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Research Article

On the Jones Polynomial in the Solid Torus

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Abstract. We introduce an infinite class of elementary knots in the solid torus, together with general recursive and explicit formulas of the values of these knots under the generalized invariant of Jones polynomial to the solid torus. These values can be used as an infinite set of initial data for this invariant. We also introduce a procedure of resolving certain knots called spiral knots into these elementary knots. We show that our explicit formulas involve exactly $n+1$ terms for an elementary knot with n crossings, which reduces the calculations needed to compute the invariant for spiral knots and arbitrary knots and links in the solid torus.

Keywords. Knots; Links; Jones polynomial; Solid torus

MSC. 57M27

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1. Introduction

The Jones polynomial invariant for knots and links, which was discovered by V. Jones in 1984, has been a leading invariant over the last three decades (see [6]). Many generalizations of the Jones polynomial were discovered, and much of work has been done to investigate the ramifications of this invariant. The first well-known generalization of the Jones polynomial appeared in [3]. These efforts created the theory of quantum invariants, which is one of the most active fields of research in knot theory.

One important generalization of the Jones polynomial was the polynomial invariant of Hoste and Przytycki $\widetilde{d}$ for 1-trivial dichromatic links in [5]. In [1], Bataineh and Hajij defined and studied this invariant for knots and links in the solid torus and generalized it to links in a handlebody. As Kauffman showed in [7], computing such polynomial invariants for one given knot or link projection with n crossings involves computing the invariant for 2^n states, and the result involves 2^n terms.

In [2], we noticed the importance of a sequence of elementary knots in the solid torus. In this article we introduce an infinite class of elementary knots including those in [2]. The importance of these elementary knots is that their appearance is very frequent in the process of computing $\widetilde{d}$ and their values can be used as initial data for $\widetilde{d}$. We give recursive and explicit formulas of the values of these elementary knots in the invariant $\widetilde{d}$. We also determine the number of terms of $\widetilde{d}$ at an elementary knot with n crossings. We show that our explicit formulas involve only $n+1$ terms for an elementary knot with n crossings, which reduces the calculations needed to compute the invariant for arbitrary knots. We define spiral knots in the solid torus, which generalize our sequences of elementary knots and we give examples to show the use of our explicit values of the sequences of elementary knots in the procedure of resolving spiral knots to compute $\widetilde{d}$ for these spiral knots.

In Section 2 we give basic concepts and terminology. In Section 3 we introduce the elementary knots and their values in $\widetilde{d}$. In Section 4 we show that these values of the elementary knots can be written explicitly as Laurent polynomials with number of terms of $n+1$ for an elementary knot with n crossings. In Section 5 we introduce spiral knots and give applications of our results on these knots.

2. Basic Concepts and Terminology

Hoste and Przytycki in [5] introduced the two-variable Laurent polynomial invariants $d(L)$ and $\widetilde{d}(L)$ of 1-trivial dichromatic links in $\mathbb{R}^3$. These invariants can be viewed as invariants of links in the solid torus.

Theorem 1. *Let D denote a figure of an oriented 1-trivial dichromatic link L in $\mathbb{R}^3$, and let $\langle |D| \rangle$ be determined by the following formulas:*

$$\left\langle \times \right\rangle = A \left\langle)(\right\rangle + A^{-1} \left\langle \asymp \right\rangle,$$

$$\left\langle |D| \cup \bigcirc \right\rangle = (-A^2 - A^{-2}) \langle |D| \rangle,$$

$$\left\langle |D| \cup \odot \right\rangle = (-A^2 - A^{-2}) h \, \langle |D| \rangle,$$

$$\left\langle \bigcirc \right\rangle = 1, \quad \left\langle \odot \right\rangle = h.$$

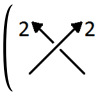
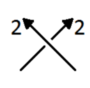
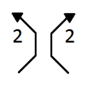
Then

(i) $d(L) = (-A^3)^{-sw(D)} \langle |D| \rangle$ *is a Laurent polynomial invariant in $\mathbb{Z}[A, A^{-1}, h]$ of unoriented links, where $sw(D)$ is the sum of the signs of those crossings between strands belonging to the same component.*

(ii) $\widetilde{d}(L) = (-A^3)^{-w(D)} \langle |D| \rangle$ *is a Laurent polynomial invariant in $\mathbb{Z}[A, A^{-1}, h]$ of oriented links, where $w(D)$ is the sum of the signs of all crossings of D. In other words $\widetilde{d}(L) = (-A^3)^{-2lk(L)} d(L)$, where $lk(L)$ denotes the sum of the linking numbers between each pair of the components.*

For the following two theorems see [4].

