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# Some Common Fixed Point on Generalized Cyclic Contraction Mappings with Implicit Relation and Its Applications

Research Article

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**Abstract.** From the concept of cyclic relation, we introduced the generalized cyclic contraction with respect to multi-valued mappings under implicit relation and obtained some common fixed point theorem in complete metric spaces. In addition, some examples and applications are presented to demonstrate our results.

**Keywords.** Cyclic contraction; Multivalued mapping; Implicit function

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## 1. Introduction

The famous theorem and most interesting in the fixed point theory is Banach contraction principle which considered on Banach's contraction that introduced by Banach [9] in 1922 for proving the existence of the solution of various problems in nonlinear analysis. There are many works focus on the extension of the contraction as a generalized of the mappings see in [1] which shown the diversified type of mappings. In the beginning, there is an enormous number of works that generalized the contraction on single-valued mappings. Afterward, Nadler [10] extended the idea of Banach to set-valued mappings in 1969.

In 2003, Kirk [14] introduced the notion of cyclic representation which are cyclic relation and cyclic contraction in metric spaces and investigated the existence and uniqueness of fixed point for cyclical condition. Many papers considered cyclic condition for different contractions and some works introduced new class of cyclic contraction mappings and further in another spaces such as Neammanee [7] extended the concept of cyclic for single-valued to multivalued mappings, Shatanawi [15] utilized the cyclic mapping for $\Omega$-distance in $G$-metric space, Nashine et al. [3] presented the new formula of cyclic contractive condition for implicit relation and proved the existence and uniqueness of fixed point for the mappings, while in 2014, Nashine [4] got some fixed point results for cyclic contraction endowed with implicit relation. In addition, Popa [13] provided cyclic for multivalued mapping, which be more generalized than [4]. Moreover, Kumari and Panthi [8] are also consider the cyclic contraction for proving fixed point theorem in the generating spaces.

In the way of extending Banach fixed point theory, there are many numbers of work considered on the contraction which are the generalizations of the well-known contraction. Recently many years, Popa introduced and used the implicit function whose strength lies rather than the various contractions to demonstrate some fixed point theorems in metric spaces. Moreover, there are many works introduced the new condition of implicit function that we shall see in [5], [11], and [12].

In the study, we shall introduce the generalized cyclic representation with respect to multi-valued mappings and implicit relation on metric spaces. As we shall see in the main results which show existence and common fixed point for such mappings.

## 2. Preliminaries

Throughout this paper for complete metric space $(X,d)$, we denote $CB(X)$ be the family of all nonempty closed bounded subsets of $X$. We recall the Hausdorff distance for multivalued as following

$$H(A,B)=\max\left\{\sup_{a\in A}\{d(a,B)\},\sup_{b\in B}\{d(b,A)\}\right\}$$

and

$$d(x,A)=\inf_{y\in A}\{d(x,y)\}.$$

For any $A,B\in CB(X)$, $k>1$ for $a\in A$, there exist $b\in B$ such that

$$d(a,b)\le kH(A,B).$$

Many papers defined the definition of implicit relation that introduced in the literature. The following functions are the example of the implicit function in variety conditions.

**Definition 2.1** ([6])**.** Let $\tau$ be the set of all real continuous functions $T:\mathbb{R}_+^6\to\mathbb{R}$ where $\mathbb{R}$ is the set of all real numbers and $\mathbb{R}_+^6=[0,\infty)$, satisfies the following conditions:

$T_1$: $T$ is non-increasing in variables $t_2,t_3,t_4,t_5,t_6$ ;

$T_2$: there exists a right continuous function $f:\mathbb{R}_+\to\mathbb{R}$, $f(0)=0$, $f(t)<t$ for $t>0$ such that for $u,v>0$,

$$T(u,v,u,v,0,u+v)\le 0 \quad\text{or}\quad T(u,v,0,0,v,v)\le 0$$

implies $u\le f(v)$ ;

$T_3$: $T(u,0,u,0,0,u) > 0$ and $T(u,u,0,0,u,u) > 0$ for all $u > 0$.

