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Research Article

# Apostol Type ($p$,$q$)-Bernoulli, ($p$,$q$)-Euler and ($p$,$q$)-Genocchi Polynomials and Numbers

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**Abstract.** The main subject of this work is to introduce and investigate a new generalizations of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials under the theory of *post quantum calculus*, denoted by $(p,q)$-calculus. We call them Apostol type $(p,q)$-Bernoulli polynomials of order $\alpha$, Apostol type $(p,q)$-Euler polynomials of order $\alpha$ and the Apostol type $(p,q)$-Genocchi polynomials of order $\alpha$. We derive some of their properties involving addition theorems, difference equations, derivative properties, recurrence relationships, and so on. Also, $(p,q)$-analogues of some familiar formulae belonging to usual Apostol-Bernoulli, Euler and Genocchi polynomials are shown. Furthermore, $(p,q)$-generalizations of Cheon's main result [G.S. Cheon, *Appl. Math. Lett.* **16** (2003) 365–368] and the formula of Srivastava and Pintér [H.M. Srivastava, A. Pintér, *Appl. Math. Lett.* **17** (2004), 375–380] are investigated.

**Keywords.** $(p,q)$-calculus; Apostol Bernoulli polynomials; Apostol Euler polynomials; Apostol Genocchi polynomials; Generating function; Cauchy product

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## 1. Introduction

In the usual notations: $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{N}$ denotes the set of the natural numbers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The $(p,q)$-numbers are defined as

$$[n]_{p,q};=p^{n-1}+p^{n-2}q+p^{n-3}q^2+\cdots+pq^{n-2}+q^{n-1}=\frac{p^n-q^n}{p-q}.$$

It is clear that $[n]_{p,q}=p^{n-1}[n]_{q/p}$, where $[n]_{q/p}$ is the $q$-number in $q$-calculus given by $[n]_{q/p}=\frac{(q/p)^n-1}{(q/p)-1}$. Thereby, it means that $(p,q)$-numbers and $q$-numbers are different, that is, we can not obtain $(p,q)$-numbers just by substituting $q$ by $q/p$ in the definition of $q$-numbers. However, if we take $p=1$ in definition of $(p,q)$-numbers, then $q$-numbers becomes a special case of $(p,q)$-numbers. In company with the introduction of these $(p,q)$-numbers, the theory of $(p,q)$-calculus has been studied and investigated extensively, since 1991, by many mathematicians and physicists. Corcino [4] studied on the $(p,q)$-extension of the binomial coefficients and also derived some properties parallel to those of the ordinary and $q$-binomial coefficients, comprised horizontal generating function, the triangular, vertical, and the horizontal recurrence relations, and the inverse and the orthogonality relationships. Duran, Acikgoz and Araci [6] introduced $(p,q)$-analogues of Bernoulli polynomials, Euler polynomials and Genocchi polynomials and acquired the $(p,q)$-analogues of known earlier formulae. Gupta [8] proposed the $(p,q)$-variant of the Baskakov-Kantorovich operators by means of $(p,q)$-integrals and also analyzed some approximation properties of them. Milovanovic, Gupta and Malik [22] introduced a new extension of Beta functions under the theory of $(p,q)$-calculus and committed the integral modification of the generalized Bernstien polynomials. Sadjang [24] developed some properties of the $(p,q)$-derivatives and the $(p,q)$-integrations and as an application, presented two $(p,q)$-Taylor formulas for polynomials. Besides, Sadjang provided the fundamental theorem of $(p,q)$-calculus and proved the formula of $(p,q)$-integration by part.

We now review briefly some concepts of the $(p,q)$-calculus.

The $(p,q)$-derivative of a function $f$ with respect to $x$ is defined by

$$D_{p,q;x}f(x):=D_{p,q}f(x)=\frac{f(px)-f(qx)}{(p-q)x}\quad (x\neq 0) \tag{1.1}$$

and $(D_{p,q}f)(0)=f'(0)$, provided that $f$ is differentiable at 0. The linear $(p,q)$-derivative operator satisfies the following properties

$$D_{p,q}(f(x)g(x))=g(px)D_{p,q}f(x)+f(qx)D_{p,q}g(x) \tag{1.2}$$

$$=f(px)D_{p,q}g(x)+g(qx)D_{p,q}f(x) \tag{1.3}$$

and

$$D_{p,q}\left(\frac{f(x)}{g(x)}\right)=\frac{g(qx)D_{p,q}f(x)-f(qx)D_{p,q}g(x)}{g(px)g(qx)} \tag{1.4}$$

$$=\frac{g(px)D_{p,q}f(x)-f(px)D_{p,q}g(x)}{g(px)g(qx)}. \tag{1.5}$$

The $(p,q)$-analogue of $(x+a)^n$ is described by

$$(x+a)^n_{p,q}:=\begin{cases}(x+a)(px+aq)\cdots(p^{n-2}x+aq^{n-2})(p^{n-1}x+aq^{n-1}), & \text{if } n\geq 1,\\ 1, & \text{if } n=0,\end{cases}$$

and equally

$$\sum_{k=0}^{n}\left[\begin{matrix} n \\ k \end{matrix}\right]_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} x^k a^{n-k},$$

where the $(p,q)$-Gauss Binomial coefficients $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{p,q}$ and $(p,q)$-factorial $[n]_{p,q}!$ are defined by $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!}$ $(n \geq k)$ and $[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q} \cdots [2]_{p,q}[1]_{p,q}$ $(n \in \mathbb{N})$.

The $(p,q)$-exponential functions,

$$e_{p,q}(x) = \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!} \quad \text{and} \quad E_{p,q}(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!}$$

satisfy

$$e_{p,q}(x)E_{p,q}(-x) = 1 \quad \text{and} \quad e_{p^{-1},q^{-1}}(x) = E_{p,q}(x), \tag{1.6}$$

and have the following $(p,q)$-derivatives

$$D_{p,q}e_{p,q}(x) = e_{p,q}(px) \quad \text{and} \quad D_{p,q}E_{p,q}(x) = E_{p,q}(qx). \tag{1.7}$$

The definite $(p,q)$-integral is defined by

$$\int_0^a f(x)d_{p,q}x = (p-q)a \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}}a\right)$$

in conjunction with

$$\int_a^b f(x)d_{p,q}x = \int_0^b f(x)d_{p,q}x - \int_0^a f(x)d_{p,q}x \quad \text{(see [24])}. \tag{1.8}$$

A more detailed statement of above, including $(p,q)$-notations, is found in [2,4–6,8,9,22,24].

T.M. Apostol [1] introduced an analogue of the classical Bernoulli polynomials and numbers (called Apostol-Bernoulli polynomials and numbers) when he studied the Lipschitz-Lerch Zeta functions and he not only gave elementary properties of these polynomials and numbers, but also acquired the recursion formula for the set of these numbers by means of the Stirling numbers of the second kind. From T.M. Apostol time to the present, Apostol type polynomials and their generalizations have been studied and investigated by many mathematicians. Luo [14] obtained two new formulas for the Apostol-Bernoulli polynomials, using the Gaussian hypergeometric functions and Hurwitz Zeta functions respectively, and also provided certain special cases and applications. Luo and Srivastava [18] gave analogs definitions of Apostol type polynomials (see [1] and [27]) for the so-called Apostol-Bernoulli numbers and polynomials of higher order. Also, they discovered their elementary properties, derived several explicit representations for them in terms of the Gaussian hypergeometric function and the Hurwitz-Zeta function, and presented some of their special cases and applications. Luo and Srivastava [17] proved several general relationships and properties including the Apostol-Bernoulli and Apostol-Euler polynomials. Wang, Jia and Wang [29] established two relationships between the generalized Apostol-Bernoulli and Apostol-Euler polynomials and presented a study on the sums of products of these polynomials. Tremblay, Gaboury and Fugére [28] investigated a new class of generalized

Apostol-Bernoulli polynomials and derived a generalization of the Srivastava-Pintér addition theorem in [26]. Ozarslan [23] provided a unified family of polynomials including the Apostol-Bernoulli, Euler and Genocchi polynomials and also acquired the explicit representation of this unified family, in terms of a Gaussian hypergeometric function. Moreover, he stated some symmetry identities and multiplication formula for this unified family. Mahmudov and Keleshteri [20] studied and investigated generalized Apostol-type Bernoulli, Euler and Genocchi polynomials and numbers and proved $q$-generalization of the Luo-Srivastava addition theorem in [17]. El-Desouky and Gomaa [7] introduced a new unified family of polynomials including unified Apostol-Bernoulli, Euler and Genocchi polynomials by applying the Mellin transformation for the generating function of the new unified family of polynomials and gave a unified class of zeta function. He, Araci, Srivastava and Acikgoz [10] gave a further investigation for the Apostol-Bernoulli polynomials and the Apostol-Genocchi polynomials and by making use of the generating function methods and summation transform techniques, established some new identities involving the products of these polynomials. Kurt [12] defined the Apostol type $q$-Frobenius-Euler polynomials and obtained some identities for these polynomials including a $q$-generalizations of the Srivastava-Pintér addition theorem between the Bernoulli polynomials and Apostol type Frobenius-Euler polynomials. Kurt [13] introduced a new unification of Apostol type polynomials and numbers and investigated some symmetry identities, explicit relations and recurrence relation for these unified family.

The Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x;\lambda)$ of order $\alpha$, the Apostol-Euler polynomials $E_n^{(\alpha)}(x;\lambda)$ of order $\alpha$ and the Apostol-Genocchi polynomials $G_n^{(\alpha)}(x;\lambda)$ of order $\alpha$ are defined by means of the following generating functions:

$$\sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!} = \left(\frac{z}{\lambda e^z - 1}\right)^{\alpha} e^{xz},$$

$$(|z| < 2\pi \text{ when } \lambda = 1; |z| < |\log\lambda| \text{ when } \lambda \neq 1)$$

$$\sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!} = \left(\frac{2}{\lambda e^z + 1}\right)^{\alpha} e^{xz},$$

$$(|z| < \pi \text{ when } \lambda = 1; |z| < |\log(-\lambda)| \text{ when } \lambda \neq 1)$$

$$\sum_{n=0}^{\infty} G_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!} = \left(\frac{2z}{\lambda e^z + 1}\right)^{\alpha} e^{xz},$$

$$(|z| < \pi \text{ when } \lambda = 1; |z| < |\log(-\lambda)| \text{ when } \lambda \neq 1).$$

Letting $x = 0$, we have $B_n^{(\alpha)}(0;\lambda) := B_n^{(\alpha)}(\lambda)$, $E_n^{(\alpha)}(0;\lambda) := E_n^{(\alpha)}(\lambda)$ and $G_n^{(\alpha)}(0;\lambda) := G_n^{(\alpha)}(\lambda)$ which are called, respectively, $n$-th Apostol-Bernoulli number of order $\alpha$, $n$-th Apostol-Euler number of order $\alpha$ and $n$-th Apostol-Genocchi number of order $\alpha$, cf. [1,7,10,12–14,17,18,20,23,27–29]. Notice that in the case $\lambda = \alpha = 1$, then these turns out to the following generating functions

$$\sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!} = \frac{z}{e^z - 1} e^{xz} \quad \text{and} \quad \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = \frac{z}{e^z - 1} \qquad (|z| < 2\pi),$$

$$\sum_{n=0}^{\infty} E_n(x)\frac{z^n}{n!} = \frac{2}{e^z+1}e^{xz} \quad \text{and} \quad \sum_{n=0}^{\infty} E_n\frac{z^n}{n!} = \frac{2}{e^z+1} \qquad (|z|<\pi),$$

$$\sum_{n=0}^{\infty} G_n(x)\frac{z^n}{n!} = \frac{2z}{e^z+1}e^{xz} \quad \text{and} \quad \sum_{n=0}^{\infty} G_n\frac{z^n}{n!} = \frac{2z}{e^z+1} \qquad (|z|<\pi),$$

where $B_n(x)$ and $B_n$, $E_n(x)$ and $E_n$, and $G_n(x)$ and $G_n$ denote the classical Bernoulli polynomials and numbers, classical Euler polynomials and numbers, and classical Genocchi polynomials and numbers, see [3,5,6,11,15,16,19,21,25,26].