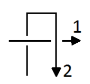
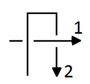
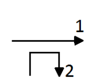
Theorem 2. *The invariant $\tilde{d}(L)$ is determined by the following skein relation and initial data*

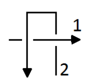
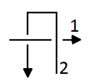
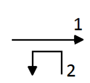
(a) $A^4\tilde{d}$ () $- A^{-4}\tilde{d}$ () $= (A^{-2} - A^2)\tilde{d}$ ().

(b) $\tilde{d}(U_1) = A^{-6}h$, $\tilde{d}(U_{-1}) = A^6 h$, $\tilde{d}(U_0) = 1$.

In the local graphs of the skein relation above, (b) is necessary if we are dealing with 1-trivial dichromatic links, but it is not necessary if we are dealing with links in the solid torus.

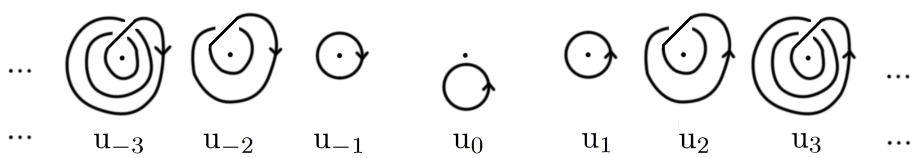
Theorem 3. *The invariant $\tilde{d}(L)$ satisfies the following clasp rules:*

(I) $A^8\tilde{d}$ () $+ A^{-8}\tilde{d}$ () $= h(A^2 + A^{-2})\tilde{d}$ ().

(II) $A^4\tilde{d}$ () $+ A^{-4}\tilde{d}$ () $= h(A^2 + A^{-2})\tilde{d}$ ().

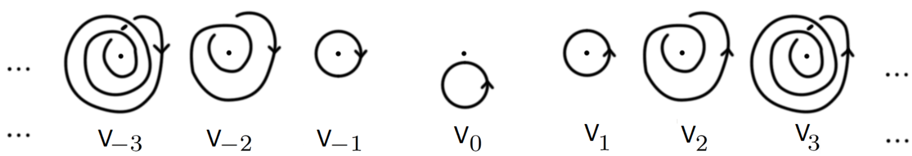
3. Elementary Sequences of Knots and their Values in $\tilde{d}$

Let $n \in \mathbb{Z}$. Let U_n be the knot with winding number equal to n given in the following figure.



And, let U_n^* be the knot with winding number equal to n given in the following figure.

Also, let V_n be the knot with winding number equal to n given in the following figure.

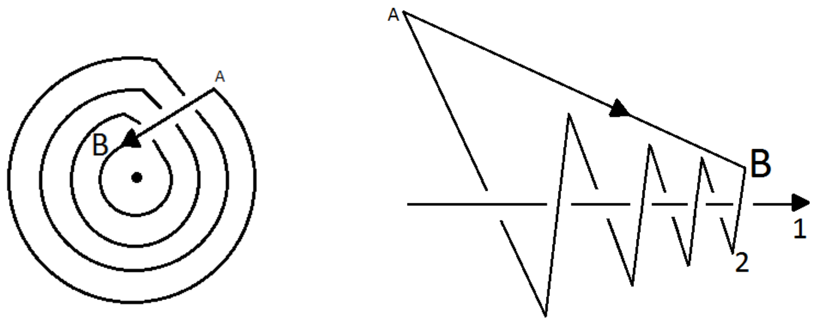


And, let V_n^* be the knot with winding number equal to n given in the following figure.

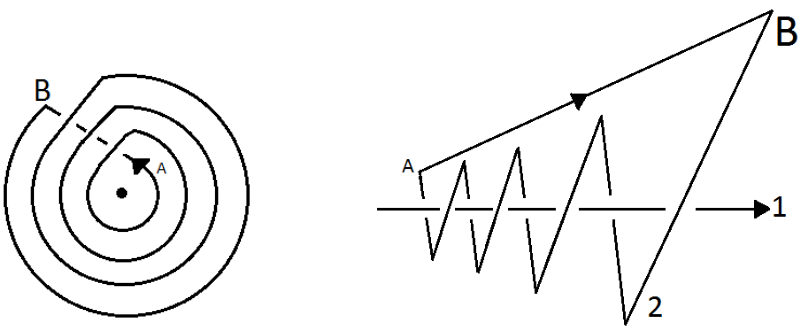
In all of the punctured diagrams above, the puncture represents an oriented trivial component, which is pointing up to the eye of the reader.

Lemma 1. *For every $n \in \mathbb{Z}$; U_n is isotopy equivalent to U_n^*, and V_n is isotopy equivalent to V_n^*.*

Proof. In the following figure, think of the knot U_n ($n = 4$ for simplicity) starting at the point B, then winding 4 times counter-clockwise around the puncture to get to the point A, then proceeding on the line segment from A to B from right to left with 3 over-crossings. To see U_4 as 1-trivial dichromatic two-component link, again start at the point B, then wind 4 times counter-clockwise around the trivial component to get to the point A, then proceed on the line segment from A to B from back to front with 3 over-crossings.