In recently, Popa [13] introduced the definition of implicit relation for multi-valued mappings.

**Definition 2.2.** Let $\tau$ be the family of all real continuous functions $T : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

$P_1$: $T$ is non-increasing in variables $t_3, t_4, t_5$ ;

$P_2$: there exists $h \in [0,1)$ and $k > 1$ such that for any $u, v, t \geq 0$, $u \leq kt$ and $T(t,v,v,u,u+v,0) \leq 0$ implies $u \leq hv$.

In addition, Kirk [14] defined the notion of cyclic representation which are cyclic relation as follow.

**Definition 2.3.** Let $(X,d)$ be a metric space. Let $p$ be a positive integer in which, $A_1, A_2, \dots, A_p$ be nonempty closed subset of $X$, $Y = \bigcup_{i=1}^{p} A_i$ and $T : Y \to Y$. Then $Y$ is said to be a cyclical representation of $Y$ with respect to $T$ if

(1) $A_i$ are nonempty closed subsets of $X$ when $i = 1, 2, \dots, p$ ;

(2) $T(A_1) \subseteq A_2, T(A_2) \subseteq A_3, \dots, T(A_{p-1}) \subseteq A_p, T(A_p) \subseteq A_1$.

Moreover, Nashine et al. [3] defined a new notion of cyclic contractive mapping and demonstrated the new result from the following mapping.

**Definition 2.4.** Let $(X,d)$ be a metric space, $p$ a positive integer, $A_1, A_2, \dots, A_p$ nonempty closed subset of $X$ and $Y = \bigcup_{i=1}^{p} A_i$. An operator $f : Y \to Y$ is called an implicit relation type cyclic contractive mapping if

(1) $Y = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of $Y$ with respect to $f$ ;

(2) for every $x, y \in A_i \times A_{i+1}$ when $i = 1, 2, \dots, p$ and $A_{p+1} = A_1$,

$$T(d(fx,fy), d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)) \leq 0$$

for some $T \in \tau_6$.

## 3. Fixed Point Theorems for Implicit Generalized Cyclic Contractive Mappings

In this section, we shall discuss on the some fixed point theorem that concern about the implicit relation on the generalized cyclic mappings. Firstly, we will introduce some definitions that extend from the literature.

**Definition 3.1.** Let $\Psi_6$ be the family of all real continuous functions $\varphi$, where $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ which satisfies the following conditions:

$M_1$: $\varphi$ is decreasing in variable $t_3, t_4, t_5$ ;

$M_2$: there exists $h \in [0,1)$ and $k > 1$ such that

if $u, v, t \geq 0$, $u \leq kt$, and $\varphi(t,v,v,u,u+v,0) \leq 0$ then $u \leq hv$ ;

$M_3$: $\varphi(u,0,u,0,0,u) > 0$, $\varphi(u,0,0,u,u,0) > 0$ for all $u > 0$.

**Example 3.2.** $\varphi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a_1t_2 - a_2t_3 - a_3t_4 - a_4t_5 - a_5t_6$, where $a_1,a_2,a_3,a_4,a_5 \geq 0$, and $a_1+a_2+a_3+a_4+a_5 < 1$.

$M_1$: Obviously.

$M_2$: Given $u,v,t \geq 0$, $u \leq kt$, where
$$1 < k < \frac{1}{a_1+a_2+a_3+a_4+a_5},$$
and $\varphi(t,v,v,u,u+v,0) = t - a_1v - a_2v - a_3u - a_4(u+v) \leq 0$.

Then $t \leq (a_1+a_2+a_4)v + (a_3+a_4)u$, that is $kt \leq k[(a_1+a_2+a_4)v + (a_3+a_4)u]$.

Therefore, $u \leq hv$, where $h = \frac{k(a_1+a_2+a_4)}{1-(a_3+a_4)}$.

