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Research Article

Several Inequalities for Khatri-Rao Products of Hilbert Space Operators

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Abstract. We establish several inequalities for Khatri-Rao products of Hilbert space operators, involving ordinary products, ordinary powers, ordinary inverses, and Moore-Penrose inverses. Kantorovich type inequalities concerning Khatri-Rao products are also investigated. Our results generalize some matrix inequalities in the literature. In our case, we must impose some mild conditions on operators such as the closeness of their ranges. Furthermore, we develop new operator inequalities by using block partitioning technique and unital positive linear maps.

Keywords. Tensor product; Khatri-Rao product; Tracy-Singh product; Moore-Penrose inverse; Unital positive linear map

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1. Introduction

In matrix and operator theory, there are various kinds of products, namely, Kronecker (tensor) product, Tracy-Singh product, and Khatri-Rao product. Denote by $M_{m,n}(\mathbb{C})$ the set of all complex matrices of order $m \times n$ and abbreviate $M_{n,n}(\mathbb{C})$ to $M_n(\mathbb{C})$. Recall that the Kronecker product of $A = [a_{ij}] \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$ is defined by

$$A \hat{\otimes} B = \left[a_{ij}B\right]_{ij} \in M_{mp,nq}(\mathbb{C}).$$

To define the Khatri-Rao product [3] of two matrices A and B with arbitrary sizes, we must partition $A = [A_{ij}]$ and $B = [B_{ij}]$ in the same block-matrix form (the sizes of A_{ij} and B_{ij} may be different). The Khatri-Rao product of A and B is given by

$$A \,\widehat{\boxdot}\, B = \left[A_{ij} \,\widehat{\otimes}\, B_{ij}\right]_{ij}.$$

Note that if A is considered of only one block, then $A \,\widehat{\boxdot}\, B = A \,\widehat{\otimes}\, B$. If both A and B are entrywise partitioned (each block is a 1×1 matrix), then their Khatri-Rao product is just their Hadamard product:

$$A \,\widehat{\odot}\, B = [a_{ij} b_{ij}].$$

Interesting matrix inequalities concerning Khatri-Rao products and Hadamard products have been established by many authors, e.g. [1, 5, 6, 15] and references therein.

As is well known, the Kronecker product of complex matrices is generalized to the tensor product of operators on a complex Hilbert space. Recently, the notions of Tracy-Singh product and Khatri-Rao product for Hilbert space operators were investigated in [11, 12, 13, 14].

In this paper, we continue developing the theory of operator products by establishing certain inequalities for Khatri-Rao products of Hilbert space operators. These inequalities involve ordinary products and powers, ordinary and Moore-Penrose inverses. We also discuss Kantorovich type inequalities concerning Khatri-Rao products. Our results generalize some matrix inequalities in [1, 5, 6, 15]. In operator case, we require some mild conditions on operators such as the closeness of their ranges. Moreover, we develop new operator inequalities by using the techniques of block partitioning and unital positive linear maps.

This paper is organized as follows. Section 2 supplies preliminaries on Tracy-Singh products, Khatri-Rao products and Moore-Penrose inverses of operators on a Hilbert space. In Section 3, we establish certain operator inequalities concerning Khatri-Rao products, ordinary products and powers. In Section 4, we derive several inequalities for Khatri-Rao products of operators involving ordinary and Moore-Penrose inverses. In Section 5 deals with Kantorovich type inequalities. Finally, conclusion is provided at the end of the paper.

2. Preliminaries

Let $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{K}$ and $\mathcal{K}'$ be complex Hilbert spaces. When $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, let $\mathbb{B}(\mathcal{X}, \mathcal{Y})$ be the Banach space of all bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$, and abbreviate $\mathbb{B}(\mathcal{X}, \mathcal{X})$ to $\mathbb{B}(\mathcal{X})$. Capital letters always denote bounded linear operators, except for the positive constant M. In particular, I and 0 stand for the identity operator and the zero operator, respectively. Denote the spectrum of an operator X by $\mathrm{Sp}(X)$. As usual, $\otimes$ and $\oplus$ denote the tensor product and the direct sum, respectively.

2.1 Tracy-Singh Products of Operators

From projection theorem, we can make Hilbert space decompositions as follows:

$$\mathcal{H} = \bigoplus_{j=1}^{n} \mathcal{H}_j, \quad \mathcal{H}' = \bigoplus_{i=1}^{m} \mathcal{H}'_i, \quad \mathcal{K} = \bigoplus_{j=1}^{q} \mathcal{K}_j, \quad \mathcal{K}' = \bigoplus_{i=1}^{p} \mathcal{K}'_i. \tag{2.1}$$

Thus each operator $A \in \mathbb{B}(\mathcal{H},\mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K},\mathcal{K}')$ can be represented uniquely as operator matrices

$$A = [A_{ij}]_{i,j=1}^{m,n} \quad \text{and} \quad B = [B_{kl}]_{k,l=1}^{p,q}$$

where $A_{ij} \in \mathbb{B}(\mathcal{H}_j,\mathcal{H}'_i)$ and $B_{kl} \in \mathbb{B}(\mathcal{K}_l,\mathcal{K}'_k)$ for each $i = 1,\dots,m$, $j = 1,\dots,n$, $k = 1,\dots,p$ and $l = 1,\dots,q$.