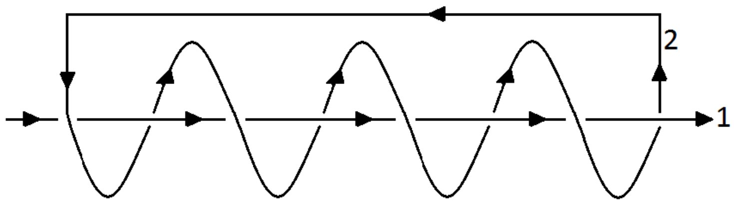


Now, it is obvious that the 1-trivial dichromatic two-component link above is isotopy equivalent to the 1-trivial dichromatic two-component link in the following figure. This can be seen by stretching the strands at the right and compressing the strands at the left. Finally, U_4^* can be obtained from the resulting 1-trivial dichromatic two-component link as in the following figure.

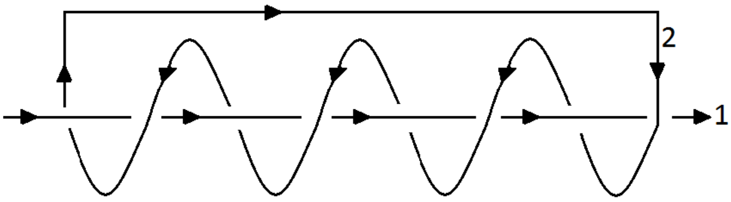


Showing that V_n is isotopy equivalent to V_n^* can be done in a similar way. □

For computing purposes, we will mostly view these knots in the solid torus as 1-trivial two-component links. See the following figure for U_4.



See also the following figure for V_4.



Next, we will accomplish our computations for $\widetilde{d}(U_n)$ and $\widetilde{d}(V_n)$ for a positive winding number n. We will see later that the values of $\widetilde{d}(U_n)$ and $\widetilde{d}(V_n)$ for a negative winding number n can be deduced easily.

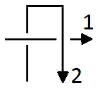
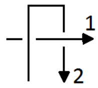
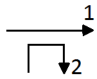
Lemma 2. *For $n \geq 0$, $\widetilde{d}(U_n)$ is given recursively by*

$$\widetilde{d}(U_n) + A^{-16}\widetilde{d}(U_{n-2}) = h\left(A^2 + A^{-2}\right)A^{-8}\widetilde{d}(U_{n-1}), \quad \textit{for } n \geq 2,$$

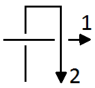
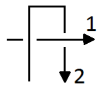
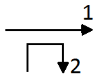
$$\widetilde{d}(U_1) = A^{-6}h,$$

$$\widetilde{d}(U_0) = 1.$$

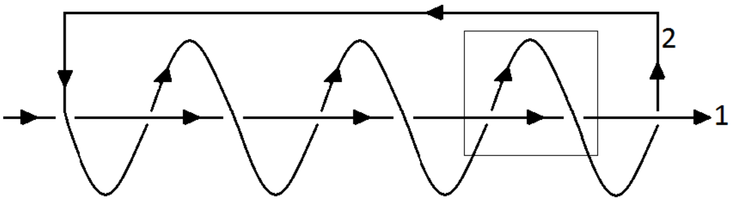
Proof. $\widetilde{d}(U_0) = 1$ and $\widetilde{d}(U_1) = A^{-6}h$ are already known as initial data.
By the clasp rule (I), we have

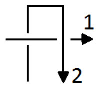
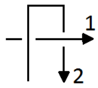
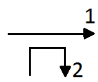
$A^8\widetilde{d}\Big($  $\Big) + A^{-8}\widetilde{d}\Big($  $\Big) = h\left(A^2 + A^{-2}\right)\widetilde{d}\Big($  $\Big).$

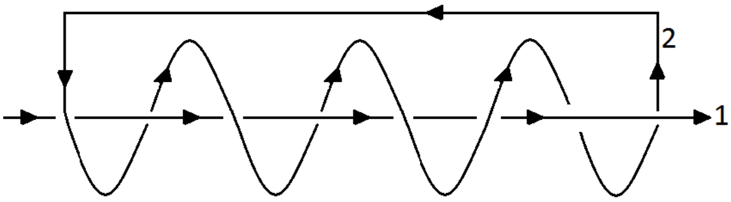
Multiplying both sides by A^{-8}, we get

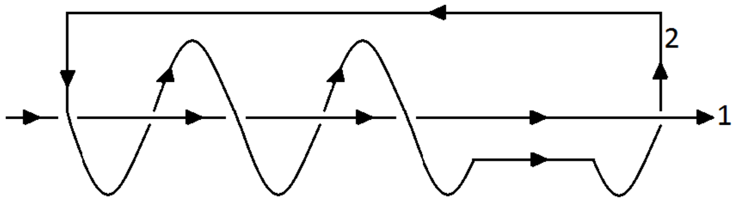
$\widetilde{d}\Big($  $\Big) + A^{-16}\widetilde{d}\Big($  $\Big) = h\left(A^2 + A^{-2}\right)A^{-8}\widetilde{d}\Big($  $\Big).$

We apply the last rule on the clasp in the rectangle of the following figure of U_n ($n = 4$ for simplification).