The following definition is new and plays an important role in deriving the main results of this paper. Now we are ready to state the following Definition 1.

**Definition 1.** Apostol type $(p,q)$-Bernoulli polynomials $\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q)$ of order $\alpha$, the Apostol type $(p,q)$-Euler polynomials $\mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q)$ of order $\alpha$ and the Apostol type $(p,q)$-Genocchi polynomials $\mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q)$ of order $\alpha$ are defined by means of the following generating functions:

$$\sum_{n=0}^{\infty} \mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q)\frac{z^n}{[n]_{p,q}!} = \left(\frac{z}{\lambda e_{p,q}(z)-1}\right)^{\alpha} e_{p,q}(xz)E_{p,q}(yz)$$

$$\left(|z|<2\pi \text{ when } \lambda=1; |z|<|\log\lambda| \text{ when } \lambda\neq 1,\ 1^{\alpha}=1\right)$$

$$\sum_{n=0}^{\infty} \mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q)\frac{z^n}{[n]_{p,q}!} = \left(\frac{2}{\lambda e_{p,q}(z)+1}\right)^{\alpha} e_{p,q}(xz)E_{p,q}(yz)$$

$$\left(|z|<\pi \text{ when } \lambda=1; |z|<|\log(-\lambda)| \text{ when } \lambda\neq 1,\ 1^{\alpha}=1\right)$$

$$\sum_{n=0}^{\infty} \mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q)\frac{z^n}{[n]_{p,q}!} = \left(\frac{2z}{\lambda e_{p,q}(z)+1}\right)^{\alpha} e_{p,q}(xz)E_{p,q}(yz)$$

$$\left(|z|<\pi \text{ when } \lambda=1; |z|<|\log(-\lambda)| \text{ when } \lambda\neq 1,\ 1^{\alpha}=1\right)$$

where $\lambda$ and $\alpha$ are suitable (real or complex) parameters and $p,q,\in\mathbb{C}$ with $0<|q|<|p|\leq 1$.

Letting $x=0$ and $y=0$ in Definition 1, we then have $\mathscr{B}_n^{(\alpha)}(0,0;\lambda:p,q):=\mathscr{B}_n^{(\alpha)}(\lambda:p,q)$, $\mathscr{E}_n^{(\alpha)}(0,0;\lambda:p,q):=\mathscr{E}_n^{(\alpha)}(\lambda:p,q)$ and $\mathscr{G}_n^{(\alpha)}(0,0;\lambda:p,q):=\mathscr{G}_n^{(\alpha)}(\lambda:p,q)$ which are called, respectively, $n$-th Apostol type $(p,q)$-Bernoulli number of order $\alpha$, $n$-th Apostol type $(p,q)$-Euler number of order $\alpha$ and $n$-th Apostol type $(p,q)$-Genocchi number of order $\alpha$. In the case $\alpha=1$, we have

$$\mathscr{B}_n^{(1)}(x,y;\lambda:p,q):=\mathscr{B}_n(x,y;\lambda:p,q),\ \mathscr{E}_n^{(1)}(x,y;\lambda:p,q):=\mathscr{E}_n(x,y;\lambda:p,q)$$

and

$$\mathscr{G}_n^{(1)}(x,y;\lambda:p,q):=\mathscr{G}_n(x,y;\lambda:p,q)$$

termed as $n$-th Apostol type $(p,q)$-Bernoulli polynomial, $n$-th Apostol type $(p,q)$-Euler polynomial and $n$-th Apostol type $(p,q)$-Genocchi polynomial.

Remark that when $p=1$, the Apostol type $(p,q)$-polynomials in Definition 1 turns out to be Apostol type $q$-polynomials defined earlier by Kurt in [12].

The paper is divided in three sections. In the second section, we develop some basic properties for Apostol type $(p,q)$-Bernoulli polynomials $\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q)$ of order $\alpha$, the Apostol type $(p,q)$-Euler polynomials $\mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q)$ of order $\alpha$ and the Apostol type $(p,q)$-Genocchi polynomials $\mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q)$ of order $\alpha$ involving addition properties, derivative properties and so on. The third section not only includes some explicit formulas and relations among $\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q)$, $\mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q)$ and $\mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q)$, but also provides a new $(p,q)$-extension of main result of Cheon's work [3] and a $(p,q)$-generalization of the formula of Srivastava-Pinter [26]. The last part examines some special cases of our results in this paper.

## 2. Properties of Apostol Type $(p,q)$-Polynomials of order $\alpha$

In this section, we provide some basic formulas and identites for aforementioned Apostol type $(p,q)$-polynomials so as to derive the main outcomes of this paper in the subsequent section. We now start with the following addition theorems for Apostol type $(p,q)$-Bernoulli, $(p,q)$-Euler and $(p,q)$-Genocchi polynomials of order $\alpha$ as Propositions 1 and 2.

**Proposition 1.** *We have*

$$\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\mathscr{B}_k^{(\alpha)}(\lambda:p,q)(x+y)_{p,q}^{n-k},$$

$$\mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\mathscr{E}_k^{(\alpha)}(\lambda:p,q)(x+y)_{p,q}^{n-k},$$

$$\mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\mathscr{G}_k^{(\alpha)}(\lambda:p,q)(x+y)_{p,q}^{n-k}.$$

**Proposition 2.** *The following relationships hold true:*

$$\begin{aligned}\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q)&=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}q^{\binom{n-k}{2}}\mathscr{B}_k^{(\alpha)}(x,0;\lambda:p,q)y^{n-k}\\&=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}p^{\binom{n-k}{2}}\mathscr{B}_k^{(\alpha)}(0,y;\lambda:p,q)x^{n-k},\\ \mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q)&=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}q^{\binom{n-k}{2}}\mathscr{E}_k^{(\alpha)}(x,0;\lambda:p,q)y^{n-k}\\&=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}p^{\binom{n-k}{2}}\mathscr{E}_k^{(\alpha)}(0,y;\lambda:p,q)x^{n-k},\\ \mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q)&=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}q^{\binom{n-k}{2}}\mathscr{G}_k^{(\alpha)}(x,0;\lambda:p,q)y^{n-k}\\&=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}p^{\binom{n-k}{2}}\mathscr{G}_k^{(\alpha)}(0,y;\lambda:p,q)x^{n-k}.\end{aligned}$$

Next, we discuss some special cases of Proposition 1 and 2 in Corollary 3 and 4.

**Corollary 3.** *Upon setting $x=0$ (or $y=0$) in Proposition 2 gives the following formulas*

$$\mathscr{B}_n^{(\alpha)}(x,0;\lambda:p,q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}}\mathscr{B}_k^{(\alpha)}(\lambda:p,q)x^{n-k},$$

$$\mathscr{B}_n^{(\alpha)}(0,y;\lambda:p,q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{\binom{n-k}{2}}\mathscr{B}_k^{(\alpha)}(\lambda:p,q)y^{n-k},$$

$$\mathscr{E}_n^{(\alpha)}(x,0;\lambda:p,q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}}\mathscr{E}_k^{(\alpha)}(\lambda:p,q)x^{n-k},$$

$$\mathscr{E}_n^{(\alpha)}(0,y;\lambda:p,q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{\binom{n-k}{2}}\mathscr{E}_k^{(\alpha)}(\lambda:p,q)y^{n-k},$$

$$\mathscr{G}_n^{(\alpha)}(x,0;\lambda:p,q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}}\mathscr{G}_k^{(\alpha)}(\lambda:p,q)x^{n-k},$$

$$\mathscr{G}_n^{(\alpha)}(0,y;\lambda:p,q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{\binom{n-k}{2}}\mathscr{G}_k^{\alpha}(\lambda:p,q)y^{n-k}.$$

**Corollary 4.** *Substituting $x=1$ (or $y=1$) in Proposition 2 yields to the following formulas:*

$$\mathscr{B}_n^{(\alpha)}(x,1;\lambda:p,q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{\binom{n-k}{2}}\mathscr{B}_k^{(\alpha)}(x,0;\lambda:p,q), \tag{2.1}$$

$$\mathscr{B}_n^{(\alpha)}(1,y;\lambda:p,q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}}\mathscr{B}_k^{(\alpha)}(0,y;\lambda:p,q), \tag{2.2}$$

$$\mathscr{E}_n^{(\alpha)}(x,1;\lambda:p,q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{\binom{n-k}{2}}\mathscr{E}_k^{(\alpha)}(x,0;\lambda:p,q), \tag{2.3}$$

$$\mathscr{E}_n^{(\alpha)}(1,y;\lambda:p,q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}}\mathscr{E}_k^{(\alpha)}(0,y;\lambda:p,q), \tag{2.4}$$

$$\mathscr{G}_n^{(\alpha)}(x,1;\lambda:p,q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{\binom{n-k}{2}}\mathscr{G}_k^{(\alpha)}(x,0;\lambda:p,q), \tag{2.5}$$

$$\mathscr{G}_n^{(\alpha)}(1,y;\lambda:p,q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}}\mathscr{G}_k^{(\alpha)}(0,y;\lambda:p,q). \tag{2.6}$$

We remark that Eqs. (2.1)-(2.6) are $(p,q)$-generalizations of the following familiar formulas:

$$B_n^{(\alpha)}(x+1;\lambda)=\sum_{k=0}^{n}\binom{n}{k}B_k^{(\alpha)}(x;\lambda),$$

$$E_n^{(\alpha)}(x+1;\lambda)=\sum_{k=0}^{n}\binom{n}{k}E_k^{(\alpha)}(x;\lambda),$$

$$G_n^{(\alpha)}(x+1;\lambda)=\sum_{k=0}^{n}\binom{n}{k}G_k^{(\alpha)}(x;\lambda).$$

Now we present the addition properties as follows.