Definition 1 ([13]). *Let $A = [A_{ij}]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{H},\mathcal{H}')$ and $B = [B_{kl}]_{k,l=1}^{p,q} \in \mathbb{B}(\mathcal{K},\mathcal{K}')$ be partitioned according to the decompositions* (2.1). *We define the Tracy-Singh product of A and B to be the bounded linear operator from $\bigoplus_{j,l=1}^{n,q} \mathcal{H}_j \otimes \mathcal{K}_l$ to $\bigoplus_{i,k=1}^{m,p} \mathcal{H}'_i \otimes \mathcal{K}'_k$ represented by an operator matrix*

$$A \boxtimes B = \big[[A_{ij} \otimes B_{kl}]_{kl}\big]_{ij}.$$

Lemma 1 ([13]). *The Tracy-Singh product for operators satisfies the following properties (provided that each term is well-defined):*

(i) *The map $(A,B) \mapsto A \boxtimes B$ is bilinear.*

(ii) *Compatibility with adjoints: $(A \boxtimes B)^* = A^* \boxtimes B^*$.*

(iii) *Compatibility with ordinary products: $(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD$.*

(iv) *Positivity: if $A \geqslant 0$ and $B \geqslant 0$, then $A \boxtimes B \geqslant 0$.*

Lemma 2 ([14]). *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be positive operators. Then for any positive real number α,*

$$(A \boxtimes B)^\alpha = A^\alpha \boxtimes B^\alpha.$$

2.2 Khatri-Rao products of operators

Throughout this paper, we fix the orthogonal decompositions of Hilbert spaces:

$$\mathcal{H} = \bigoplus_{j=1}^{n} \mathcal{H}_j, \quad \mathcal{H}' = \bigoplus_{i=1}^{m} \mathcal{H}'_i, \quad \mathcal{K} = \bigoplus_{j=1}^{n} \mathcal{K}_j, \quad \mathcal{K}' = \bigoplus_{i=1}^{m} \mathcal{K}'_i. \tag{2.2}$$

Definition 2 ([11]). *Let $A = \big[A_{ij}\big]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{H},\mathcal{H}')$ and $B = \big[B_{ij}\big]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{K},\mathcal{K}')$ be partitioned according to the decompositions* (2.2). *The Khatri-Rao product of A and B is defined to be the bounded linear operator from $\bigoplus_{j=1}^{n} \mathcal{H}_j \otimes \mathcal{K}_j$ to $\bigoplus_{i=1}^{m} \mathcal{H}'_i \otimes \mathcal{K}'_i$ represented by an operator matrix*

$$A \boxdot B = \big[A_{ij} \otimes B_{ij}\big]_{i,j=1}^{m,n}.$$

Lemma 3 ([11]). *The Khatri-Rao product for operators satisfies the following properties:*

(i) *Compatibility with adjoints: $(A \boxdot B)^* = A^* \boxdot B^*$.*

(ii) *Positivity: if $A \geqslant 0$ and $B \geqslant 0$, then $A \boxdot B \geqslant 0$.*

Lemma 4 ([11]). *There are isometries Z_1 and Z_2 such that $Z_i Z_i^* \leqslant I$ for $i = 1,2$ and for any $A \in \mathbb{B}(\mathcal{H},\mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K},\mathcal{K}')$, we have*

$$A \boxdot B = Z_1^*(A \boxtimes B)Z_2.$$

In particular, if $\mathcal{H} = \mathcal{H}'$ *and* $\mathcal{K} = \mathcal{K}'$, *then* $Z_1 = Z_2 := Z$ *and*

$$A \boxdot B = Z^*(A \boxtimes B)Z$$

for any $A \in \mathbb{B}(\mathcal{H})$ *and* $B \in \mathbb{B}(\mathcal{K})$.

We call (Z_1, Z_2) the ordered pair of selection operators associated with the ordered tuple $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$ of Hilbert spaces and call Z the selection operator associated with the ordered tuple $(\mathcal{H}, \mathcal{K})$.

Lemma 5 ([11]). *Let* (Z_1, Z_2) *be the ordered pair of selection operators associated with the ordered tuple* $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$ *of Hilbert spaces. Then there are bounded linear operators* V *and* W *such that*

$$Z_1^* V = 0, \quad Z_2^* W = 0, \quad Z_1 Z_1^* + VV^* = I, \quad Z_2 Z_2^* + WW^* = I.$$

2.3 Moore-Penrose inverse of operators

Recall that a *Moore-Penrose inverse* of $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$ is an operator $A^\dagger \in \mathbb{B}(\mathcal{K}, \mathcal{H})$ satisfying the following conditions ([10])

(i) $AA^\dagger A = A$,

(ii) $A^\dagger A A^\dagger = A^\dagger$,

(iii) $(AA^\dagger)^* = AA^\dagger$,

(iv) $(A^\dagger A)^* = A^\dagger A$.

It is well known (see *e.g.* [2]) that a Moore-Penrose inverse of $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$ exists if and only if the range of A is closed. In this case, $A^\dagger$ is uniquely determined. If A is invertible, then $A^\dagger = A^{-1}$.

The next lemma provides a block-matrix technique for deriving operator inequalities.

Lemma 6 ([16]). *Consider an operator in* $\mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ *represented by an operator matrix*

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^* & R_{22} \end{bmatrix},$$

where R_{11} *and* R_{22} *are Hermitian, and* R_{11} *has a closed range. Then* $R \geqslant 0$ *if and only if*

(i) $R_{11} \geqslant 0$,

(ii) $R_{12} = R_{11} R_{11}^\dagger R_{12}$,

(iii) $R_{22} \geqslant R_{12}^* R_{11}^\dagger R_{12}$.

Lemma 7 ([13]). *Let* $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ *and* $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$. *If* A *and* B *have closed ranges, then so is* $A \boxtimes B$ *and the following property holds:*

$$(A \boxtimes B)^\dagger = A^\dagger \boxtimes B^\dagger.$$

3. Inequalities involving Ordinary Products and Powers

In this section, we derive certain inequalities for Khatri-Rao products involving ordinary products and powers of operators.

Proposition 1. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be Hermitian operators. Then*

$$(A \boxdot B)^2 \leqslant A^2 \boxdot B^2.$$

Proof. The operators $A \boxdot B$ and $A^2 \boxdot B^2$ are Hermitian due to the property (i) in Lemma 3. Let Z be the selection operator associated with the ordered tuple $(\mathcal{H},\mathcal{K})$. Lemmas 1(iii) and 4 imply that

$$\begin{aligned}(A \boxdot B)^2 &= (Z^*(A \boxtimes B)Z)^2 \\ &= Z^*(A \boxtimes B)ZZ^*(A \boxtimes B)Z \\ &\leqslant Z^*(A \boxtimes B)I(A \boxtimes B)Z \\ &= Z^*(A^2 \boxtimes B^2)Z \\ &= A^2 \boxdot B^2.\end{aligned}$$

The above inequality holds since $A \boxtimes B$ is Hermitian (by Lemma 1(ii)). □

The notion of unital positive linear map is useful in later discussions.

