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Research Article

A New Study on Generalized Absolute Matrix Summability

Hikmet Seyhan Özarslan

Department of Mathematics, Erciyes University, Kayseri, Turkey
seyhan@erciyes.edu.tr; hseyhan38@gmail.com

Abstract. In this paper, a general theorem on $|A,p_n;\delta|_k$ summability factors, which generalizes a theorem of Bor [4] on $|\bar{N},p_n|_k$ summability factors, has been proved by using almost increasing sequences.

Keywords. Summability factors; Absolute matrix summability; Almost increasing sequence; Infinite series; Hölder inequality; Minkowski inequality

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1. Introduction

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Obviously, every increasing sequence is almost increasing sequence but the converse need not be true as can be seen from the example $b_n = ne^{(-1)^n}$. Let $\sum a_n$ be a given infinite series with the partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0,\ i \geq 1). \tag{1.1}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.2}$$

defines the sequence (σ_n) of the Riesz mean or simply the $(\bar{N},p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [5]).

The series $\sum a_n$ is said to be summable $|\bar{N},p_n|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\Delta\sigma_{n-1}|^k < \infty, \tag{1.3}$$

where

$$\Delta\sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1. \tag{1.4}$$

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0,1,\dots\ . \tag{1.5}$$

The series $\sum a_n$ is said to be summable $|A,p_n;\delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [6])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |\bar{\Delta} A_n(s)|^k < \infty, \tag{1.6}$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

If we set $\delta = 0$, then $|A,p_n;\delta|_k$ summability reduces to $|A,p_n|_k$ summability (see [8]). If we take $a_{nv} = \frac{p_v}{P_n}$ and $\delta = 0$, then $|A,p_n;\delta|_k$ summability reduces to $|\bar{N},p_n|_k$ summability. In the special case $\delta = 0$ and $p_n = 1$ for all n, $|A,p_n;\delta|_k$ summability is the same as $|A|_k$ summability (see [9]). Also if we take $a_{nv} = \frac{p_v}{P_n}$, then $|A,p_n;\delta|_k$ summability is the same as $|\bar{N},p_n;\delta|_k$ summability (see [3]).

Before stating the main theorem we must first introduce some further notations.

Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\widehat{A} = (\widehat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n,v = 0,1,\dots \tag{1.7}$$

and

$$\widehat{a}_{00} = \bar{a}_{00} = a_{00}, \ \ \widehat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1,2,\dots\ . \tag{1.8}$$

It may be noted that $\bar{A}$ and $\widehat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{1.9}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \widehat{a}_{nv} a_v\,. \tag{1.10}$$

2. Known Result

In [4], the following theorem dealing with $|\bar{N},p_n|_k$ summability factors of infinite series has already been proved.

Theorem 2.1. *Let (X_n) be an almost increasing sequence and let there be sequences (β_n) and (λ_n) such that*

$$|\Delta\lambda_n| \leq \beta_n, \tag{2.1}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{2.2}$$

$$\sum_{n=1}^{\infty} n\left|\Delta\beta_n\right| X_n < \infty, \tag{2.3}$$

$$|\lambda_n| X_n = O(1). \tag{2.4}$$

If

$$\sum_{n=1}^{m} \frac{|\lambda_n|}{n} = O(1) \quad as \quad m \to \infty, \tag{2.5}$$

$$\sum_{n=1}^{m} \frac{1}{n}|t_n|^k = O(X_m) \quad as \quad m \to \infty \tag{2.6}$$

and (p_n) is a sequence such that

$$\sum_{n=1}^{m} \frac{p_n}{P_n}|t_n|^k = O(X_m) \quad as \quad m \to \infty, \tag{2.7}$$

where (t_n) is the nth $(C,1)$ mean of the sequence (na_n), then the series $\sum a_n\lambda_n$ is summable $|\bar{N},p_n|_k$, $k \geq 1$.

3. Main Result

The aim of this paper is to generalize Theorem 2.1 to $|A,p_n;\delta|_k$ summability. Now, we shall prove the following theorem.

Theorem 3.1. *Let $A = (a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0} = 1, \quad n = 0,1,\ldots, \tag{3.1}$$

$$a_{n-1,v} \geq a_{nv}, \quad for\ n \geq v+1, \tag{3.2}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{3.3}$$

and (X_n) be an almost increasing sequence. If the conditions (2.1)-(2.5) of Theorem 2.1 and the conditions

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k-1} |t_n|^k = O(X_m) \quad as \quad m \to \infty, \tag{3.4}$$

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{|t_n|^k}{n} = O(X_m) \quad as \quad m \to \infty, \tag{3.5}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta_v \hat{a}_{nv}| = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k-1}\right\} \quad as \ \ m \to \infty, \tag{3.6}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\hat{a}_{n,v+1}| = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k}\right\} \quad as \ \ m \to \infty \tag{3.7}$$

are satisfied, then the series $\sum a_n \lambda_n$ *is summable* $|A, p_n; \delta|_k$, $k \geq 1$ *and* $0 \leq \delta < 1/k$.

We need the following lemma for the proof of Theorem 3.1.