**Proposition 5** (Addition properties)**.** *Let $n \in \mathbb{N}$ and $\alpha, \beta$ be real or complex numbers. Then we have*

$$\mathscr{B}_n^{(\alpha+\beta)}(x,y;\lambda:p,q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathscr{B}_{n-k}^{(\alpha)}(x,0;\lambda:p,q)\mathscr{B}_k^{(\beta)}(0,y;\lambda:p,q),$$

$$\mathscr{E}_n^{(\alpha+\beta)}(x,y;\lambda:p,q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathscr{E}_{n-k}^{(\alpha)}(x,0;\lambda:p,q)\mathscr{E}_k^{(\beta)}(0,y;\lambda:p,q),$$

$$\mathscr{G}_n^{(\alpha+\beta)}(x,y;\lambda:p,q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathscr{G}_{n-k}^{(\alpha)}(x,0;\lambda:p,q)\mathscr{G}_k^{(\beta)}(0,y;\lambda:p,q).$$

Here are $(p,q)$-derivatives of the polynomials $\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q)$, $\mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q)$ and $\mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q)$, with respect to $x$ and $y$, as follows.

**Proposition 6** (Derivative property)**.** *We have*

$$D_{p,q;x}\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q) = [n]_{p,q}\mathscr{B}_{n-1}^{(\alpha)}(px,y;\lambda:p,q),$$

$$D_{p,q;y}\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q) = [n]_{p,q}\mathscr{B}_{n-1}^{(\alpha)}(x,qy;\lambda:p,q),$$

$$D_{p,q;x}\mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q) = [n]_{p,q}\mathscr{E}_{n-1}^{(\alpha)}(px,y;\lambda:p,q),$$

$$D_{p,q;y}\mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q) = [n]_{p,q}\mathscr{E}_{n-1}^{(\alpha)}(x,qy;\lambda:p,q),$$

$$D_{p,q;x}\mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q) = [n]_{p,q}\mathscr{G}_{n-1}^{(\alpha)}(px,y;\lambda:p,q),$$

$$D_{p,q;y}\mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q) = [n]_{p,q}\mathscr{G}_{n-1}^{(\alpha)}(x,qy;\lambda:p,q).$$

**Proposition 7.** *For $n \in \mathbb{N}_0$ and $x, y \in \mathbb{C}$, the following relationships hold true:*

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathscr{B}_k^{(\alpha)}(x,0;\lambda:p,q)\mathscr{B}_{n-k}^{(-\alpha)}(p,q) = p^{\binom{n}{2}}x^n,$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathscr{B}_k^{(\alpha)}(0,y;\lambda:p,q)\mathscr{B}_{n-k}^{(-\alpha)}(p,q) = q^{\binom{n}{2}}y^n,$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathscr{E}_k^{(\alpha)}(x,0;\lambda:p,q)\mathscr{E}_{n-k}^{(-\alpha)}(p,q) = p^{\binom{n}{2}}x^n,$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathscr{E}_k^{(\alpha)}(0,y;\lambda:p,q)\mathscr{E}_{n-k}^{(-\alpha)}(p,q) = q^{\binom{n}{2}}y^n,$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathscr{G}_k^{(\alpha)}(x,0;\lambda:p,q)\mathscr{G}_{n-k}^{(-\alpha)}(p,q) = p^{\binom{n}{2}}x^n,$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathscr{G}_k^{(\alpha)}(0,y;\lambda:p,q)\mathscr{G}_{n-k}^{(-\alpha)}(p,q) = q^{\binom{n}{2}}y^n.$$

Note that

$$\begin{aligned}
&\mathscr{B}_n^{(0)}(x,y;\lambda:p,q) = \mathscr{E}_n^{(0)}(x,y;\lambda:p,q) = \mathscr{G}_n^{(0)}(x,y;\lambda:p,q) = (x+y)_{p,q}^n\,, \\
&\mathscr{B}_n^{(0)}(0,y;\lambda:p,q) = \mathscr{E}_n^{(0)}(0,y;\lambda:p,q) = \mathscr{G}_n^{(0)}(0,y;\lambda:p,q) = q^{n(n-1)/2}y^n\,, \\
&\mathscr{B}_n^{(0)}(x,0;\lambda:p,q) = \mathscr{E}_n^{(0)}(x,0;\lambda:p,q) = \mathscr{G}_n^{(0)}(x,0;\lambda:p,q) = p^{n(n-1)/2}x^n\,.
\end{aligned} \tag{2.7}$$

The following results are arised directly from the Definition 1.

**Proposition 8.** *For $n \in \mathbb{N}_0$ and $x,y,a \in \mathbb{C}$, the following relationships hold true:*

$$\mathscr{B}_n^{(\alpha)}(x+a,y;\lambda:p,q) = \sum_{u=0}^{n} \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \mathscr{B}_{n-u}^{(\alpha)}(0,y;\lambda:p,q) p^{\binom{u}{2}} \sum_{s=0}^{u} \binom{u}{s} x^s a^{u-s},$$

$$\mathscr{B}_n^{(\alpha)}(x,y+a;\lambda:p,q) = \sum_{u=0}^{n} \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \mathscr{B}_{n-u}^{(\alpha)}(x,0;\lambda:p,q) q^{\binom{u}{2}} \sum_{s=0}^{u} \binom{u}{s} y^s a^{u-s},$$

$$\mathscr{E}_n^{(\alpha)}(x+a,y;\lambda:p,q) = \sum_{u=0}^{n} \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \mathscr{E}_{n-u}^{(\alpha)}(0,y;\lambda:p,q) p^{\binom{u}{2}} \sum_{s=0}^{u} \binom{u}{s} x^s a^{u-s},$$

$$\mathscr{E}_n^{(\alpha)}(x,y+a;\lambda:p,q) = \sum_{u=0}^{n} \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \mathscr{E}_{n-u}^{(\alpha)}(x,0;\lambda:p,q) q^{\binom{u}{2}} \sum_{s=0}^{u} \binom{u}{s} y^s a^{u-s},$$

$$\mathscr{G}_n^{(\alpha)}(x+a,y;\lambda:p,q) = \sum_{u=0}^{n} \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \mathscr{G}_{n-u}^{(\alpha)}(0,y;\lambda:p,q) p^{\binom{u}{2}} \sum_{s=0}^{u} \binom{u}{s} x^s a^{u-s},$$

$$\mathscr{G}_n^{(\alpha)}(x,y+a;\lambda:p,q) = \sum_{u=0}^{n} \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \mathscr{G}_{n-u}^{(\alpha)}(x,0;\lambda:p,q) q^{\binom{u}{2}} \sum_{s=0}^{u} \binom{u}{s} y^s a^{u-s}.$$

Some new connections among the polynomials $\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q)$, $\mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q)$ and $\mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q)$ are given as follows.

**Proposition 9.** *For $n \in \mathbb{N}_0$ and $x,y \in \mathbb{C}$, the following relationships hold true:*

$$\begin{aligned}
\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q) &= \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} \mathscr{B}_{n-l}^{(\alpha)}(0,y;\lambda:p,q) \sum_{k=0}^{l} \begin{bmatrix} l \\ k \end{bmatrix}_{p,q} \mathscr{E}_k^{(\alpha)}(x,0;\lambda:p,q) \mathscr{E}_{l-k}^{(-\alpha)}(\lambda:p,q) \\
&= \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} \mathscr{B}_{n-l}^{(\alpha)}(0,y;\lambda:p,q) \sum_{k=0}^{l} \begin{bmatrix} l \\ k \end{bmatrix}_{p,q} \mathscr{G}_k^{(\alpha)}(x,0;\lambda:p,q) \mathscr{G}_{l-k}^{(-\alpha)}(\lambda:p,q), \\
\mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q) &= \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} \mathscr{E}_{n-l}^{(\alpha)}(0,y;\lambda:p,q) \sum_{k=0}^{l} \begin{bmatrix} l \\ k \end{bmatrix}_{p,q} \mathscr{B}_k^{(\alpha)}(x,0;\lambda:p,q) \mathscr{B}_{l-k}^{(-\alpha)}(\lambda:p,q) \\
&= \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} \mathscr{E}_{n-l}^{(\alpha)}(0,y;\lambda:p,q) \sum_{k=0}^{l} \begin{bmatrix} l \\ k \end{bmatrix}_{p,q} \mathscr{G}_k^{(\alpha)}(x,0;\lambda:p,q) \mathscr{G}_{l-k}^{(-\alpha)}(\lambda:p,q), \\
\mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q) &= \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} \mathscr{G}_{n-l}^{(\alpha)}(0,y;\lambda:p,q) \sum_{k=0}^{l} \begin{bmatrix} l \\ k \end{bmatrix}_{p,q} \mathscr{B}_k^{(\alpha)}(x,0;\lambda:p,q) \mathscr{B}_{l-k}^{(-\alpha)}(\lambda:p,q) \\
&= \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} \mathscr{G}_{n-l}^{(\alpha)}(0,y;\lambda:p,q) \sum_{k=0}^{l} \begin{bmatrix} l \\ k \end{bmatrix}_{p,q} \mathscr{E}_k^{(\alpha)}(x,0;\lambda:p,q) \mathscr{E}_{l-k}^{(-\alpha)}(\lambda:p,q).
\end{aligned}$$

## 3. Main Results

In this section, we state some explicit formulas and relations among $\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q)$, $\mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q)$ and $\mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q)$. Also, we present some $(p,q)$-extensions of known results. Moreover, we give new theorems and some of their special cases, which include $(p,q)$-generalizations of the formulas in Kurt [11], Kurt [12], Srivatava and Pintér [26], Cheon [3], Mahmudov ( [19] and [21]) and others. We only prove the facts for one of them. Clearly, others can be proved by applying the similar proof technique.

The $(p,q)$-integral representations of $\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q)$, $\mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q)$ and $\mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q)$ are given by the following theorem.