Definition 3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces. A map $\Phi : \mathbb{B}(\mathcal{X}) \to \mathbb{B}(\mathcal{Y})$ is said to be unital if $\Phi(I) = I$. The map Φ is said to be positive if $\Phi(A) \geqslant 0$ whenever $A \geqslant 0$.*

Theorem 1. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be positive operators. Then for $\alpha \in [1,2]$, we have*

$$(A \boxdot B)^\alpha \leqslant A^\alpha \boxdot B^\alpha. \tag{3.1}$$

For $\alpha \in (0,1]$, inequality (3.1) will be reversed.

Proof. Note first that $A \boxdot B \geqslant 0$ by Lemma 3(ii). For any unital positive linear map Φ and a positive operator X, the following holds for $\alpha \in [1,2]$ (see [9])

$$\Phi(X)^\alpha \leqslant \Phi(X^\alpha). \tag{3.2}$$

When $\alpha \in (0,1]$, the inequality (3.2) will be reversed. Consider a map

$$\Phi(X) = Z^* X Z$$

where Z is the selection operator in Lemma 4, associated with the ordered tuple $(\mathcal{H},\mathcal{K})$. Then Φ is linear, unital and positive. It follows from Lemmas 2 and 4 that for any $\alpha \in [1,2]$,

$$\begin{aligned}(A \boxdot B)^\alpha &= \left(Z^*(A \boxtimes B)Z\right)^\alpha \leqslant Z^*(A \boxtimes B)^\alpha Z \\ &= Z^*(A^\alpha \boxtimes B^\alpha)Z = A^\alpha \boxdot B^\alpha.\end{aligned}$$

Similarly, the inequality (3.1) will be reversed in the case $\alpha \in (0,1]$. □

Lemma 8. *Let $S \in \mathbb{B}(\mathcal{H})$ be a positive operator such that $\operatorname{Sp}(S) \subseteq [m,M]$ for some constants $m,M > 0$. Then*

$$S^2 \leqslant (m+M)S - mMI. \tag{3.3}$$

Proof. Since $mI \leqslant S \leqslant MI$, we have $(MI - S)(mI - S) \leqslant 0$ and hence

$$(MI - S)(mI - S)S^{-1} = S^{-\frac{1}{2}}(MI - S)(mI - S)S^{-\frac{1}{2}} \leqslant 0.$$

It follows that $S \leqslant (m + M)I - mMS^{-1}$ and hence the inequality (3.3) holds. □

Lemma 9. *Let* $S \in \mathbb{B}(\mathcal{H})$ *be a Hermitian operator such that* $\mathrm{Sp}(S) \subseteq [m, M]$ *for some constants* $m, M \in \mathbb{R}$. *For any isometry* $X \in \mathbb{B}(\mathcal{K}, \mathcal{H})$, *we have*

$$4mM(X^*S^2X) \leqslant (m + M)^2(X^*SX)^2.$$

Proof. The case $mM \leqslant 0$ is trivial. Consider the case $M > m > 0$, i.e., $S > 0$. Since $X^*X = I$, it follows from Lemma 8 that

$$\begin{aligned} X^*S^2X &\leqslant X^*((m + M)S - mMI)X \\ &= \frac{(m + M)^2}{4mM}(X^*SX)^2 - \left(\frac{m + M}{2\sqrt{mM}}X^*SX - \sqrt{mM}I\right)^2 \\ &\leqslant \frac{(m + M)^2}{4mM}(X^*SX)^2. \end{aligned}$$

Finally, consider the case $m < M < 0$, i.e., $S < 0$. This case is done by applying the previous case to $-S$. □

Note that in Lemma 9, there always exists two constants m, M for which $\mathrm{Sp}(S) \subseteq [m, M]$. For example, one can take

$$m = \inf_{\|x\|=1} \langle Sx, x\rangle \quad \text{and} \quad M = \sup_{\|x\|=1} \langle Sx, x\rangle.$$

For the case of Hermitian matrices, m and M reduce to the smallest and the largest eigenvalues of S, respectively.

Theorem 2. *Let* $A \in \mathbb{B}(\mathcal{H})$ *and* $B \in \mathbb{B}(\mathcal{K})$ *be Hermitian operators such that* $\mathrm{Sp}(A \boxtimes B) \subseteq [m, M]$. *Then*

$$4mM(A^2 \boxdot B^2) \leqslant (m + M)^2(A \boxdot B)^2. \tag{3.4}$$

Proof. Substitute $S = A \boxtimes B$ and $X = Z$ in Lemma 9, where Z is the selection operator. It follows from Lemmas 2 and 4 that

$$\begin{aligned} 4mM(A^2 \boxdot B^2) &= 4mMZ^*(A^2 \boxtimes B^2)Z \\ &= 4mMZ^*(A \boxtimes B)^2Z \\ &\leqslant (m + M)^2\left(Z^*(A \boxtimes B)Z\right)^2 \\ &= (m + M)^2(A \boxdot B)^2. \end{aligned}$$
□

Corollary 1. *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$ *be Hermitian matrices with eigenvalues* $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ *and* $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n$, *respectively. Then*

$$4\lambda_1\mu_1\lambda_n\mu_n(A^2 \widehat{\odot} B^2) \leqslant (\lambda_1\mu_1 + \lambda_n\mu_n)^2(A \widehat{\odot} B)^2.$$

Proof. Consider a matrix case in Theorem 2. When we partition matrices A and B entrywise, their Khatri-Rao product and their Tracy-Singh product reduce to the Hadamard product $A \hat{\odot} B$ and the Kronecker product $A \hat{\otimes} B$, respectively. Note that the smallest and the largest eigenvalues of $A \hat{\otimes} B$ are given by $\lambda_n \mu_n$ and $\lambda_1 \mu_1$, respectively. $\square$