Note that the local graph inside this rectangle is . The graphs of  and  applied on U_n are given respectively by:





which are isotopy equivalent to U_{n-2} and U_{n-1}, respectively. By substituting in the rule above, the recursion follows. □

For V_n, the following lemma can be proved in a similar way.

Lemma 3. *For $n \geq 0$, $\widetilde{d}(V_n)$ is given recursively by*

$$\widetilde{d}(V_n) + A^{-8}\widetilde{d}(V_{n-2}) = h\left(A^2 + A^{-2}\right)A^{-4}\widetilde{d}(V_{n-1}), \quad \textit{for } n \geq 2,$$
$$\widetilde{d}(V_1) = A^{-6}h,$$
$$\widetilde{d}(V_0) = 1.$$

Theorem 4. *For $n \geq 2$, $\widetilde{d}(U_n)$ is given explicitly by*

$$\widetilde{d}(U_n) = \frac{A^{-6}h - R_2}{A^{-8}\sqrt{h^2\left(A^2+A^{-2}\right)^2 - 4}} R_1^n + \frac{R_1 - A^{-6}h}{A^{-8}\sqrt{h^2\left(A^2+A^{-2}\right)^2 - 4}} R_2^n,$$

where

$$R_1, R_2 = \frac{1}{2}h\left(A^2 + A^{-2}\right)A^{-8} \pm \frac{1}{2}A^{-8}\sqrt{h^2\left(A^2+A^{-2}\right)^2 - 4}.$$

Proof. From the previous lemma, for $n \geq 2$, we have the recursive equation

$$\widetilde{d}(U_n) + A^{-16}\widetilde{d}(U_{n-2}) = h\left(A^2 + A^{-2}\right)A^{-8}\widetilde{d}(U_{n-1}).$$

Let $\widetilde{d}(U_n) = R^n$ to get

$$R^n + A^{-16}R^{n-2} = h\left(A^2 + A^{-2}\right)A^{-8}R^{n-1}$$

or

$$R^2 - h\left(A^2 + A^{-2}\right)A^{-8}R + A^{-16} = 0$$

Solving for R, we get

$$R_1 = \frac{1}{2}h\left(A^2 + A^{-2}\right)A^{-8} + \frac{1}{2}A^{-8}\sqrt{h^2\left(A^2+A^{-2}\right)^2 - 4},$$
$$R_2 = \frac{1}{2}h\left(A^2 + A^{-2}\right)A^{-8} - \frac{1}{2}A^{-8}\sqrt{h^2\left(A^2+A^{-2}\right)^2 - 4}.$$

Therefore the solution of the recursive equation is

$$\widetilde{d}(U_n) = \alpha R_1^n + \beta R_2^n.$$

To determine the values of α and β, we use the initial data

$$\widetilde{d}(U_0) = 1, \quad \widetilde{d}(U_1) = A^{-6}h$$

which gives the following two equations:

$$\alpha + \beta = 1,$$
$$\alpha R_1 + \beta R_2 = A^{-6}h.$$

The solution of this system is

$$\alpha = \frac{A^{-6}h - R_2}{R_1 - R_2}, \quad \beta = \frac{R_1 - A^{-6}h}{R_1 - R_2}.$$

Note that $R_1 - R_2 = A^{-8}\sqrt{h^2\left(A^2 + A^{-2}\right)^2 - 4}$.
Therefore

$$\alpha = \frac{A^{-6}h - R_2}{A^{-8}\sqrt{h^2\left(A^2 + A^{-2}\right)^2 - 4}}, \quad \beta = \frac{R_1 - A^{-6}h}{A^{-8}\sqrt{h^2\left(A^2 + A^{-2}\right)^2 - 4}},$$

and hence

$$\widetilde{d}(U_n) = \frac{A^{-6}h - R_2}{A^{-8}\sqrt{h^2\left(A^2 + A^{-2}\right)^2 - 4}}R_1^n + \frac{R_1 - A^{-6}h}{A^{-8}\sqrt{h^2\left(A^2 + A^{-2}\right)^2 - 4}}R_2^n,$$

where

$$R_1 = \frac{1}{2}h\left(A^2 + A^{-2}\right)A^{-8} + \frac{1}{2}A^{-8}\sqrt{h^2\left(A^2 + A^{-2}\right)^2 - 4},$$

$$R_2 = \frac{1}{2}h\left(A^2 + A^{-2}\right)A^{-8} - \frac{1}{2}A^{-8}\sqrt{h^2\left(A^2 + A^{-2}\right)^2 - 4}.$$

□

The following theorem for V_n can be proved in a similar way.