**Theorem 10.** *We have*

$$\int_a^b \mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q)d_{p,q}x = p\frac{\mathscr{B}_{n+1}^{(\alpha)}\left(\frac{b}{p},y;\lambda:p,q\right)-\mathscr{B}_{n+1}^{(\alpha)}\left(\frac{a}{p},y;\lambda:p,q\right)}{[n+1]_{p,q}},$$

$$\int_a^b \mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q)d_{p,q}y = p\frac{\mathscr{B}_{n+1}^{(\alpha)}\left(x,\frac{b}{q};\lambda:p,q\right)-\mathscr{B}_{n+1}^{(\alpha)}\left(x,\frac{a}{q};\lambda:p,q\right)}{[n+1]_{p,q}},$$

$$\int_a^b \mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q)d_{p,q}x = p\frac{\mathscr{E}_{n+1}^{(\alpha)}\left(\frac{b}{p},y;\lambda:p,q\right)-\mathscr{E}_{n+1}^{(\alpha)}\left(\frac{a}{p},y;\lambda:p,q\right)}{[n+1]_{p,q}},$$

$$\int_a^b \mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q)d_{p,q}y = p\frac{\mathscr{E}_{n+1}^{(\alpha)}\left(x,\frac{b}{q};\lambda:p,q\right)-\mathscr{E}_{n+1}^{(\alpha)}\left(x,\frac{a}{q};\lambda:p,q\right)}{[n+1]_{p,q}},$$

$$\int_a^b \mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q)d_{p,q}x = p\frac{\mathscr{G}_{n+1}^{(\alpha)}\left(\frac{b}{p},y;\lambda:p,q\right)-\mathscr{G}_{n+1}^{(\alpha)}\left(\frac{a}{p},y;\lambda:p,q\right)}{[n+1]_{p,q}},$$

$$\int_a^b \mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q)d_{p,q}y = p\frac{\mathscr{G}_{n+1}^{(\alpha)}\left(x,\frac{b}{q};\lambda:p,q\right)-\mathscr{G}_{n+1}^{(\alpha)}\left(x,\frac{a}{q};\lambda:p,q\right)}{[n+1]_{p,q}}.$$

*Proof.* Since

$$\int_a^b D_{p,q}f(x)d_{p,q}x = pf(b)-f(a) \qquad \text{(see [24])} \tag{3.1}$$

in view of Proposition 6 and Eqs. (1.7) and (1.8), we obtain

$$\begin{aligned}\int_a^b \mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q)d_{p,q}x &= p\frac{1}{[n+1]_{p,q}}\int_a^b D_{p,q}\mathscr{B}_{n+1}^{(\alpha)}\left(\frac{x}{p},y;\lambda:p,q\right)d_{p,q}x\\ &= p\frac{\mathscr{B}_{n+1}^{(\alpha)}\left(\frac{b}{p},y;\lambda:p,q\right)-\mathscr{B}_{n+1}^{(\alpha)}\left(\frac{a}{p},y;\lambda:p,q\right)}{[n+1]_{p,q}}.\end{aligned}$$

Also, the $(p,q)$-integral representation of $\mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q)$ and $\mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q)$ can be shown in a like manner. Therefore, we complete the proof of this theorem. □

The equations in Theorem 10 are $(p,q)$-generalizations of the formulas (see [27])

$$\int_a^b \mathscr{B}_n^{(\alpha)}(x,\lambda)dx = p\frac{\mathscr{B}_{n+1}^{(\alpha)}(b;\lambda)-\mathscr{B}_{n+1}^{(\alpha)}(a;\lambda)}{n+1},$$

$$\int_a^b \mathscr{E}_n^{(\alpha)}(x,\lambda)dx = p\frac{\mathscr{E}_{n+1}^{(\alpha)}(b;\lambda)-\mathscr{E}_{n+1}^{(\alpha)}(a;\lambda)}{n+1},$$

$$\int_a^b \mathscr{G}_n^{(\alpha)}(x,\lambda)dx = p\frac{\mathscr{G}_{n+1}^{(\alpha)}(b;\lambda)-\mathscr{G}_{n+1}^{(\alpha)}(a;\lambda)}{n+1}.$$

We give the following theorem involving in difference equations for $\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q)$, $\mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q)$ and $\mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q)$.

**Theorem 11** (Difference equations)**.** *We have*

$$\lambda\mathscr{B}_n^{(\alpha)}(1,y;\lambda:p,q)-\mathscr{B}_n^{(\alpha)}(0,y;\lambda:p,q)=[n]_{p,q}\mathscr{B}_{n-1}^{(\alpha-1)}(0,y;\lambda:p,q), \tag{3.2}$$

$$\lambda\mathscr{B}_n^{(\alpha)}(x,0;\lambda:p,q)-\mathscr{B}_n^{(\alpha)}(x,-1;\lambda:p,q)=[n]_{p,q}\mathscr{B}_{n-1}^{(\alpha-1)}(x,-1;\lambda:p,q),$$

$$\lambda\mathscr{E}_n^{(\alpha)}(1,y;\lambda:p,q)+\mathscr{E}_n^{(\alpha)}(0,y;\lambda:p,q)=2\mathscr{E}_n^{(\alpha-1)}(0,y;\lambda:p,q), \tag{3.3}$$

$$\lambda\mathscr{E}_n^{(\alpha)}(x,0;\lambda:p,q)+\mathscr{E}_n^{(\alpha)}(x,-1;\lambda:p,q)=2\mathscr{E}_n^{(\alpha-1)}(x,-1;\lambda:p,q),$$

$$\lambda\mathscr{G}_n^{(\alpha)}(1,y;\lambda:p,q)+\mathscr{G}_n^{(\alpha)}(0,y;\lambda:p,q)=2[n]_{p,q}\mathscr{G}_{n-1}^{(\alpha-1)}(0,y;\lambda:p,q), \tag{3.4}$$

$$\lambda\mathscr{G}_n^{(\alpha)}(x,0;\lambda:p,q)+\mathscr{G}_n^{(\alpha)}(x,-1;\lambda:p,q)=2[n]_{p,q}\mathscr{G}_{n-1}^{(\alpha-1)}(x,-1;\lambda:p,q).$$

*Proof.* From Definition 1 and by using Cauchy product, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathscr{B}_n^{(\alpha-1)}(0,y;\lambda:p,q)\frac{z^n}{[n]_{p,q}!}&=\left(\frac{z}{\lambda e_{p,q}(z)-1}\right)^{\alpha-1}E_{p,q}(yz)\\&=\frac{1}{z}\left(\frac{z}{\lambda e_{p,q}(z)-1}\right)^{\alpha}\big(\lambda e_{p,q}(z)-1\big)E_{p,q}(yz)\\&=\sum_{n=0}^{\infty}\left[\lambda\mathscr{B}_n^{(\alpha)}(1,y;\lambda:p,q)-\mathscr{B}_n^{(\alpha)}(0,y;\lambda:p,q)\right]\frac{z^{n-1}}{[n]_{p,q}!}.\end{aligned}$$

Checking against the coefficients of $\frac{z^n}{[n]_{p,q}}$, then we have the first formula of this theorem. The others in this theorem can be proved similarly. □

By the Eqs. (2.2) and (3.2), Eqs. (2.4) and (3.3), Eqs. (2.6) and (3.4), we acquire the following formulas in Corollary 12.

**Corollary 12.** *We have*

$$\mathscr{B}_n^{(\alpha-1)}(0,y;\lambda:p,q)=\frac{1}{[n+1]_{p,q}}\left[\lambda\sum_{k=0}^{n+1}\begin{bmatrix}n+1\\k\end{bmatrix}_{p,q}p^{\binom{n+1-k}{2}}\mathscr{B}_k^{(\alpha)}(0,y;\lambda:p,q)-\mathscr{B}_{n+1}^{(\alpha)}(0,y;\lambda:p,q)\right], \tag{3.5}$$

$$\mathscr{E}_n^{(\alpha-1)}(0,y;\lambda:p,q)=\frac{1}{2}\left[\lambda\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{n-k}{2}}\mathscr{E}_k^{(\alpha)}(0,y;\lambda:p,q)+\mathscr{E}_n^{(\alpha)}(0,y;\lambda:p,q)\right], \tag{3.6}$$

$$\mathscr{G}_n^{(\alpha-1)}(0,y;\lambda:p,q)=\frac{1}{2[n+1]_{p,q}}\left[\lambda\sum_{k=0}^{n+1}\begin{bmatrix}n+1\\k\end{bmatrix}_{p,q}p^{\binom{n+1-k}{2}}\mathscr{G}_k^{(\alpha)}(0,y;\lambda:p,q)+\mathscr{G}_{n+1}^{(\alpha)}(0,y;\lambda:p,q)\right]. \tag{3.7}$$

By using Proposition 7 and Eq. (2.7), if we choose $\alpha=1$ in Corollary 12, then we get the following expressions:

$$y^n=\frac{q^{-\binom{n}{2}}}{[n+1]_{p,q}}\left[\lambda\sum_{k=0}^{n+1}\begin{bmatrix}n+1\\k\end{bmatrix}_{p,q}p^{\binom{n+1-k}{2}}\mathscr{B}_k(0,y;\lambda:p,q)-\mathscr{B}_{n+1}(0,y;\lambda:p,q)\right],$$

$$= q^{-\binom{n}{2}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathscr{B}_k^{(\alpha)}(0,y;\lambda:p,q)\mathscr{B}_{n-k}^{(-\alpha)}(\lambda:p,q),$$

$$y^n = \frac{q^{-\binom{n}{2}}}{2}\left[\lambda \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} \mathscr{E}_k(0,y;\lambda:p,q) + \mathscr{E}_n(0,y;\lambda:p,q)\right],$$

$$= q^{-\binom{n}{2}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathscr{E}_k^{(\alpha)}(0,y;\lambda:p,q)\mathscr{E}_{n-k}^{(-\alpha)}(\lambda:p,q),$$

$$y^n = \frac{q^{-\binom{n}{2}}}{2[n+1]_{p,q}}\left[\lambda \sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} p^{\binom{n+1-k}{2}} \mathscr{G}_k(0,y;\lambda:p,q) + \mathscr{G}_{n+1}(0,y;\lambda:p,q)\right],$$

$$= q^{-\binom{n}{2}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathscr{G}_k^{(\alpha)}(0,y;\lambda:p,q)\mathscr{G}_{n-k}^{(-\alpha)}(\lambda:p,q).$$

The first formulas of the last binary expressions are $(p,q)$-extensions of the formulas

$$y^n = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k(y),$$

$$y^n = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}E_k(y) + E_n(y),$$

$$y^n = \frac{1}{2(n+1)}\sum_{k=0}^{n+1}\binom{n+1}{k}G_k(y) + G_{n+1}(y),$$

respectively.

We give the recurrence relations for $\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q)$, $\mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q)$ and $\mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q)$ as follows.