Theorem 3. *Let $A, B \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and let α, β be any real scalars not both zero. Then*

$$\begin{aligned} &\alpha^2 AA^* \boxdot BB^* + \alpha\beta\left(AB^* \boxdot BA^* + BA^* \boxdot AB^*\right) + \beta^2 BB^* \boxdot AA^* \\ &\qquad\qquad \geqslant (\alpha A \boxdot B + \beta B \boxdot A)(\alpha A^* \boxdot B^* + \beta B^* \boxdot A^*). \end{aligned} \tag{3.5}$$

Proof. Let (Z_1, Z_2) be the ordered pair of selection operators associated with $(\mathcal{H}, \mathcal{H}', \mathcal{H}, \mathcal{H}')$. Set

$$T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} Z_2 & 0 \\ 0 & Z_1 \end{bmatrix}$$

where $T_1 = I \boxtimes I$ and $T_2 = \alpha A \boxtimes B + \beta B \boxtimes A$. It follows that

$$\begin{aligned} 0 &\leqslant TT^* \\ &= \begin{bmatrix} T_1 T_1^* & T_1 T_2^* \\ T_2 T_1^* & T_2 T_2^* \end{bmatrix} \\ &= \begin{bmatrix} I & \alpha A^* \boxtimes B^* + \beta B^* \boxtimes A^* \\ \alpha A \boxtimes B + \beta B \boxtimes A & X \end{bmatrix}, \end{aligned}$$

where $X = \alpha^2 AA^* \boxtimes BB^* + \alpha\beta(AB^* \boxtimes BA^* + BA^* \boxtimes AB^*) + \beta^2 BB^* \boxtimes AA^*$. Using Lemma 4, we obtain

$$\begin{aligned} 0 &\leqslant R^* TT^* R \\ &= \begin{bmatrix} I & \alpha A^* \boxdot B^* + \beta B^* \boxdot A^* \\ \alpha A \boxdot B + \beta B \boxdot A & Y \end{bmatrix}, \end{aligned}$$

where $Y = \alpha^2 AA^* \boxdot BB^* + \alpha\beta(AB^* \boxdot BA^* + BA^* \boxdot AB^*) + \beta^2 BB^* \boxdot AA^*$. Note that Y is Hermitian by Lemma 3(i). The proof is done by applying Lemma 6. $\square$

Corollary 2. *For any $A, B \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$, we have*

$$AA^* \boxdot BB^* \geqslant (A \boxdot B)(A^* \boxdot B^*).$$

Proof. This is a special case $\alpha = 1, \beta = 0$ of Theorem 3. $\square$

Visick [15, Theorem 11] show that for any $m \times n$ complex matrices A and B, and for any $\gamma \in [-1, 1]$, we have

$$AA^* \hat{\odot} BB^* + \gamma(AB^* \hat{\odot} BA^*) \geqslant (1+\gamma)(A \hat{\odot} B)(A \hat{\odot} B)^*.$$

The next result generalizes this fact to Khatri-Rao product of operators.

Corollary 3. *Let $A, B \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $\gamma \in [-1, 1]$. Suppose $A \boxdot B = B \boxdot A$, $AB^* \boxdot BA^* = BA^* \boxdot AB^*$ and $AA^* \boxdot BB^* = BB^* \boxdot AA^*$. Then*

$$AA^* \boxdot BB^* + \gamma(AB^* \boxdot BA^*) \geqslant (1+\gamma)(A \boxdot B)(A^* \boxdot B^*). \tag{3.6}$$

Proof. Since $|\gamma| \leqslant 1$, we can write $\gamma = 2\alpha\beta/\left(\alpha^2+\beta^2\right)$ for some real numbers α and β not both zero. It follows from Theorem 3 that

$$\left(\alpha^2+\beta^2\right)\left(AA^* \boxdot BB^*\right) + 2\alpha\beta(AB^* \boxdot BA^*) \geqslant (\alpha+\beta)^2 (A \boxdot B)(A \boxdot B)^*.$$

Dividing both sides with $\alpha^2+\beta^2$ yields the desired result. $\square$

The next result generalizes the matrix result for Hadamard products provided in [15, Corollary 12] to the Khatri-Rao product of operators.

Corollary 4. *Let $A,B \in \mathbb{B}(\mathcal{H},\mathcal{H}')$. Suppose that $A \boxdot B = B \boxdot A$, $AB^* \boxdot BA^* = BA^* \boxdot AB^*$ and $AA^* \boxdot BB^* = BB^* \boxdot AA^*$. We have*

$$AA^* \boxdot BB^* \geqslant \pm(AB^* \boxdot BA^*), \tag{3.7}$$

$$2(AA^* \boxdot BB^*) \geqslant AA^* \boxdot BB^* + AB^* \boxdot BA^* \geqslant 2(A \boxdot B)(A^* \boxdot B^*). \tag{3.8}$$

Moreover, the following statements are equivalent:

(i) $AA^* \boxdot BB^* + AB^* \boxdot BA^* = 2(A \boxdot B)(A^* \boxdot B^*)$,

(ii) $Z_1^*(A \boxtimes B + B \boxtimes A)W = 0$,

(iii) $AC \boxdot BD + BC \boxdot AD = 2(A \boxdot B)(C \boxdot D)$ *for all* $C,D \in \mathbb{B}(\mathcal{H}',\mathcal{H})$.

Here, the operators Z_1 and W are described in Lemmas 4 and 5, respectively.

Proof. By taking $\gamma = -1$ in inequality (3.6), we get

$$AA^* \boxdot BB^* \geqslant AB^* \boxdot BA^*.$$

Letting $\gamma = 1$ in (3.6) yields

$$AA^* \boxdot BB^* + AB^* \boxdot BA^* \geqslant 2(A \boxdot B)(A \boxdot B)^* \geqslant 0.$$

Hence we obtain the inequalities (3.7) and (3.8).