Theorem 5. *For $n \geq 2$, $\widetilde{d}(V_n)$ is given explicitly by*

$$\widetilde{d}(V_n) = \frac{A^{-6}h - S_2}{A^{-4}\sqrt{h^2\left(A^2 + A^{-2}\right)^2 - 4}}S_1^n + \frac{S_1 - A^{-6}h}{A^{-4}\sqrt{h^2\left(A^2 + A^{-2}\right)^2 - 4}}S_2^n,$$

where

$$S_1 = \frac{1}{2}h\left(A^2 + A^{-2}\right)A^{-4} + \frac{1}{2}A^{-4}\sqrt{h^2\left(A^2 + A^{-2}\right)^2 - 4},$$

$$S_2 = \frac{1}{2}h\left(A^2 + A^{-2}\right)A^{-4} - \frac{1}{2}A^{-4}\sqrt{h^2\left(A^2 + A^{-2}\right)^2 - 4}.$$

4. Explicit Formulas as Laurent Polynomials

One disadvantage of the last two theorems is that they do not give us $\widetilde{d}(U_n)$ and $\widetilde{d}(U_n)$ as Laurent polynomials, because of the radical in the denominators in the formulas. Next, we will explore more explicit formulas for $\widetilde{d}(U_n)$ and $\widetilde{d}(V_n)$ as Laurent polynomials in A and h.

Let $\delta = \left(A^2 + A^{-2}\right)$, then $R_1, R_2 = \frac{1}{2}h\delta A^{-8} \pm \frac{1}{2}A^{-8}\sqrt{h^2\delta^2 - 4}$. In fact δ is a commonly used notation for $\left(A^2 + A^{-2}\right)$.

Theorem 6. *For $n \geq 2$, $\widetilde{d}(U_n)$ is given explicitly as a Laurent polynomial in A and h as:*

$$\widetilde{d}(U_n) = hA^{-6}\sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor}\binom{n}{2j-1}\left(\frac{1}{4}A^{-16}\left(h^2\delta^2 - 4\right)\right)^{j-1}\left(\frac{1}{2}h\delta A^{-8}\right)^{n-2j+1}$$

$$- A^{-16}\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor}\binom{n-1}{2j-1}\left(\frac{1}{4}A^{-16}\left(h^2\delta^2 - 4\right)\right)^{j-1}\left(\frac{1}{2}h\delta A^{-8}\right)^{n-2j},$$

where $\delta = A^2 + A^{-2}$.

Proof. To simplify the calculations we will let $X = \frac{1}{2}h\delta A^{-8}$, and $Z = \frac{1}{2}A^{-8}\sqrt{h^2\delta^2 - 4}$, then $R_1, R_2 = X \pm Z$. Therefore, by the last theorem, we have

$$\begin{aligned}\widetilde{d}(U_n) &= \frac{A^{-6}h - R_2}{2Z}R_1^n + \frac{R_1 - A^{-6}h}{2Z}R_2^n \\ &= \frac{A^{-6}hR_1 - R_1R_2}{2Z}R_1^{n-1} + \frac{R_1R_2 - A^{-6}hR_2}{2Z}R_2^{n-1}.\end{aligned}$$

Recall that $R^2 - h\left(A^2 + A^{-2}\right)A^{-8}R + A^{-16} = (R - R_1)(R - R_1) = 0$, hence $R_1R_2 = A^{-16}$. Therefore

$$\begin{aligned}\widetilde{d}(U_n) &= \frac{A^{-6}hR_1 - A^{-16}}{2Z}R_1^{n-1} + \frac{A^{-16} - A^{-6}hR_2}{2Z}R_2^{n-1} \\ &= \frac{1}{2Z}\left[\left(A^{-6}hR_1 - A^{-16}\right)R_1^{n-1} + \left(A^{-16} - A^{-6}hR_2\right)R_2^{n-1}\right] \\ &= \frac{1}{2Z}\left[A^{-6}hR_1^n - A^{-16}R_1^{n-1} + A^{-16}R_2^{n-1} - A^{-6}hR_2^n\right] \\ &= \frac{1}{2Z}\left[A^{-6}h\left(R_1^n - R_2^n\right) + A^{-16}\left(R_2^{n-1} - R_1^{n-1}\right)\right] \\ &= \frac{1}{2Z}\left[A^{-6}h\left(R_1^n - R_2^n\right) - A^{-16}\left(R_1^{n-1} - R_2^{n-1}\right)\right].\end{aligned}$$

Therefore

$$\widetilde{d}(U_n) = \frac{hA^{-6}}{2Z}\left(R_1^n - R_2^n\right) - \frac{A^{-16}}{2Z}\left(R_1^{n-1} - R_2^{n-1}\right).$$