**Theorem 13** (Recurrence relationships). *$\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q)$, $\mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q)$ and $\mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q)$ fulfill the following difference relations:*

$$\lambda \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} m^k \mathscr{B}_k^{(\alpha)}(x,0;\lambda:p,q) - \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} m^k \mathscr{B}_k^{(\alpha)}(x,-1;\lambda:p,q)$$
$$= [n]_{p,q} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\binom{n-1-k}{2}} m^{k+1} \mathscr{B}_k^{(\alpha-1)}(x,-1;\lambda:p,q), \tag{3.8}$$

$$\lambda \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} m^k \mathscr{E}_k^{(\alpha)}(x,0;\lambda:p,q) + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} m^k \mathscr{E}_k^{(\alpha)}(x,-1;\lambda:p,q)$$
$$= 2 \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} m^{k} \mathscr{E}_k^{(\alpha-1)}(x,-1;\lambda:p,q), \tag{3.9}$$

$$\lambda \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} m^k \mathscr{G}_k^{(\alpha)}(x,0;\lambda:p,q) + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} m^k \mathscr{G}_k^{(\alpha)}(x,-1;\lambda:p,q)$$
$$= 2[n]_{p,q} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\binom{n-1-k}{2}} m^{k+1} \mathscr{G}_k^{(\alpha-1)}(x,-1;\lambda:p,q). \tag{3.10}$$

*Proof.* From Definition 1 and by using Cauchy product, then we acquire

$$\lambda\left(\sum_{n=0}^{\infty}\mathscr{B}_n^{(\alpha)}(x,0;\lambda:p,q)\frac{(mz)^n}{[n]_{p,q}!}\right)\left(\sum_{n=0}^{\infty}p^{\binom{n}{2}}\frac{z^n}{[n]_{p,q}!}\right)$$
$$-\left(\sum_{n=0}^{\infty}\mathscr{B}_n^{(\alpha)}(x,-1;\lambda:p,q)\frac{(mz)^n}{[n]_{p,q}!}\right)\left(\sum_{n=0}^{\infty}p^{\binom{n}{2}}\frac{z^n}{[n]_{p,q}!}\right)$$
$$=\lambda\left(\frac{mz}{\lambda e_{p,q}(mz)-1}\right)^{\alpha}e_{p,q}(xmz)e_{p,q}(z)-\left(\frac{mz}{\lambda e_{p,q}(mz)-1}\right)^{\alpha}E_{p,q}(-mz)e_{p,q}(xmz)e_{p,q}(z)$$
$$=mz\left(\frac{mz}{\lambda e_{p,q}(mz)-1}\right)^{\alpha-1}e_{p,q}(xmz)E_{p,q}(-mz)e_{p,q}(z)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\mathscr{B}_k^{(\alpha-1)}(x,-1;\lambda:p,q)m^{k+1}p^{\binom{n-k}{2}}\right)\frac{z^{n+1}}{[n]_{p,q}!}.$$

Checking against the coefficients of $\frac{z^n}{[n]_{p,q}}$, then we have desired result (3.8). The others in this theorem can be proved in a like manner. □

Now we are in a position to state some recurrence relationships for the Apostol type $(p,q)$-Bernoulli polynomials, the Apostol type $(p,q)$-Euler polynomials and the Apostol type $(p,q)$-Genocchi polynomials.

**Theorem 14.** *For $n\in\mathbb{N}_0$ and $x,y\in\mathbb{C}$, $\mathscr{B}_n(x,y;\lambda:p,q)$, $\mathscr{E}_n(x,y;\lambda:p,q)$ and $\mathscr{G}_n(x,y;\lambda:p,q)$ satisfy the following recurrence relations:*

$$\begin{aligned}\mathscr{B}_n(x,y;\lambda:p,q)&=xq^{n-1}\mathscr{B}_{n-1}\left(\frac{p}{q}x,\frac{p}{q}y;\lambda:p,q\right)+yq^{n-1}\mathscr{B}_{n-1}(x,y;\lambda:p,q)\\&\quad+\frac{q^{n-1}}{[n]_{p,q}}\mathscr{B}_n\left(\frac{p}{q}x,\frac{p}{q}y;\lambda:p,q\right)\\&\quad-\frac{\lambda}{[n]_{p,q}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\mathscr{B}_{n-k}\left(\frac{p}{q},0;\lambda:p,q\right)q^{n-1-k}p^k\mathscr{B}_k(x,y;\lambda:p,q),\\
\mathscr{E}_n(x,y;\lambda:p,q)&=xq^{n-1}\mathscr{E}_{n-1}\left(\frac{p}{q}x,\frac{p}{q}y;\lambda:p,q\right)+yq^{n-1}\mathscr{E}_{n-1}(x,y;\lambda:p,q)\\&\quad-\frac{\lambda}{2}\sum_{k=0}^{n-1}\begin{bmatrix}n-1\\k\end{bmatrix}_{p,q}\mathscr{E}_{n-1-k}\left(\frac{p}{q},0;\lambda:p,q\right)q^{n-1-k}p^k\mathscr{E}_k(x,y;\lambda:p,q),\\
\mathscr{G}_n(x,y;\lambda:p,q)&=xq^{n-1}\mathscr{G}_{n-1}\left(\frac{p}{q}x,\frac{p}{q}y;\lambda:p,q\right)+yq^{n-1}\mathscr{G}_{n-1}(x,y;\lambda:p,q)\\&\quad+\frac{q^{n-1}}{[n]_{p,q}}\mathscr{G}_n\left(\frac{p}{q}x,\frac{p}{q}y;\lambda:p,q\right)\\&\quad-\frac{\lambda}{2[n]_{p,q}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\mathscr{G}_{n-k}\left(\frac{p}{q},0;\lambda:p,q\right)q^{n-1-k}q^k\mathscr{G}_k(x,y;\lambda:p,q).\end{aligned}$$

*Proof.* For $\alpha=1$ in Definition 1, by applying $(p,q)$-derivative to the Apostol type $(p,q)$-Euler polynomials $\mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q)$ with respect to $z$, we readily derive

$$\begin{aligned}
D_{p,q;z}\left\{\sum_{n=0}^{\infty}\mathscr{E}_n(x,y;\lambda:p,q)\frac{z^n}{[n]_{p,q}!}\right\} &= D_{p,q;z}\left\{\frac{2}{\lambda e_{p,q}(z)+1}e_{p,q}(xz)E_{p,q}(yz)\right\}\\
&= 2x\frac{e_{p,q}(pxz)E_{p,q}(pyz)}{\lambda e_{p,q}(qz)+1}+2y\frac{e_{p,q}(qxz)E_{p,q}(qyz)}{\lambda e_{p,q}(qz)+1}\\
&\quad -\frac{\lambda}{2}\frac{2e_{p,q}(pxz)E_{p,q}(pyz)}{\lambda e_{p,q}(qz)+1}\frac{2e_{p,q}(pz)}{\lambda e_{p,q}(pz)+1}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathscr{E}_{n+1}(x,y;\lambda:p,q)\frac{z^n}{[n]_{p,q}!}\\
&\quad = x\sum_{n=0}^{\infty}\mathscr{E}_n\left(\frac{p}{q}x,\frac{p}{q}y;\lambda:p,q\right)q^n\frac{z^n}{[n]_{p,q}!}+y\sum_{n=0}^{\infty}\mathscr{E}_n(x,y;\lambda:p,q)q^n\frac{z^n}{[n]_{p,q}!}\\
&\quad -\frac{\lambda}{2}\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\mathscr{E}_{n-k}\left(\frac{p}{q},0;\lambda:p,q\right)q^{n-k}p^k\mathscr{E}_k(x,y;\lambda:p,q)\right\}\frac{z^n}{[n]_{p,q}!}.
\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{[n]_{p,q}!}$ gives the desired result for Apostol type $(p,q)$-Euler polynomials. Similarly, the others in this theorem can be shown. □

We now state the following recurrence relations.

**Theorem 15.** *For $n\in\mathbb{N}_0$ and $x,y\in\mathbb{C}$, the following formulas are valid:*

$$\begin{aligned}
\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q) &= \frac{1}{[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix}n+1\\u\end{bmatrix}_{p,q}\mathscr{B}_{n-u+1}(0,my;\lambda:p,q)m^{u-n}\\
&\quad\cdot\left(\lambda\sum_{s=0}^{u}\begin{bmatrix}u\\s\end{bmatrix}_{p,q}\mathscr{B}_s^{(\alpha)}(x,0;\lambda:p,q)m^{s-u}-\mathscr{B}_u^{(\alpha)}(x,0;\lambda:p,q)\right),\\
\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q) &= \frac{1}{[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix}n+1\\u\end{bmatrix}_{p,q}\mathscr{B}_{n-u+1}(mx,0;\lambda:p,q)m^{u-n}\\
&\quad\cdot\left(\lambda\sum_{s=0}^{u}\begin{bmatrix}u\\s\end{bmatrix}_{p,q}\mathscr{B}_s^{(\alpha)}(0,y;\lambda:p,q)m^{s-u}p^{\binom{u-s}{2}}-\mathscr{B}_u^{(\alpha)}(0,y;\lambda:p,q)\right),\\
\mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q) &= \frac{1}{2}\sum_{u=0}^{n}\begin{bmatrix}n\\u\end{bmatrix}_{p,q}\mathscr{E}_{n-u}(0,my;\lambda:p,q)m^{u-n}\\
&\quad\cdot\left(\lambda\sum_{s=0}^{u}\begin{bmatrix}u\\s\end{bmatrix}_{p,q}\mathscr{E}_s^{(\alpha)}(x,0;\lambda:p,q)m^{s-u}p^{\binom{u-s}{2}}+\mathscr{E}_u^{(\alpha)}(x,0;\lambda:p,q)\right),\\
\mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q) &= \frac{1}{2}\sum_{u=0}^{n}\begin{bmatrix}n\\u\end{bmatrix}_{p,q}\mathscr{E}_{n-u}(mx,0;\lambda:p,q)m^{u-n}\\
&\quad\cdot\left(\lambda\sum_{s=0}^{u}\begin{bmatrix}u\\s\end{bmatrix}_{p,q}\mathscr{E}_s^{(\alpha)}(0,y;\lambda:p,q)m^{s-u}p^{\binom{u-s}{2}}+\mathscr{E}_u^{(\alpha)}(0,y;\lambda:p,q)\right),\\
\mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q) &= \frac{1}{2[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix}n+1\\u\end{bmatrix}_{p,q}\mathscr{G}_{n-u+1}(0,my;\lambda:p,q)m^{u-n}
\end{aligned}$$

$$\cdot\left(\lambda\sum_{s=0}^{u}\begin{bmatrix}u\\s\end{bmatrix}_{p,q}\mathscr{G}_s^{(\alpha)}(x,0;\lambda:p,q)m^{s-u}p^{\binom{u-s}{2}}+\mathscr{G}_u^{(\alpha)}(x,0;\lambda:p,q)\right),$$

$$\mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q)=\frac{1}{2[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix}n+1\\u\end{bmatrix}_{p,q}\mathscr{G}_{n-u+1}(mx,0;\lambda:p,q)m^{u-n}$$

$$\cdot\left(\lambda\sum_{s=0}^{u}\begin{bmatrix}u\\s\end{bmatrix}_{p,q}\mathscr{G}_s^{(\alpha)}(0,y;\lambda:p,q)m^{s-u}p^{\binom{u-s}{2}}+\mathscr{G}_u^{(\alpha)}(0,y;\lambda:p,q)\right).$$

*Proof.* Indeed

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q)\frac{z^n}{[n]_{p,q}!}\\
&=\left(\frac{2z}{\lambda e_{p,q}(z)+1}\right)^{\alpha}e_{p,q}(xz)\frac{\lambda e_{p,q}\left(\frac{z}{m}\right)+1}{2\frac{z}{m}}\frac{\frac{z}{m}2}{\lambda e_{p,q}\left(\frac{z}{m}\right)+1}E_{p,q}\left(my\frac{z}{m}\right)\\
&=\frac{m}{2}\sum_{n=0}^{\infty}\sum_{u=0}^{n}\begin{bmatrix}n\\u\end{bmatrix}_{p,q}\left\{\lambda\sum_{s=0}^{u}\begin{bmatrix}u\\s\end{bmatrix}_{p,q}\mathscr{G}_s^{(\alpha)}(x,0;\lambda:p,q)m^{s-u}p^{\binom{u-s}{2}}+\mathscr{G}_u^{(\alpha)}(x,0;\lambda:p,q)\right\}\\
&\quad\cdot\mathscr{G}_{n-u}(0,my;\lambda:p,q)m^{u-n}\frac{z^{n-1}}{[n]_{p,q}!}.
\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{[n]_{p,q}!}$, we get desired result for fifth equation. The others in this theorem can be proved in a like manner. □

Here are new correlations for the polynomials $\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q)$, $\mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q)$ and $\mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q)$ as follows.