It is clear that (iii)⇒(i). To prove (i)⇒(ii), note that the condition $A \boxdot B = B \boxdot A$ implies $A^* \boxdot B^* = B^* \boxdot A^*$ via Lemma 3(i). Note that the pair (Z_1,Z_2) is associated to $(\mathcal{H},\mathcal{H}',\mathcal{H},\mathcal{H}')$ while the pair (Z_2,Z_1) is associated to $(\mathcal{H}',\mathcal{H},\mathcal{H}',\mathcal{H})$. It follows from Lemmas 1(iii), 4 and 5 that

$$\begin{aligned}
&4(A \boxdot B)(A^* \boxdot B^*) \\
&= (2A \boxdot B)(2A^* \boxdot B^*) \\
&= (A \boxdot B + B \boxdot A)(A^* \boxdot B^* + B^* \boxdot A^*) \\
&= Z_1^*(A \boxtimes B + B \boxtimes A)Z_2 Z_2^*(A^* \boxtimes B^* + B^* \boxtimes A^*)Z_1 \\
&= Z_1^*(A \boxtimes B + B \boxtimes A)(I - WW^*)(A^* \boxtimes B^* + B^* \boxtimes A^*)Z_1 \\
&= Z_1^*(AA^* \boxtimes BB^* + AB^* \boxtimes BA^* + BA^* \boxtimes AB^* + BB^* \boxtimes AA^*)Z_1 \\
&\quad - \left[Z_1^*(A \boxtimes B + B \boxtimes A)W\right]\left[Z_1^*(A \boxtimes B + B \boxtimes A)W\right]^* \\
&= AA^* \boxdot BB^* + AB^* \boxdot BA^* + BA^* \boxdot AB^* + BB^* \boxdot AA^*
\end{aligned}$$

$$
\begin{aligned}
&- \left[Z_1^*(A \boxtimes B + B \boxtimes A)W\right]\left[Z_1^*(A \boxtimes B + B \boxtimes A)W\right]^* \\
&= 2(AA^* \boxdot BB^* + AB^* \boxdot BA^*) - \left[Z_1^*(A \boxtimes B + B \boxtimes A)W\right]\left[Z_1^*(A \boxtimes B + B \boxtimes A)W\right]^*.
\end{aligned}
$$

Then (i) holds only if

$$\left[Z_1^*(A \boxtimes B + B \boxtimes A)W\right]\left[Z_1^*(A \boxtimes B + B \boxtimes A)W\right]^* = 0,$$

i.e., $Z_1^*(A \boxtimes B + B \boxtimes A)W = 0$.

Now, suppose that (ii) holds. We have by Lemma 5 that

$$Z_1^*(A \boxtimes B + B \boxtimes A)(I - Z_2 Z_2^*) = Z_1^*(A \boxtimes B + B \boxtimes A)WW^* = 0$$

or $Z_1^*(A \boxtimes B + B \boxtimes A) = Z_1^*(A \boxtimes B + B \boxtimes A)Z_2 Z_2^*$. It follows from Lemmas 1(iii) and 4 that for any $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$ and $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$,

$$
\begin{aligned}
AC \boxdot BD + BC \boxdot AD &= Z_1^*(AC \boxtimes BD + BC \boxtimes AD)Z_1 \\
&= Z_1^*(A \boxtimes B + B \boxtimes A)(C \boxtimes D)Z_1 \\
&= Z_1^*(A \boxtimes B + B \boxtimes A)Z_2 Z_2^*(C \boxtimes D)Z_1 \\
&= 2(A \boxdot B + B \boxdot A)(C \boxdot D).
\end{aligned}
$$

□

A simple form of inequality (3.8) is obtained for the case of Hermitian operators.

Corollary 5. *Let $A, B \in \mathbb{B}(\mathcal{H})$ be Hermitian operators. Suppose $A \boxdot B = B \boxdot A$, $A^2 \boxdot B^2 = B^2 \boxdot A^2$ and $AB \boxdot BA = BA \boxdot AB$. Then*

$$(A \boxdot B)^2 \leqslant \frac{1}{2}\left(A^2 \boxdot B^2 + AB \boxdot BA\right) \leqslant A^2 \boxdot B^2. \tag{3.9}$$

The inequality (3.9) for Hadamard product of matrices was obtained in [15, Corollary 13] as follows: For any Hermitian matrices $A, B \in \mathbb{M}_n(\mathbb{C})$, we have

$$(A \mathbin{\hat{\circ}} B)^2 \leqslant \frac{1}{2}\left(A^2 \mathbin{\hat{\circ}} B^2 + AB \mathbin{\hat{\circ}} BA\right) \leqslant A^2 \mathbin{\hat{\circ}} B^2.$$

Note that in this case we do not impose commutativity conditions since the Hadamard product of matrices is always commutative.

4. Inequalities involving Ordinary and Moore-Penrose Inverses

This section deals with operator inequalities for Khatri-Rao products involving ordinary and Moore-Penrose inverses.

Lemma 10 ([4]). *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces. For any unital positive linear map $\Phi : \mathbb{B}(\mathcal{X}) \to \mathbb{B}(\mathcal{Y})$ and for any operator $T \in \mathbb{B}(\mathcal{X})$ such that $\mathrm{Sp}(T) \subseteq [m, M] \subseteq (0, \infty)$, we have*

$$\Phi(T)^{-1} \leqslant \Phi(T^{-1}) \leqslant \frac{(m+M)^2}{4mM}\Phi(T)^{-1}. \tag{4.1}$$

Theorem 4. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be positive operators. Suppose that there are positive constants $m, M > 0$ such that $\mathrm{Sp}(A \boxtimes B) \subseteq [m, M]$. Then*

$$(A \boxdot B)^{-1} \leqslant A^{-1} \boxdot B^{-1} \leqslant \frac{(m+M)^2}{4mM}(A \boxdot B)^{-1}. \tag{4.2}$$

Proof. Consider a unital positive linear map $\Phi : T \mapsto Z^*TZ$ where Z is the selection operator described in Lemma 4, associated with the ordered tuple $(\mathcal{H},\mathcal{K})$. Using Lemmas 1(iv), 4 and 10, we get

$$\begin{aligned}(A \boxdot B)^{-1} &= \big(Z^*(A \boxtimes B)Z\big)^{-1} \leqslant Z^*(A \boxtimes B)^{-1}Z \\ &= Z^*(A^{-1} \boxtimes B^{-1})Z = A^{-1} \boxdot B^{-1}.\end{aligned}$$

