,k)-Gamma function, *Moroccan Journal of Pure and Applied Analysis* **3**(1) (2017), 63 – 69.

[8] K. Nantomah, Generalized Hölder's and Minkowski's inequalities for Jackson's q-integral and some applications to the incomplete q-Gamma function, *Abstract and Applied Analysis* **2017** (2017), Article ID 9796873, 6 pages.

[9] K. Nantomah, Remarks on some inequalities for analogues of the polygamma function, *Mathematical Sciences and Applications E-Notes* **6**(1) (2018), 1 – 6.