Now, by Binomial Theorem, we have

$$\begin{aligned}R_1^n - R_2^n &= (X+Z)^n - (X-Z)^n \\ &= \sum_{i=0}^{n}\binom{n}{i}Z^iX^{n-i} - \sum_{i=0}^{n}\binom{n}{i}(-1)^iZ^iX^{n-i} \\ &= \sum_{i=0}^{n}\binom{n}{i}Z^iX^{n-i} - \binom{n}{i}(-1)^iZ^iX^{n-i} \\ &= \sum_{i=0}^{n}\left(\binom{n}{i} - \binom{n}{i}(-1)^i\right)Z^iX^{n-i}.\end{aligned}$$

Note that

$$\binom{n}{i} - \binom{n}{i}(-1)^i = \begin{cases} 2\binom{n}{i}, & \text{if } i \text{ is odd,} \\ 0, & \text{if } i \text{ is even.}\end{cases}$$

Therefore

$$R_1^n - R_2^n = \begin{cases} 2\binom{n}{1}Z^1X^{n-1} + 2\binom{n}{3}Z^3X^{n-3} + \cdots + 2\binom{n}{n}Z^nX^0, & \text{if } n \text{ is odd,} \\ 2\binom{n}{1}Z^1X^{n-1} + 2\binom{n}{3}Z^3X^{n-3} + \cdots + 2\binom{n}{n-1}Z^{n-1}X^1, & \text{if } n \text{ is even.}\end{cases}$$

Therefore

$$R_1^n - R_2^n = \sum_{j=1}^{\lfloor\frac{n+1}{2}\rfloor} 2\binom{n}{2j-1}Z^{2j-1}X^{n-2j+1},$$

because

$$\left\lfloor\frac{n+1}{2}\right\rfloor = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd,} \\ \frac{n}{2}, & \text{if } n \text{ is even.}\end{cases}$$

Similarly

$$R_1^{n-1} - R_2^{n-1} = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} 2\binom{n-1}{2j-1} Z^{2j-1} X^{n-2j}.$$

Therefore

$$\begin{aligned}
\widetilde{d}(U_n) &= \frac{hA^{-6}}{2Z}(R_1^n - R_2^n) - \frac{A^{-16}}{2Z}(R_1^{n-1} - R_2^{n-1}) \\
&= \frac{hA^{-6}}{2Z} \sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} 2\binom{n}{2j-1} Z^{2j-1} X^{n-2j+1} - \frac{A^{-16}}{2Z} \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} 2\binom{n-1}{2j-1} Z^{2j-1} X^{n-2j} \\
&= hA^{-6} \sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n}{2j-1} Z^{2j-2} X^{n-2j+1} - A^{-16} \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1}{2j-1} Z^{2j-2} X^{n-2j}.
\end{aligned}$$

Note that Z has an even power in both terms, which will eliminate the square root from the expressions, and gives us the Laurent polynomial explicitly.

$$\begin{aligned}
\widetilde{d}(U_n) &= hA^{-6} \sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n}{2j-1} (Z^2)^{j-1} X^{n-2j+1} - A^{-16} \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1}{2j-1} (Z^2)^{j-1} X^{n-2j} \\
&= hA^{-6} \sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n}{2j-1} \left(\frac{1}{4} A^{-16}(h^2\delta^2 - 4)\right)^{j-1} \left(\frac{1}{2} h\delta A^{-8}\right)^{n-2j+1} \\
&\quad - A^{-16} \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1}{2j-1} \left(\frac{1}{4} A^{-16}(h^2\delta^2 - 4)\right)^{j-1} \left(\frac{1}{2} h\delta A^{-8}\right)^{n-2j}.
\end{aligned}$$

This completes the proof. □

The following theorem for V_n can be proved in a similar way.

Theorem 7. *For $n \geq 2$, $\widetilde{d}(V_n)$ is given explicitly as a Laurent polynomial in A and h as:*

$$\begin{aligned}
\widetilde{d}(V_n) &= hA^{-6} \sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n}{2j-1} \left(\frac{1}{4} A^{-8}(h^2\delta^2 - 4)\right)^{j-1} \left(\frac{1}{2} h\delta A^{-4}\right)^{n-2j+1} \\
&\quad - A^{-8} \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1}{2j-1} \left(\frac{1}{4} A^{-8}(h^2\delta^2 - 4)\right)^{j-1} \left(\frac{1}{2} h\delta A^{-4}\right)^{n-2j},
\end{aligned}$$

where $\delta = (A^2 + A^{-2})$.

The number of terms in calculating $\widetilde{d}$ of U_n or V_n is 2^n, while our formulas give explicitly the values of $\widetilde{d}(U_n)$ and $\widetilde{d}(V_n)$ in only n terms as illustrated in the following corollary.