Similarly, we obtain the right-hand side of (4.2). □

Corollary 6. *Let $A,B \in \mathbb{M}_n(\mathbb{C})$ be positive definite matrices with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ and $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n$, respectively. Then*

$$(A \,\widehat{\odot}\, B)^{-1} \leqslant A^{-1} \,\widehat{\odot}\, B^{-1} \leqslant \frac{(\lambda_1\mu_1 + \lambda_n\mu_n)^2}{4\lambda_1\mu_1\lambda_n\mu_n}(A \,\widehat{\odot}\, B)^{-1}.$$

Proof. Consider a matrix case in Theorem 4. Partition A and B entrywise and then take m and M to be the smallest and the largest eigenvalues of $A \boxtimes B = A \,\widehat{\otimes}\, B$, respectively. □

Theorem 5. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be positive invertible operators. Suppose that $\mathrm{Sp}((A^{-1} \boxtimes B) \oplus (A \boxtimes B^{-1})) \subseteq [m,M] \subseteq (0,\infty)$. Then we have the following bounds:*

$$2I \leqslant A \boxdot B^{-1} + A^{-1} \boxdot B \leqslant \frac{m+M}{\sqrt{mM}}I. \tag{4.3}$$

Proof. The operator $T := A \boxtimes B^{-1}$ is positive by Lemma 1(v). The spectral mapping theorem implies that $T + T^{-1} \geqslant 2I$. Applying Lemma 1, we have

$$A \boxtimes B^{-1} + A^{-1} \boxtimes B = A \boxtimes B + \big(A \boxtimes B^{-1}\big)^{-1} \geqslant 2I.$$

Lemma 4 then implies that $A \boxdot B^{-1} + A^{-1} \boxdot B \geqslant 2I$. Let Z be the selection operator associated with $(\mathcal{H},\mathcal{K})$. Denote

$$S = \big(A^{-1} \boxtimes B\big) \oplus \big(A \boxtimes B^{-1}\big) \quad \text{and} \quad X = \frac{1}{\sqrt{2}}\begin{bmatrix} Z \\ Z \end{bmatrix}.$$

Since Z is an isometry, we have $X^*X = I$. It follows that the map $\Phi : T \mapsto X^*TX$ is a unital positive linear map. Using Lemma 4 again, we obtain

$$\begin{aligned}X^*SX &= \frac{1}{2}\begin{bmatrix} Z^* & Z^* \end{bmatrix}\begin{bmatrix} A^{-1} \boxtimes B & 0 \\ 0 & A \boxtimes B^{-1} \end{bmatrix}\begin{bmatrix} Z \\ Z \end{bmatrix} \\ &= \frac{1}{2}Z^*(A^{-1} \boxtimes B + A \boxtimes B^{-1})Z \\ &= \frac{1}{2}\big(A \boxdot B^{-1} + A^{-1} \boxdot B\big).\end{aligned}$$

The property (iv) of Lemma 1 implies that

$$\begin{aligned}X^*S^{-1}X &= \frac{1}{2}\begin{bmatrix} Z^* & Z^* \end{bmatrix}\begin{bmatrix} A \boxtimes B^{-1} & 0 \\ 0 & A^{-1} \boxtimes B \end{bmatrix}\begin{bmatrix} Z \\ Z \end{bmatrix} \\ &= \frac{1}{2}\big(A \boxdot B^{-1} + A^{-1} \boxdot B\big).\end{aligned}$$

Now, Lemma 10 assures that

$$A \boxdot B^{-1} + A^{-1} \boxdot B \leqslant \frac{(m+M)^2}{mM} \left(A \boxdot B^{-1} + A^{-1} \boxdot B\right)^{-1}.$$

Thus we obtain the right-hand side of (4.3). □

Corollary 7. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive definite matrices with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ and $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n$, respectively. Then*

$$2I \leqslant A \,\widehat{\odot}\, B^{-1} + A^{-1} \,\widehat{\odot}\, B \leqslant \frac{m+M}{\sqrt{mM}} I,$$

where $m = \min\left\{\frac{\mu_n}{\lambda_1}, \frac{\lambda_n}{\mu_1}\right\}$ *and* $M = \max\left\{\frac{\mu_1}{\lambda_n}, \frac{\lambda_1}{\mu_n}\right\}$.

Proof. When we partition A and B entrywise, the Khatri-Rao product $A \,\widehat{\boxdot}\, B$ reduces to the Hadamard product $A \,\widehat{\odot}\, B$. In this case, their Tracy-Singh product reduces to the Kronecker product. We have

$$\begin{aligned}
\operatorname{Sp}\begin{bmatrix} A \,\widehat{\boxtimes}\, B^{-1} & 0 \\ 0 & A^{-1} \,\widehat{\boxtimes}\, B \end{bmatrix} &= \operatorname{Sp}(A^{-1} \,\widehat{\otimes}\, B) \cup \operatorname{Sp}(A \,\widehat{\otimes}\, B^{-1}) \\
&= \{\lambda^{-1}\mu : \lambda \in \operatorname{Sp}(A), \mu \in \operatorname{Sp}(B)\} \cup \{\lambda\mu^{-1} : \lambda \in \operatorname{Sp}(A), \mu \in \operatorname{Sp}(B)\} \\
&\subseteq [m, M].
\end{aligned}$$

Now, the result follows from Theorem 5. □

Proposition 2. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be positive operators. If both A and B have closed ranges, then*

$$A \boxdot B^{\dagger} + A^{\dagger} \boxdot B \geqslant 2AA^{\dagger} \boxdot BB^{\dagger}. \tag{4.4}$$

Proof. Since the ranges of A and B are closed, the Moore-Penrose inverses $A^{\dagger}$ and $B^{\dagger}$ exists and are unique. The positivity of A and B implies that $B^{\dagger}B = BB^{\dagger}$ and $S := A \boxtimes B^{\dagger} \geqslant 0$. The spectral mapping theorem implies that $S + S^{\dagger} \geqslant 2SS^{\dagger}$. It follows from Lemmas 1(iii) and 7 that

$$\begin{aligned}
A \boxtimes B^{\dagger} + A^{\dagger} \boxtimes B &= A \boxtimes B^{\dagger} + (A \boxtimes B^{\dagger})^{\dagger} \\
&\geqslant 2(A \boxtimes B^{\dagger})(A \boxtimes B^{\dagger})^{\dagger} \\
&= 2(A \boxtimes B^{\dagger})(A^{\dagger} \boxtimes B) \\
&= 2AA^{\dagger} \boxtimes B^{\dagger}B.
\end{aligned}$$

We get the desired result by pre- and post-multiplying the above inequality with Z^* and Z. □

We mention that Proposition 2 is an operator extension of [6, Theorem 6].