**Theorem 16.** *For $n\in\mathbb{N}_0$ and $x,y\in\mathbb{C}$, the following correlations hold true:*

$$\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q)=\frac{1}{2}\sum_{u=0}^{n}\begin{bmatrix}n\\u\end{bmatrix}_{p,q}\mathscr{E}_{n-u}(0,my;\lambda:p,q)m^{u-n}$$

$$\cdot\left(\lambda\sum_{s=0}^{u}\begin{bmatrix}u\\s\end{bmatrix}_{p,q}\mathscr{B}_s^{(\alpha)}(x,0;\lambda:p,q)m^{s-u}p^{\binom{u-s}{2}}+\mathscr{B}_u^{(\alpha)}(x,0;\lambda:p,q)\right),$$

$$\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q)=\frac{1}{2[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix}n+1\\u\end{bmatrix}_{p,q}\mathscr{G}_{n-u+1}(0,my;\lambda:p,q)m^{u-n}$$

$$\cdot\left(\lambda\sum_{s=0}^{u}\begin{bmatrix}u\\s\end{bmatrix}_{p,q}\mathscr{B}_s^{(\alpha)}(x,0;\lambda:p,q)m^{s-u}p^{\binom{u-s}{2}}+\mathscr{B}_u^{(\alpha)}(x,0;\lambda:p,q)\right),$$

$$\mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q)=\frac{1}{[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix}n+1\\u\end{bmatrix}_{p,q}\mathscr{B}_{n-u+1}(0,my;\lambda:p,q)m^{u-n}$$

$$\cdot\left(\lambda\sum_{s=0}^{u}\begin{bmatrix}u\\s\end{bmatrix}_{p,q}\mathscr{E}_s^{(\alpha)}(x,0;\lambda:p,q)m^{s-u}p^{\binom{u-s}{2}}-\mathscr{E}_u^{(\alpha)}(x,0;\lambda:p,q)\right),$$

$$\mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q)=\frac{1}{2[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix}n+1\\u\end{bmatrix}_{p,q}\mathscr{G}_{n-u+1}(0,my;\lambda:p,q)m^{u-n}$$

$$\cdot\left(\lambda\sum_{s=0}^{u}\begin{bmatrix}u\\s\end{bmatrix}_{p,q}\mathscr{E}_s^{(\alpha)}(x,0;\lambda:p,q)m^{s-u}p^{\binom{u-s}{2}}+\mathscr{E}_u^{(\alpha)}(x,0;\lambda:p,q)\right),$$

$$\mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q)=\frac{1}{[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix}n+1\\u\end{bmatrix}_{p,q}\mathscr{B}_{n-u+1}(0,my;\lambda:p,q)m^{u-n}$$

$$\cdot\left(\lambda\sum_{s=0}^{u}\begin{bmatrix}u\\s\end{bmatrix}_{p,q}\mathscr{G}_s^{(\alpha)}(x,0;\lambda:p,q)m^{s-u}p^{\binom{u-s}{2}}-\mathscr{G}_u^{(\alpha)}(x,0;\lambda:p,q)\right),$$

$$\mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q)=\frac{1}{2}\sum_{u=0}^{n}\begin{bmatrix}n\\u\end{bmatrix}_{p,q}\mathscr{E}_{n-u}(0,my;\lambda:p,q)m^{u-n}$$

$$\cdot\left(\lambda\sum_{s=0}^{u}\begin{bmatrix}u\\s\end{bmatrix}_{p,q}\mathscr{G}_s^{(\alpha)}(x,0;\lambda:p,q)m^{s-u}p^{\binom{u-s}{2}}+\mathscr{G}_u^{(\alpha)}(x,0;\lambda:p,q)\right).$$

*Proof.* By using Definition 1, in fact

$$\sum_{n=0}^{\infty}\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q)\frac{z^n}{[n]_{p,q}!}$$

$$=\left(\frac{z}{\lambda e_{p,q}(z)-1}\right)^{\alpha}e_{p,q}(xz)\frac{\lambda e_{p,q}\left(\frac{z}{m}\right)+1}{2\frac{z}{m}}\frac{\frac{z}{m}2}{\lambda e_{p,q}\left(\frac{z}{m}\right)+1}E_{p,q}\left(my\frac{z}{m}\right)$$

$$=\frac{m}{2z}\left[\lambda\sum_{n=0}^{\infty}\mathscr{B}_n^{(\alpha)}(x,0;\lambda:p,q)\frac{z^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}p^{\binom{n}{2}}m^{-n}\frac{z^n}{[n]_{p,q}!}\right.$$

$$\left.+\sum_{n=0}^{\infty}\mathscr{B}_n^{(\alpha)}(x,0:p,q)\frac{z^n}{[n]_{p,q}!}\right]\sum_{n=0}^{\infty}\mathscr{G}_n(0,my;\lambda:p,q)\frac{1}{m^n}\frac{z^n}{[n]_{p,q}!}$$

$$=\frac{m}{2}\sum_{n=0}^{\infty}\sum_{u=0}^{n}\begin{bmatrix}n\\u\end{bmatrix}_{p,q}\left\{\sum_{s=0}^{u}\binom{u}{s}\mathscr{B}_s^{(\alpha)}(x,0;\lambda:p,q)m^{s-u}p^{\binom{u-s}{2}}+\mathscr{B}_u^{(\alpha)}(x,0;\lambda:p,q)\right\}$$

$$\cdot\mathscr{G}_{n-u}(0,my;\lambda:p,q)m^{u-n}\frac{z^{n-1}}{[n]_{p,q}!}.$$

Equating the coefficients of $\frac{z^n}{[n]_{p,q}}$, we get desired result for second equation. The others can be proved by utilizing tha same proof technique. □

We have the following theorem.

**Theorem 17.** *For $n\in\mathbb{N}_0$, the following relationships holds true:*

$$\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q)=\frac{1}{2m^n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\left[[k]_{p,q}\sum_{s=0}^{k-1}\begin{bmatrix}k-1\\s\end{bmatrix}_{p,q}p^{\binom{k-1-s}{2}}m^{s+1}\mathscr{B}_s^{(\alpha-1)}(x,-1;\lambda:p,q)\right.$$

$$\left.+m^k\mathscr{B}_k^{(\alpha)}(x,0;\lambda:p,q)+\sum_{s=0}^{k}\begin{bmatrix}k\\s\end{bmatrix}_{p,q}p^{\binom{k-s}{2}}m^s\mathscr{B}_s^{(\alpha)}(x,-1;\lambda:p,q)\right]$$

$$\cdot\mathscr{E}_{n-k}(0,my;\lambda:p,q),$$

$$\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q)=\frac{1}{2}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\frac{1}{m^n[k+1]_{p,q}}$$

$$\cdot\left[m^{k+1}\mathscr{B}_{k+1}^{(\alpha)}(x,0;\lambda:p,q)+\sum_{s=0}^{k+1}\begin{bmatrix}k+1\\s\end{bmatrix}_{p,q}p^{\binom{k+1-s}{2}}m^s\mathscr{B}_s^{(\alpha)}(x,-1;\lambda:p,q)\right.$$
$$\left.+[k+1]_{p,q}\sum_{s=0}^{k}\begin{bmatrix}k\\s\end{bmatrix}_{p,q}p^{\binom{k-s}{2}}m^{s+1}\mathscr{B}_s^{(\alpha-1)}(x,-1;\lambda:p,q)\right]\mathscr{G}_{n-k}(0,my;\lambda:p,q),$$

$$\mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\frac{1}{m^n[k+1]_{p,q}}\left[2\sum_{s=0}^{k+1}\begin{bmatrix}k+1\\s\end{bmatrix}_{p,q}p^{\binom{k+1-s}{2}}m^s\mathscr{E}_s^{(\alpha-1)}(x,-1;\lambda:p,q)\right.$$
$$\left.-\sum_{s=0}^{k+1}\begin{bmatrix}k+1\\s\end{bmatrix}_{p,q}p^{\binom{k+1-s}{2}}m^s\mathscr{E}_s^{(\alpha)}(x,-1;\lambda:p,q)-m^{k+1}\mathscr{E}_{k+1}^{(\alpha)}(x,0;\lambda:p,q)\right]$$
$$\cdot\mathscr{B}_{n-k}(0,my:p,q),$$

$$\mathscr{E}_n^{(\alpha)}(x,y;\lambda:p,q)=\frac{1}{2}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\frac{1}{m^n[k+1]_{p,q}}\left[2\sum_{s=0}^{k+1}\begin{bmatrix}k+1\\s\end{bmatrix}_{p,q}p^{\binom{k+1-s}{2}}m^s\mathscr{E}_s^{(\alpha-1)}(x,-1;\lambda:p,q)\right.$$
$$\left.-\sum_{s=0}^{k+1}\begin{bmatrix}k+1\\s\end{bmatrix}_{p,q}p^{\binom{k+1-s}{2}}m^s\mathscr{E}_s^{(\alpha)}(x,-1;\lambda:p,q)+m^{k+1}\mathscr{E}_{k+1}^{(\alpha)}(x,0;\lambda:p,q)\right]$$
$$\cdot\mathscr{G}_{n-k}(0,my;\lambda:p,q),$$

$$\mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\frac{1}{m^n[k+1]_{p,q}}\left[2[k+1]_{p,q}\sum_{s=0}^{k}\begin{bmatrix}k\\s\end{bmatrix}_{p,q}p^{\binom{k-s}{2}}m^{s+1}\mathscr{G}_s^{(\alpha-1)}(x,-1;\lambda:p,q)\right.$$
$$\left.-\sum_{s=0}^{k+1}\begin{bmatrix}k+1\\s\end{bmatrix}_{p,q}p^{\binom{k+1-s}{2}}m^s\mathscr{G}_s^{(\alpha)}(x,-1;\lambda:p,q)-m^{k+1}\mathscr{G}_{k+1}^{(\alpha)}(x,0;\lambda:p,q)\right]$$
$$\cdot\mathscr{B}_{n-k}(0,my;\lambda:p,q),$$

$$\mathscr{G}_n^{(\alpha)}(x,y;\lambda:p,q)=\frac{1}{2m^n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\left[2[k]_{p,q}\sum_{s=0}^{k-1}\begin{bmatrix}k-1\\s\end{bmatrix}_{p,q}p^{\binom{k-1-s}{2}}m^{s+1}\mathscr{G}_s^{(\alpha-1)}(x,-1;\lambda:p,q)\right.$$
$$\left.+m^k\mathscr{G}_k^{(\alpha)}(x,0;\lambda:p,q)-\sum_{s=0}^{k}\begin{bmatrix}k\\s\end{bmatrix}_{p,q}p^{\binom{k-s}{2}}m^s\mathscr{G}_s^{(\alpha)}(x,-1;\lambda:p,q)\right]$$
$$\cdot\mathscr{E}_{n-k}(0,my;\lambda:p,q).$$