Lemma 3.2 ([4]). *Under the conditions on* (X_n), (β_n) *and* (λ_n) *as taken in the statement of Theorem* 3.1, *the following conditions hold:*

$$n\beta_n X_n = O(1) \quad as \ \ n \to \infty, \tag{3.8}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{3.9}$$

4. Proof of Theorem 3.1

Let (I_n) denotes A-transform of the series $\sum a_n \lambda_n$. Then, by (1.9) and (1.10), we have

$$\begin{aligned}\bar{\Delta} I_n &= \sum_{v=1}^{n} \hat{a}_{nv} a_v \lambda_v \\ &= \sum_{v=1}^{n} \frac{\hat{a}_{nv}\lambda_v}{v} v a_v .\end{aligned}$$

Using Abel's transformation, we have that

$$\begin{aligned}\bar{\Delta} I_n &= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv}\lambda_v}{v}\right) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn}\lambda_n}{n} \sum_{r=1}^{n} r a_r \\ &= \frac{n+1}{n} a_{nn} \lambda_n t_n + \sum_{v=1}^{n-1} \frac{v+1}{v} \Delta_v(\hat{a}_{nv}) \lambda_v t_v + \sum_{v=1}^{n-1} \frac{v+1}{v} \hat{a}_{n,v+1} \Delta\lambda_v t_v + \sum_{v=1}^{n-1} \frac{1}{v} \hat{a}_{n,v+1} \lambda_{v+1} t_v \\ &= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4} .\end{aligned}$$

To complete the proof of Theorem 3.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |I_{n,r}|^k < \infty, \quad \text{for } r = 1,2,3,4.$$

First, we have that

$$\begin{aligned}\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |I_{n,1}|^k &= O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |\lambda_n|^k |t_n|^k a_{nn}^k \\ &= O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} |\lambda_n| \, |\lambda_n|^{k-1} |t_n|^k \\ &= O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} |\lambda_n| \, |t_n|^k \\ &= O(1) \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{r=1}^{n} \left(\frac{P_r}{p_r}\right)^{\delta k - 1} |t_r|^k + O(1)|\lambda_m| \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} |t_n|^k\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + O(1)|\lambda_m| X_m \\
&= O(1)\sum_{n=1}^{m-1} \beta_n X_n + O(1)|\lambda_m| X_m \\
&= O(1) \quad \text{as} \;\; m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.2.

Now, when $k > 1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, as in $I_{n,1}$, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |I_{n,2}|^k &= O(1)\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\widehat{a}_{nv})|\, |\lambda_v|\, |t_v|\right)^k \\
&= O(1)\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\widehat{a}_{nv})|\, |\lambda_v|^k |t_v|^k\right) \left(\sum_{v=1}^{n-1} |\Delta_v(\widehat{a}_{nv})|\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} \left(\sum_{v=1}^{n-1} |\Delta_v(\widehat{a}_{nv})|\, |\lambda_v|^k |t_v|^k\right) \\
&= O(1)\sum_{v=1}^{m} |\lambda_v|^{k-1} |\lambda_v|\, |t_v|^k \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta_v(\widehat{a}_{nv})| \\
&= O(1)\sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k-1} |\lambda_v|\, |t_v|^k \\
&= O(1) \quad \text{as} \;\; m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.2. Again, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left|I_{n,3}\right|^k &= O(1)\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left(\sum_{v=1}^{n-1} |\widehat{a}_{n,v+1}|\, |\Delta\lambda_v|\, |t_v|\right)^k \\
&= O(1)\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left(\sum_{v=1}^{n-1} |\widehat{a}_{n,v+1}| \beta_v |t_v|^k\right) \left(\sum_{v=1}^{n-1} |\widehat{a}_{n,v+1}| \beta_v\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} \left(\sum_{v=1}^{n-1} |\widehat{a}_{n,v+1}| \beta_v |t_v|^k\right) \\
&= O(1)\sum_{v=1}^{m} \beta_v |t_v|^k \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\widehat{a}_{n,v+1}| \\
&= O(1)\sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k} v\beta_v \frac{|t_v|^k}{v} \\
&= O(1)\sum_{v=1}^{m-1} \left|\Delta\left(v\beta_v\right)\right| \sum_{r=1}^{v} \left(\frac{P_r}{p_r}\right)^{\delta k} \frac{|t_r|^k}{r} + O(1) m\beta_m \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{|t_v|^k}{v} \\
&= O(1)\sum_{v=1}^{m-1} v|\Delta\beta_v| X_v + O(1)\sum_{v=1}^{m-1} \beta_v X_v + O(1) m\beta_m X_m \\
&= O(1) \quad \text{as} \;\; m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.2.

Finally, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|I_{n,4}\right|^k &\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\,|\lambda_{v+1}|\frac{|t_v|}{v}\right)^k\\
&\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\,|\lambda_{v+1}|\frac{|t_v|^k}{v}\right)\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\frac{|\lambda_{v+1}|}{v}\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}\frac{|\lambda_{v+1}|}{v}|t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|\lambda_{v+1}|}{v}|t_v|^k\\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_{v+1}|\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\delta k}\frac{|t_r|^k}{r}+O(1)|\lambda_{m+1}|\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|t_v|^k}{v}\\
&= O(1)\sum_{v=1}^{m-1}\beta_{v+1}X_{v+1}+O(1)|\lambda_{m+1}|X_{m+1}\\
&= O(1)\quad \text{as}\quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.2.

This completes the proof of Theorem 3.1. If we take $a_{nv}=\frac{p_v}{P_n}$ and $\delta=0$ in Theorem 3.1, then we get Theorem 2.1. Also, if we take $\delta=0$ in Theorem 3.1, then we obtain a known theorem on $|A,p_n|_k$ summability method (see [7]).

5. Conclusions

In this study, we have generalized a known theorem dealing with absolute summability method to absolute matrix summability method by using almost increasing sequences. And so it has been brought a different perspective and studying field.

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Competing Interests

The author declares that she has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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