Remark 1. *If A and B are strictly positive, the inequality* (4.4) *reduces to the left-hand side of* (4.3).

Theorem 6. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be positive operators. If A,B and $A \boxdot B$ have closed ranges, then*

$$\left(A \boxdot BB^{\dagger} + AA^{\dagger} \boxdot B\right)(A \boxdot B)^{\dagger}\left(A \boxdot BB^{\dagger} + AA^{\dagger} \boxdot B\right) \leqslant A \boxdot B^{\dagger} + A^{\dagger} \boxdot B + 2AA^{\dagger} \boxdot BB^{\dagger}. \tag{4.5}$$

Proof. Since the ranges of A,B and $A \boxdot B$ are closed, the operators $A^{\dagger}$, $B^{\dagger}$ and $(A \boxdot B)^{\dagger}$ are well-defined. Let Z be the selection operator and denote

$$S = \begin{bmatrix} A^{1/2} \boxtimes B^{1/2} & A^{1/2} \boxtimes \left(B^{\dagger}\right)^{1/2} + \left(A^{\dagger}\right)^{1/2} \boxtimes B^{1/2} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}.$$

Using Lemma 1, we get

$$0 \leqslant S^{*}S = \begin{bmatrix} A \boxtimes B & A \boxtimes BB^{\dagger} + AA^{\dagger} \boxtimes B \\ A \boxtimes BB^{\dagger} + AA^{\dagger} \boxtimes B & A \boxtimes B^{\dagger} + A^{\dagger} \boxtimes B + 2AA^{\dagger} \boxtimes BB^{\dagger} \end{bmatrix}.$$

Using Lemma 4, we obtain

$$\begin{aligned} 0 &\leqslant X^{*}S^{*}SX \\ &= \begin{bmatrix} Z^{*} & 0 \\ 0 & Z^{*} \end{bmatrix} \begin{bmatrix} A \boxtimes B & A \boxtimes BB^{\dagger} + AA^{\dagger} \boxtimes B \\ A \boxtimes BB^{\dagger} + AA^{\dagger} \boxtimes B & A \boxtimes B^{\dagger} + A^{\dagger} \boxtimes B + 2AA^{\dagger} \boxtimes BB^{\dagger} \end{bmatrix} \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix} \\ &= \begin{bmatrix} A \boxdot B & A \boxdot BB^{\dagger} + AA^{\dagger} \boxdot B \\ A \boxdot BB^{\dagger} + AA^{\dagger} \boxdot B & Y \end{bmatrix}, \end{aligned}$$

where $Y = A \boxdot B^{\dagger} + A^{\dagger} \boxdot B + 2AA^{\dagger} \boxdot BB^{\dagger}$. Note that $A \boxdot B \geqslant 0$ by Lemma 3(ii). The operator Y is Hermitian by Lemma 3(i). Now, we get the desired result by applying Lemma 6. □

Theorem 6 is a generalization of a matrix version given in [6, Theorem 5].

5. Kantorovich type inequalities

In this section, we generalize some well-known Kantorovich type inequalities for Khatri-Rao products of matrices to that of operators. Moreover, we establish new operator inequalities.

The following lemma is an operator extension of [7].

Lemma 11. *Let $S \in \mathbb{B}(\mathcal{K})$ be a positive operator with $\mathrm{Sp}(S) \subseteq [m,M] \subseteq (0,\infty)$ and let $X \in \mathbb{B}(\mathcal{H},\mathcal{K})$ be an isometry. Then*

$$X^{*}SX - (X^{*}S^{-1}X)^{-1} \leqslant (\sqrt{M} - \sqrt{m})^{2} I, \tag{5.1}$$

$$X^{*}S^{2}X - (X^{*}SX)^{2} \leqslant \frac{1}{4}(M-m)^{2} I, \tag{5.2}$$

$$(X^{*}S^{2}X)^{1/2} - X^{*}SX \leqslant \frac{(M-m)^{2}}{4(M+m)} I. \tag{5.3}$$

Proof. The proof is similar to that of matrix versions in [7]. □

In [5, Theorem 8], Liu gave certain matrix inequalities involving Khatri-Rao products. Now, we extend some Liu's results to Khatri-Rao product of operators.

Proposition 3. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be positive invertible operators and $0 < mI \leqslant A \boxtimes B \leqslant MI$. Then*

$$A \boxdot B - (A^{-1} \boxdot B^{-1})^{-1} \leqslant (\sqrt{M} - \sqrt{m})^2 I, \tag{5.4}$$

$$A^2 \boxdot B^2 - (A \boxdot B)^2 \leqslant \frac{1}{4}(M-m)^2 I, \tag{5.5}$$

$$(A^2 \boxdot B^2)^{1/2} - A \boxdot B \leqslant \frac{(M-m)^2}{4(M+m)} I. \tag{5.6}$$

Proof. From (5.1), set $S = A \boxtimes B$ and $X = Z$, where Z is the associated selection operator. It follows from Lemmas 1 and 4 that

$$\begin{aligned} A \boxdot B - (A^{-1} \boxdot B^{-1})^{-1} &= Z^*(A \boxtimes B)Z - (Z^*(A^{-1} \boxtimes B^{-1})Z)^{-1} \\ &= Z^*(A \boxtimes B)Z - (Z^*(A \boxtimes B)^{-1}Z)^{-1} \\ &\leqslant (\sqrt{M} - \sqrt{m})^2 I. \end{aligned}$$

Thereby, from (5.2) and (5.3), we obtain (5.5) and (5.6). □

The next result is a new Kantorovich-type inequality involving Khatri-Rao products.