*Proof.* Since

$$\left(\frac{z}{\lambda e_{p,q}(z)-1}\right)^{\alpha}e_{p,q}(xz)E_{p,q}(yz)$$
$$=\frac{2}{\lambda e_{p,q}\left(\frac{z}{m}\right)+1}E_{p,q}\left(\frac{z}{m}my\right)\frac{\lambda e_{p,q}\left(\frac{z}{m}\right)+1}{2}\left(\frac{z}{\lambda e_{p,q}(z)-1}\right)^{\alpha}e_{p,q}(zx),$$

we have

$$\sum_{n=0}^{\infty}\mathscr{B}_n^{(\alpha)}(x,y;\lambda:p,q)\frac{z^n}{[n]_{p,q}!}$$
$$=\frac{1}{2}\left[\lambda\sum_{n=0}^{\infty}p^{\binom{n}{2}}\frac{z^n}{m^n[n]_{p,q}!}\sum_{n=0}^{\infty}\mathscr{B}_n^{(\alpha)}(x,0;\lambda:p,q)\frac{z^n}{[n]_{p,q}!}+\sum_{n=0}^{\infty}\mathscr{B}_n^{(\alpha)}(x,0;\lambda:p,q)\frac{z^n}{[n]_{p,q}!}\right]$$

$$
\begin{aligned}
&\cdot \sum_{n=0}^{\infty} \mathscr{E}_n(0,my;\lambda:p,q)\frac{z^n}{m^n[n]_{p,q}!}\\
&= \frac{1}{2}\sum_{n=0}^{\infty}\left(\lambda\sum_{s=0}^{n}\begin{bmatrix} n\\ s\end{bmatrix}_{p,q}\mathscr{B}_s^{(\alpha)}(x,0;\lambda:p,q)p^{\binom{n-s}{2}}m^{s-n}+\mathscr{B}_n^{(\alpha)}(x,0;\lambda:p,q)\right)\frac{z^n}{[n]_{p,q}!}\\
&\cdot \sum_{n=0}^{\infty} \mathscr{E}_n(0,my;\lambda:p,q)\frac{z^n}{m^n[n]_{p,q}!}\\
&= \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}\left[\lambda\sum_{s=0}^{k}\begin{bmatrix} k\\ s\end{bmatrix}_{p,q} m^s p^{\binom{k-s}{2}}\mathscr{B}_s^{(\alpha)}(x,0;\lambda:p,q)+m^k\mathscr{B}_k^{(\alpha)}(x,0;\lambda:p,q)\right]\right.\\
&\left.\cdot \mathscr{E}_{n-k}(0,my;\lambda:p,q)m^{-n}\right)\frac{z^n}{[n]_{p,q}!}.
\end{aligned}
$$

Using Theorem 13, we get asserted result for first equation. The others in this theorem can be proved in a like manner. □

Some special cases of Theorem 17 are examined in the subsequent Corollaries.

**Corollary 18.** *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, the relationships*

$$
\begin{aligned}
\mathscr{B}_{n,q}^{(\alpha)}(x,y;\lambda) &= \frac{1}{2m^n}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\left[\sum_{s=0}^{k}\begin{bmatrix} k\\ s\end{bmatrix}_q m^s\mathscr{B}_{s,q}^{(\alpha)}(x,-1;\lambda)\right.\\
&\left.+[k]_q\sum_{s=0}^{k-1}\begin{bmatrix} k-1\\ s\end{bmatrix}_q m^{s+1}\mathscr{B}_{s,q}^{(\alpha-1)}(x,-1;\lambda)+m^k\mathscr{B}_{k,q}^{(\alpha)}(x,0;\lambda)\right]\mathscr{E}_{n-k,q}(0,my;\lambda),\\
\mathscr{B}_{n,q}^{(\alpha)}(x,y;\lambda) &= \frac{1}{2}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\frac{1}{m^n[k+1]_q}\left[\sum_{s=0}^{k+1}\begin{bmatrix} k+1\\ s\end{bmatrix}_q m^s\mathscr{B}_{s,q}^{(\alpha)}(x,-1;\lambda)\right.\\
&\left.+[k+1]_q\sum_{s=0}^{k}\begin{bmatrix} k\\ s\end{bmatrix}_q m_s^{s+1}\mathscr{B}_{s,q}^{(\alpha-1)}(x,-1;\lambda)+m^{k+1}\mathscr{B}_{k+1,q}^{(\alpha)}(x,0;\lambda)\right]\mathscr{G}_{n-k,q}(0,my;\lambda),\\
\mathscr{E}_{n,q}^{(\alpha)}(x,y;\lambda) &= \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\frac{1}{m^n[k+1]_q}\left[2\sum_{s=0}^{k+1}\begin{bmatrix} k+1\\ s\end{bmatrix}_q m^s\mathscr{E}_{s,q}^{(\alpha-1)}(x,-1;\lambda)\right.\\
&\left.-\sum_{s=0}^{k+1}\begin{bmatrix} k+1\\ s\end{bmatrix}_q m^s\mathscr{E}_{s,q}^{(\alpha)}(x,-1;\lambda)-m^{k+1}\mathscr{E}_{k+1,q}^{(\alpha)}(x,0;\lambda)\right]\mathscr{B}_{n-k,q}(0,my;\lambda),\\
\mathscr{E}_{n,q}^{(\alpha)}(x,y;\lambda) &= \frac{1}{2}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\frac{1}{m^n[k+1]_q}\left[2\sum_{s=0}^{k+1}\begin{bmatrix} k+1\\ s\end{bmatrix}_q m^s\mathscr{E}_{s,q}^{(\alpha-1)}(x,-1;\lambda)\right.\\
&\left.-\sum_{s=0}^{k+1}\begin{bmatrix} k+1\\ s\end{bmatrix}_q m^s\mathscr{E}_{s,q}^{(\alpha)}(x,-1;\lambda)+m^{k+1}\mathscr{E}_{k+1,q}^{(\alpha)}(x,0;\lambda)\right]\mathscr{G}_{n-k,q}(0,my;\lambda),\\
\mathscr{G}_{n,q}^{(\alpha)}(x,y;\lambda) &= \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\frac{1}{m^n[k+1]_q}\left[2[k+1]_q\sum_{s=0}^{k}\begin{bmatrix} k\\ s\end{bmatrix}_q m^{s+1}\mathscr{G}_{s,q}^{(\alpha-1)}(x,-1;\lambda)\right.\\
&\left.-\sum_{s=0}^{k+1}\begin{bmatrix} k+1\\ s\end{bmatrix}_q m^s\mathscr{G}_{s,q}^{(\alpha)}(x,-1;\lambda)-m^{k+1}\mathscr{G}_{k+1,q}^{(\alpha)}(x,0;\lambda)\right]\mathscr{B}_{n-k,q}(0,my;\lambda),
\end{aligned}
$$

$$\mathscr{G}_{n,q}^{(\alpha)}(x,y;\lambda)=\frac{1}{2m^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\left[2[k]_{p,q}\sum_{s=0}^{k-1}\begin{bmatrix} k-1 \\ s \end{bmatrix}_{p,q}p^{\binom{k-1-s}{2}}m^{s+1}\mathscr{G}_{s,q}^{(\alpha-1)}(x,-1;\lambda)\right.$$
$$\left.+m^k\mathscr{G}_{k,q}^{(\alpha)}(x,0;\lambda)-\sum_{s=0}^{k}\begin{bmatrix} k \\ s \end{bmatrix}_{p,q}p^{\binom{k-s}{2}}m^s\mathscr{G}_{s,q}^{(\alpha)}(x,-1;\lambda)\right]\mathscr{E}_{n-k,q}(0,my;\lambda).$$

*hold true among the Apostol type $q$-Bernoulli polynomials, Apostol type $q$-Euler polynomials and Apostol type $q$-Genocchi polynomials of order $\alpha$, see* [12].

**Corollary 19.** *For $n\in\mathbb{N}_0$ and $m\in\mathbb{N}$, the following relationships are valid:*

$$\mathscr{B}_n(x,y;\lambda:p,q)=\frac{1}{2m^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\left[\sum_{s=0}^{k}\begin{bmatrix} k \\ s \end{bmatrix}_{p,q}p^{\binom{k-s}{2}}m^s\mathscr{B}_s(x,-1;\lambda:p,q)\right.$$
$$\left.+[k]_{p,q}\sum_{s=0}^{k-1}\begin{bmatrix} k-1 \\ s \end{bmatrix}_{p,q}p^{\binom{k-1-s}{2}}m^{s+1}(x-1)_{p,q}^s+m^k\mathscr{B}_k(x,0;\lambda:p,q)\right]$$
$$\cdot\mathscr{E}_{n-k}(0,my;\lambda:p,q),$$

$$\mathscr{B}_n(x,y;\lambda:p,q)=\frac{1}{2}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\frac{1}{m^n[k+1]_{p,q}}\left[+\sum_{s=0}^{k+1}\begin{bmatrix} k+1 \\ s \end{bmatrix}_{p,q}p^{\binom{k+1-s}{2}}m^s\mathscr{B}_s(x,-1;\lambda:p,q)\right.$$
$$\left.+m^{k+1}\mathscr{B}_{k+1}(x,0;\lambda:p,q)+[k+1]_{p,q}\sum_{s=0}^{k}\begin{bmatrix} k \\ s \end{bmatrix}_{p,q}p^{\binom{k-s}{2}}m^{s+1}(x-1)_{p,q}^s\right]$$
$$\cdot\mathscr{G}_{n-k}(0,my;\lambda:p,q),$$

$$\mathscr{E}_n(x,y;\lambda:p,q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\frac{1}{m^n[k+1]_{p,q}}\left[2\sum_{s=0}^{k+1}\begin{bmatrix} k+1 \\ s \end{bmatrix}_{p,q}p^{\binom{k+1-s}{2}}m^s(x-1)_{p,q}^s\right.$$
$$\left.-\sum_{s=0}^{k+1}\begin{bmatrix} k+1 \\ s \end{bmatrix}_{p,q}p^{\binom{k+1-s}{2}}m^s\mathscr{E}_s(x,-1;\lambda:p,q)-m^{k+1}\mathscr{E}_{k+1}(x,0;\lambda:p,q)\right]$$
$$\cdot\mathscr{B}_{n-k}(0,my;\lambda:p,q),$$