Corollary 1. *The explicit formulas of the values of $\widetilde{d}(U_n)$ and $\widetilde{d}(V_n)$ involve n terms.*

Proof. It is obvious that in the explicit formula of the value of $\widetilde{d}(U_n)$ or $\widetilde{d}(V_n)$ we have $\lfloor \frac{n+1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor = n$ terms, because

$$\left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n+1}{2} + \frac{n-1}{2}, & \text{if } n \text{ is odd}, \\ \frac{n}{2} + \frac{n}{2}, & \text{if } n \text{ is even}. \end{cases}$$

□

Note that the number of terms is more than the number of crossings by 1, because each of U_n or V_n involves $n-1$ crossings.

The following theorem gives the values of $\widetilde{d}(U_n)$ and $\widetilde{d}(V_n)$ for negative values of n.

Theorem 8. *For $n \geq 0$, we have*

$$\widetilde{d}(U_{-n}) = A^{12n}\widetilde{d}(U_n),$$
$$\widetilde{d}(V_{-n}) = A^{12n}\widetilde{d}(V_n).$$

Proof. We prove the first formula only, as the proof of the second formula is similar.
By Theorem 1, note that

$$\begin{aligned}\widetilde{d}(U_n) &= \left(-A^3\right)^{-2lk(L)} d(U_n)\\ &= \left(-A^3\right)^{-2n} d(U_n)\\ &= A^{-6n} d(U_n).\end{aligned}$$

Therefore

$$d(U_n) = A^{6n}\widetilde{d}(U_n).$$

On the other hand

$$\begin{aligned}\widetilde{d}(U_{-n}) &= \left(-A^3\right)^{-2lk(L)} d(U_{-n})\\ &= \left(-A^3\right)^{2n} d(U_{-n}).\end{aligned}$$

However, since d is an invariant of unoriented links, we have $d(U_{-n}) = d(U_n)$. Therefore

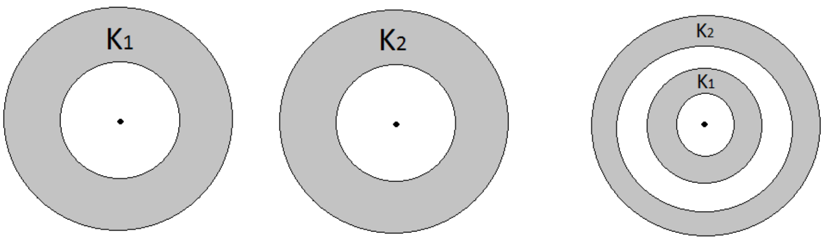
$$\begin{aligned}\widetilde{d}(U_{-n}) &= \left(-A^3\right)^{2n} d(U_n)\\ &= A^{6n} d(U_n).\end{aligned}$$

Consequently

$$\widetilde{d}(U_{-n}) = A^{12n}\widetilde{d}(U_n).$$ □

5. Disjoint Connected Sums and Spiral Knots

Definition 1. If K_1 and K_2 are two knots (or links) in the solid torus realized by their punctured diagrams, then their disjoint connected sum $K_1\#K_2$ is defined to be the link resulting from their disjoint union with a common puncture as in the following figure.



The following formula follows from Theorem 5.1 and Theorem 5.2 in [4].

Theorem 9. *If K_1 and K_2 are two knots (or links) in the solid torus, then*

$$\widetilde{d}(K_1\#K_2) = -\delta\widetilde{d}(K_1)\widetilde{d}(K_2) = -\left(A^2 + A^{-2}\right)\widetilde{d}(K_1)\widetilde{d}(K_2).$$

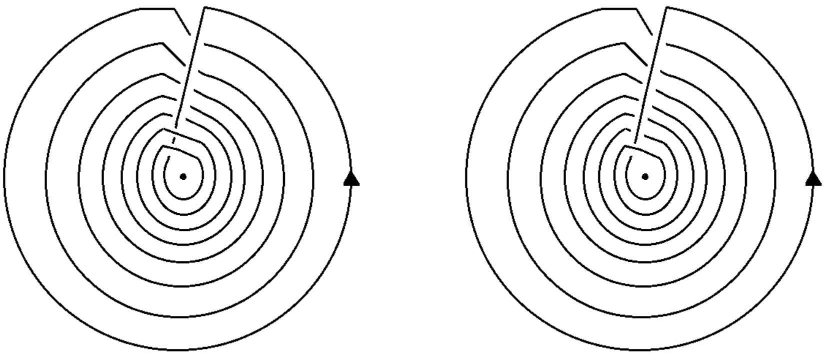
Obviously, this formula is very useful now in computing $\widetilde{d}$ for disjoint connected sums of knots from the families $\{U_n\}$ and $\{V_n\}$.

We define spiral knots in the solid torus as follows.

Definition 2. A spiral knot K in the solid torus is defined to be the result of changing some of the crossings of a knot from $\{U_n\} \cup \{V_n\}$ in their diagrams given in Section 2.