Theorem 7. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be positive invertible operators such that $mI \leqslant A \boxtimes B \leqslant MI$ for some positive constants m, M. Then*

$$(A^{-1} \boxdot B^{-1})(A \boxdot B) + (A \boxdot B)(A^{-1} \boxdot B^{-1}) \leqslant \frac{(m+M)^2}{2mM} I. \tag{5.7}$$

Proof. From [4], we have for every unital positive linear map Φ,

$$\Phi(X^{-1})\Phi(X) + \Phi(X)\Phi(X^{-1}) \leqslant \frac{(M+m)^2}{2Mm} I. \tag{5.8}$$

provided that $0 < mI \leqslant X \leqslant MI$. Consider $\Phi(X) = Z^*XZ$ where Z is the selection operator associated with $(\mathcal{H}, \mathcal{K})$. Lemma 1 yields

$$\begin{aligned} &(A^{-1} \boxdot B^{-1})(A \boxdot B) + (A \boxdot B)(A^{-1} \boxdot B^{-1}) \\ &\quad = Z^*(A^{-1} \boxtimes B^{-1})ZZ^*(A \boxtimes B)Z + Z^*(A \boxtimes B)ZZ^*(A^{-1} \boxtimes B^{-1})Z \\ &\quad = Z^*(A \boxtimes B)^{-1}ZZ^*(A \boxtimes B)Z + Z^*(A \boxtimes B)ZZ^*(A \boxtimes B)^{-1}Z \\ &\quad \leqslant \frac{(m+M)^2}{2mM} I. \end{aligned}$$

□

Corollary 8. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive definite matrices with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ and $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n$, respectively. Then*

$$(A \,\hat{\odot}\, B)(A^{-1} \,\hat{\odot}\, B^{-1}) \leqslant \frac{(\lambda_1\mu_1 + \lambda_n\mu_n)^2}{4\lambda_1\mu_1\lambda_n\mu_n} I.$$

Proof. Apply Theorem 7 to matrices A and B partitioned entrywise. □

Lemma 12. *Let $S \in \mathbb{B}(\mathcal{K})$ be a positive operator such that $\mathrm{Sp}(S) \subseteq [m,M] \subseteq (0,\infty)$ and let $X \in \mathbb{B}(\mathcal{H},\mathcal{K})$ have a closed range. Then*

$$X^*SXX^\dagger SX \leqslant X^*S^2X \leqslant \frac{(m+M)^2}{4mM} X^*SXX^\dagger SX, \tag{5.9}$$

$$X^*S^2X - X^*SXX^\dagger SX \leqslant \frac{(M-m)^2}{4} X^*X. \tag{5.10}$$

Proof. The closeness of the range of X implies the existence and uniqueness of $X^\dagger$. Since $XX^\dagger$ is Hermitian and idempotent, it is a projection and thus $XX^\dagger \leqslant I$. Now, the proof can be proceed using the same technique as in [8, Propositions 3.3 and 3.4]. □

The final result is an operator extension of some Khatri-Rao product inequalities in [6, Theorem 1]. Here, we must impose the closeness of the range of a certain operator.

Proposition 4. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be positive operators such that $\mathrm{Sp}(A \boxtimes B) \subseteq [m,M] \subseteq (0,\infty)$. Let $U \in \mathbb{B}(\mathcal{K},\mathcal{H})$, $V \in \mathbb{B}(\mathcal{H},\mathcal{K})$ and let Z be the selection operator associated with the ordered tuple $(\mathcal{K},\mathcal{H})$. If $(U \boxtimes V)Z$ has a closed range, then $U^*U \boxdot V^*V$ has a close range and the following hold:*

$$\begin{aligned}
&(U^*AU \boxdot V^*BV)(U^*U \boxdot V^*V)^\dagger (U^*AU \boxdot V^*BV) \\
&\qquad \leqslant U^*A^2U \boxdot V^*B^2V \\
&\qquad \leqslant \frac{(m+M)^2}{4mM} \left(U^*AU \boxdot V^*BV\right)\left(U^*U \boxdot V^*V\right)^\dagger \left(U^*AU \boxdot V^*BV\right), \\
&U^*A^2U \boxdot V^*B^2V - \left(U^*AU \boxdot V^*BV\right)\left(U^*U \boxdot V^*V\right)^\dagger \left(U^*AU \boxdot V^*BV\right) \\
&\qquad \leqslant \frac{(M-m)^2}{4} (U^*U \boxdot V^*V).
\end{aligned}$$

Proof. Denote $S = A \boxtimes B$ and $X = (U \boxtimes V)Z$. Then S is positive by Lemma 1(v). Since the range of X is closed, $X^\dagger$ exists. It follows that $(X^*X)^\dagger$ exists, i.e., X^*X has a closed range. Using Lemmas 1, 2 and 4, we have

$$\begin{aligned}
X^*SX &= Z^*(U^* \boxtimes V^*)(A \boxtimes B)(U \boxtimes V)Z = U^*AU \boxdot V^*BV, \\
X^*S^2X &= Z^*(U^* \boxtimes V^*)(A^2 \boxtimes B^2)(U \boxtimes V)Z = U^*A^2U \boxdot V^*B^2V, \\
X^\dagger &= \left(X^*X\right)^\dagger X^* = (U^*U \boxdot V^*V)^\dagger Z^*(U^* \boxtimes V^*), \\
X^\dagger SX &= (U^*U \boxdot V^*V)^\dagger Z^*(U^* \boxtimes V^*)(A \boxtimes B)(U \boxtimes V)Z \\
&= (U^*U \boxdot V^*V)^\dagger (U^*AU \boxdot V^*BV).
\end{aligned}$$

Substitution in (5.9) and (5.10), we get the results. □

6. Conclusions

Relations between the Khatri-Rao product of Hilbert space operators and ordinary products, powers, ordinary inverses, and Moore-Penrose inverses are established in terms of inequalities.

In particular, such relations hold for the tensor product of operators, the Khatri-Rao product and the Hadamard product of complex matrices.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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