$$\mathscr{E}_n(x,y;\lambda:p,q)=\frac{1}{2}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\frac{1}{m^n[k+1]_{p,q}}\left[2\sum_{s=0}^{k+1}\begin{bmatrix} k+1 \\ s \end{bmatrix}_{p,q}p^{\binom{k+1-s}{2}}m^s(x-1)_{p,q}^s\right.$$
$$\left.-\sum_{s=0}^{k+1}\begin{bmatrix} k+1 \\ s \end{bmatrix}_{p,q}p^{\binom{k+1-s}{2}}m^s\mathscr{E}_s(x,-1;\lambda:p,q)+m^{k+1}\mathscr{E}_{k+1}(x,0;\lambda:p,q)\right]$$
$$\cdot\mathscr{G}_{n-k}(0,my;\lambda:p,q),$$

$$\mathscr{G}_n(x,y;\lambda:p,q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\frac{1}{m^n[k+1]_{p,q}}\left[2[k+1]_{p,q}\sum_{s=0}^{k}\begin{bmatrix} k \\ s \end{bmatrix}_{p,q}p^{\binom{k-s}{2}}m^{s+1}(x-1)_{p,q}^s\right.$$
$$\left.-\sum_{s=0}^{k+1}\begin{bmatrix} k+1 \\ s \end{bmatrix}_{p,q}p^{\binom{k+1-s}{2}}m^s\mathscr{G}_s(x,-1;\lambda:p,q)-m^{k+1}\mathscr{G}_{k+1}(x,0;\lambda:p,q)\right]$$
$$\cdot\mathscr{B}_{n-k}(0,my;\lambda:p,q),$$

$$\mathscr{G}_n(x,y;\lambda:p,q)=\frac{1}{2m^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\left[2[k]_{p,q}\sum_{s=0}^{k-1}\begin{bmatrix} k-1 \\ s \end{bmatrix}_{p,q}p^{\binom{k-1-s}{2}}m^{s+1}(x-1)_{p,q}^s\right.$$

$$-\sum_{s=0}^{k}\begin{bmatrix}k\\s\end{bmatrix}_{p,q}p^{\binom{k-s}{2}}m^{s}\mathscr{G}_{s}(x,-1;\lambda:p,q)+m^{k}\mathscr{G}_{k}(x,0;\lambda:p,q)\Bigg]$$
$$\cdot\mathscr{E}_{n-k}(0,my;\lambda:p,q).$$

**Corollary 20.** *For $n\in\mathbb{N}_0$ and $m\in\mathbb{N}$, the following relationships hold true:*

$$\begin{aligned}
B_n(x+y;\lambda)&=\sum_{k=0}^{n}\binom{n}{k}\frac{E_{n-k}(my;\lambda)}{2m^n}\\
&\quad\cdot\left[m^kB_k(x;\lambda)+m^kB_k\left(x-1+\frac{1}{m};\lambda\right)+km(1+m(x-1))^{k-1}\right]\\
&=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\frac{G_{n-k}(my;\lambda)}{m^n(k+1)}\\
&\quad\cdot\left[m^{k+1}B_{k+1}(x;\lambda)+m^{k+1}B_{k+1}\left(x-1+\frac{1}{m};\lambda\right)+(k+1)m(1+m(x-1))^k\right],\\
E_n(x+y;\lambda)&=\sum_{k=0}^{n}\binom{n}{k}\frac{B_{n-k}(my;\lambda)}{m^n(k+1)}\\
&\quad\cdot\left[2(1+m(x-1))^{k+1}-m^{k+1}E_{k+1}\left(x-1+\frac{1}{m};\lambda\right)-m^{k+1}E_{k+1}(x;\lambda)\right]\\
&=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\frac{G_{n-k}(my;\lambda)}{m^n(k+1)}\\
&\quad\cdot\left[2(1+m(x-1))^{k+1}-m^{k+1}E_{k+1}\left(x-1+\frac{1}{m};\lambda\right)-m^{k+1}E_{k+1}(x;\lambda)\right],\\
G_n(x+y;\lambda)&=\sum_{k=0}^{n}\binom{n}{k}\frac{E_{n-k}(my;\lambda)}{2m^n}\\
&\quad\cdot\left[m^kG_k(x;\lambda)+2km(1+m(x-1))^{k-1}-m^kG_k\left(x-1+\frac{1}{m};\lambda\right)\right]\\
&=\sum_{k=0}^{n}\binom{n}{k}\frac{B_{n-k}(my;\lambda)}{m^n(k+1)}\\
&\quad\cdot\left[2(k+1)m(1+m(x-1))^k-m^{k+1}G_{k+1}\left(x-1+\frac{1}{m};\lambda\right)-m^{k+1}G_{k+1}(x;\lambda)\right].
\end{aligned}$$

**Corollary 21.** *For $n\in\mathbb{N}_0$, the following relationships hold true:*

$$\mathscr{B}_n(0,y;\lambda:p,q)=\sum_{\substack{k=0\\k\neq1}}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\mathscr{B}_k(\lambda:p,q)\mathscr{E}_{n-k}(0,y;\lambda:p,q)+[n]_{p,q}\left(\mathscr{B}_1(\lambda:p,q)+\frac{1}{2}\right)\mathscr{E}_{n-1}(0,y;\lambda:p,q), \tag{3.11}$$

$$\begin{aligned}
\mathscr{B}_n(0,y;\lambda:p,q)&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\frac{1}{[k+1]_{p,q}}\mathscr{B}_{k+1}(\lambda:p,q)\mathscr{G}_{n-k}(0,y;\lambda:p,q)+\mathscr{G}_n(0,y;\lambda:p,q),\\
\mathscr{E}_n(0,y;\lambda:p,q)&=-2\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\frac{\mathscr{E}_{k+1}(\lambda:p,q)\mathscr{B}_{n-k}(0,y;\lambda:p,q)}{[k+1]_{p,q}}, \qquad (3.12)\\
\mathscr{G}_n(0,y;\lambda:p,q)&=-\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\frac{1}{[k+1]_{p,q}}\mathscr{G}_{k+1}(\lambda:p,q)\mathscr{B}_{n-k}(0,y;\lambda:p,q)+2\mathscr{B}_n(0,y;\lambda:p,q).
\end{aligned}$$

Since $\mathscr{B}_1(1:p,q) = -\frac{p}{[2]_{p,q}}$, the formula 3.11 is a $(p,q)$-extension of main relationship of Cheon's work [3]. The formula 3.12 is a $(p,q)$-generalization of the formula of Srivastava-Pinter [26].

**Corollary 22.** *For $n \in \mathbb{N}_0$, the following relationships hold true:*

$$\mathscr{B}_n(\lambda:p,q) = \sum_{\substack{k=0\\k\neq 1}}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathscr{B}_k(\lambda:p,q)\mathscr{E}_{n-k}(\lambda:p,q) + [n]_{p,q}\left(\mathscr{B}_1(\lambda:p,q)+\frac{1}{2}\right)\mathscr{E}_{n-1}(\lambda:p,q),$$

$$\mathscr{B}_n(\lambda:p,q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \frac{1}{[k+1]_{p,q}}\mathscr{B}_{k+1}(\lambda:p,q)\mathscr{G}_{n-k}(\lambda:p,q) + \mathscr{G}_n(\lambda:p,q),$$

$$\mathscr{E}_n(\lambda:p,q) = -\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \frac{2}{[k+1]_{p,q}}\mathscr{E}_{k+1}(\lambda:p,q)\mathscr{B}_{n-k}(\lambda:p,q),$$

$$\mathscr{G}_n(\lambda:p,q) = -\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \frac{1}{[k+1]_{p,q}}\mathscr{G}_{k+1}(\lambda:p,q)\mathscr{B}_{n-k}(\lambda:p,q) + 2\mathscr{B}_n(\lambda:p,q).$$

**Corollary 23.** *For $n \in \mathbb{N}_0$, the following relationships*

$$\begin{aligned} B_n(x;\lambda) &= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(\lambda)\left[2B_k(x;\lambda)+kx^{k-1}\right] \\ &= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\frac{G_{n-k}(\lambda)}{(k+1)}\left[2B_{k+1}(x;\lambda)+(k+1)x^{k}\right], \end{aligned}$$

$$E_n(x;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\frac{B_{n-k}(\lambda)}{k+1}\left[2x^{k+1}-2E_{k+1}(x;\lambda)\right],$$

$$G_n(x;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\frac{B_{n-k}(\lambda)}{k+1}\left[2(k+1)x^{k}-2G_{k+1}(x;\lambda)\right]$$

*hold true for classical Apostol-Bernoulli polynomials, classical Apostol-Euler polynomials and classical Apostol-Genocchi polynomials.*

**Corollary 24.** *For $n \in \mathbb{N}_0$, the following relationships*

$$\begin{aligned} B_n(\lambda) &= \sum_{\substack{k=0\\k\neq 1}}^{n}\binom{n}{k}E_{n-k}(\lambda)B_k(\lambda) + nE_{n-1}(\lambda)\left(B_1(\lambda)+\frac{1}{2}\right) \\ &= \sum_{k=0}^{n}\binom{n}{k}\frac{G_{n-k}(\lambda)}{(k+1)}B_{k+1}(\lambda)+\frac{1}{2}G_n(\lambda), \end{aligned}$$

$$E_n(\lambda) = -\sum_{k=0}^{n}\binom{n}{k}\frac{2}{k+1}B_{n-k}(\lambda)E_{k+1}(\lambda),$$

$$G_n(\lambda) = -\sum_{k=0}^{n}\binom{n}{k}\frac{2}{k+1}B_{n-k}(\lambda)G_{k+1}(\lambda)+2B_n(\lambda)$$

*are true for classical Apostol-Bernoulli numbers, classical Apostol-Euler numbers and classical Apostol-Genocchi numbers.*

## 4. Conclusion

In this paper, we have introduced $(p,q)$-extension of the generalized Apostol-Bernoulli, the generalized Apostol-Euler and the generalized Apostol-Genocchi polynomials and numbers. Then we have analyzed some of their behaviours including addition theorems, difference equations, derivative properties, recurrence relationships, and so on. We have also derived $(p,q)$-analogues of some familiar formulae belonging to usual Apostol-Bernoulli, Euler and Genocchi polynomials. Moreover, we have given $(p,q)$-generalizations of Cheon's main result [3] and the formula of Srivastava and Pintér [26]. We notice that the results obtained here reduce to known results of $q$-polynomials when $p = 1$. Also, when $q \to p = 1$, our results in this paper turn into the ordinary results belonging to familiar Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials and numbers.

### Competing Interests

The authors declare that they have no competing interests.

### Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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