In the following example, we illustrate a procedure of computing $\widetilde{d}$ for a spiral knot getting use of the results in Section 4.

Example 1. Let the following two spiral knots be denoted by K and L, respectively.



We compute $\widetilde{d}(K)$ using the skein relation and our results as follows. At first we apply the skein relation on the second closest crossing to the puncture in K to get

$$A^4\widetilde{d}(K) - A^{-4}\widetilde{d}(L) = \left(A^{-2} - A^2\right)\widetilde{d}(U_6\#V_2).$$

Therefore

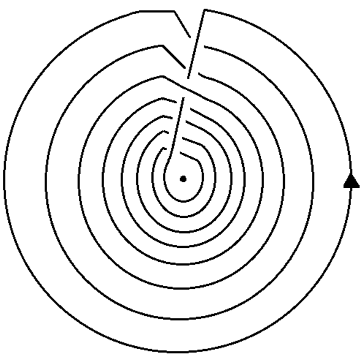
$$\widetilde{d}(K) = A^{-8}\widetilde{d}(L) + A^{-4}\left(A^{-2} - A^2\right)\widetilde{d}(U_6\#V_2).$$

Now we apply the skein relation on the closest crossing to the puncture in L to get

$$\begin{aligned}\widetilde{d}(K) &= A^{-8}\left[A^{-8}\widetilde{d}(U_8) + A^{-4}\left(A^{-2} - A^2\right)\widetilde{d}(U_7\#V_1)\right] + A^{-4}\left(A^{-2} - A^2\right)\widetilde{d}(U_6\#V_2) \\ &= A^{-16}\widetilde{d}(U_8) - A^{-12}\left(A^{-2} - A^2\right)\delta\widetilde{d}(U_7)\widetilde{d}(V_1) - A^{-4}\left(A^{-2} - A^2\right)\delta\widetilde{d}(U_6)\widetilde{d}(V_2).\end{aligned}$$

Finally, the value of $\widetilde{d}(K)$ can be obtained by substituting for $\widetilde{d}(U_8)$, $\widetilde{d}(U_7)$, $d(U_6)$, $\widetilde{d}(V_2)$ and $\widetilde{d}(V_1)$ from Theorem 6 and Theorem 7.

Example 2. Let the following spiral knot be denoted by M.



We compute $\widetilde{d}(M)$ using the skein relation and our results as follows. At first we apply the skein relation on the fifth closest crossing to the puncture in M to get

$$A^4\widetilde{d}(M)-A^{-4}\widetilde{d}(U_8)=\left(A^{-2}-A^2\right)\widetilde{d}(U_5\#U_3).$$

Therefore

$$\begin{aligned}\widetilde{d}(M)&=A^{-8}\widetilde{d}(U_8)+A^{-4}\left(A^{-2}-A^2\right)\widetilde{d}(U_5\#U_3)\\&=A^{-8}\widetilde{d}(U_8)-\delta A^{-4}\left(A^{-2}-A^2\right)\widetilde{d}(U_5)\widetilde{d}(V_3).\end{aligned}$$

Finally, the value of $\widetilde{d}(M)$ can be obtained by substituting for $\widetilde{d}(U_8)$, $\widetilde{d}(U_5)$ and $\widetilde{d}(U_3)$ from Theorem 6 and Theorem 7.

6. Conclusion

Unlike knots in the three spheres, knots in the solid torus have infinitely many homotopy classes. We characterized the simplest knots in these classes and gave their values explicitly in Jones polynomial. These values can work as an infinite class of initial data for spiral and arbitrary knots.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

References

[1] K. Bataineh and M. Hajij, Jones polynomial for links in the handlebody, *Rocky Mountain Journal of Mathematics*, **43** (3) (2013), 737–753.

[2] K. Bataineh and H. Belkhirat, The derivatives of the Hoste and Przytycki polynomial for oriented links in the solid torus, accepted for publication in *Houston Journal of Mathematics*.

[3] P. Freyd, D. Yetter, J. Hoste, W. Lickorish, K. Millett and A. Ocneanu, A new polynomial invariant of knots and links, *Bulletin of the American Mathematical Society*, **12** (2) (1985), 239–246.

[4] J. Hoste and M. Kidwell, Dichromatic link invariants, *Trans. Amer. Math. Soc.*, **321** (1) (1990), 197–229.

[5] J. Hoste and J. Przytycki, An invariant of dichromatic links, *Proc. Amer. Math. Soc.*, **105** (1989), 1003–1007.

[6] V. Jones, A polynomial invariant for knots via von Neumann algebra, *Bull. Amer. Math. Soc.*, **12** (1985), 103–111.

[7] L. Kauffman, New invariants in the theory of knots, *American Mathematical Monthly*, **95** (1988), 195–242.

[8] A. Kawauchi, *A Survey of Knot Theory*, Birkhauser-Verlag (1996).