As an inspiration from [15], [2] and other, we shall define the generalization of cyclic relation on two multivalued mappings.

**Definition 3.3.** Let $(X,d)$ be a metric space, $p$ is a positive integer, and $A_1,A_2,\ldots,A_p$ be nonempty closed subset of $X$. If $Y = \bigcup_{i=1}^{p} A_i$, $F: Y \to CB(X)$, and $G: Y \to CB(X)$, then we called $Y$ a generalized cyclic representation with respect to $F$ and $G$ if it satisfies the following conditions:

$C_1$: $A_i$ are nonempty closed subsets of $X$ when $i = 1,2,\ldots,p$ ;

$C_2$: $F(A_1) \subseteq A_2, G(A_2) \subseteq A_3, F(A_3) \subseteq A_4, \ldots,$

when $p$ is even number, $G(A_p) \subseteq A_1$

when $p$ is odd number, $F(A_p) \subseteq A_1$.

**Definition 3.4.** Let $(X,d)$ be a complete metric space and $A_1,A_2,\ldots,A_p$ be nonempty closed subsets of $X$ when $p$ is a positive integer. Then $F: Y \to CB(X)$ and $G: Y \to CB(X)$ are called generalized implicit cyclic contractive mappings if it satisfies the following conditions:

$I_1$: $Y = \bigcup_{i=1}^{p} A_i$ is a generalized cyclic representation of $Y$ with respect to $F$ and $G$ ;

$I_2$: For any $(x,y) \in A_i \times A_{i+1}$ when $i = 1,2,\ldots,p$ and $A_{p+1} = A_1$,
$$\varphi(H(Fx,Gy),d(x,y),d(x,Fx),d(y,Gy),d(x,Gy),d(y,Fx)) \leq 0 \tag{3.1}$$
for some $\varphi \in \Psi_6$.

Next theorem is the main result of this work that focus on the common fixed point of mappings which satisfies the properties of generalized cyclic on implicit relation.

**Theorem 3.5.** *Let $(X,d)$ be a complete metric space and $A_1,A_2,\ldots,A_p$ be nonempty closed subsets of $X$ when $p$ be a positive integer. If $Y = \bigcup_{i=1}^{p} A_i$ such that $F: Y \to CB(X)$ and $G: Y \to CB(X)$ are generalized implicit cyclic contractive mappings for some $\varphi \in \Psi_6$, then there are common fixed point of $F$ and $G$ in $\bigcap_{i=1}^{p} A_i$.*

*Proof.* Consider $x_0 \in A_1$ and $F(A_1) \subseteq A_2$ from $C_2$, then there exists $x_1 \in Fx_0$ which implies that $x_1 \in A_2$. And since $G(A_2) \subseteq A_3$, then we can choose $x_2 \in Gx_1$, and $k > 1$ such that

$$d(x_1, x_2) \leq kH(Fx_0, Gx_1). \tag{3.2}$$

Similarly, there exists $x_3 \in Fx_2$ which implies that $x_3 \in A_4$ such that

$$d(x_2, x_3) \leq kH(Gx_1, Fx_2). \tag{3.3}$$

Following this process, then we can consider into 2 cases for the ending of the process.

*Case* 1: If $p$ is even number, then $F(A_{p-1}) \subseteq A_p$ and $G(A_p) \subseteq A_1$, then there exists $x_{p-1} \in Fx_{p-2}$ (when $x_{p-2} \in A_{p-1}$) such that

$$d(x_{p-2}, x_{p-1}) \leq kH(Gx_{p-3}, Fx_{p-2}) \tag{3.4}$$

and there exists $x_p \in Gx_{p-1} \subseteq A_{p+1} = A_1$ such that

$$d(x_{p-1}, x_p) \leq kH(Fx_{p-2}, Gx_{p-1}). \tag{3.5}$$

*Case* 2: If $p$ is odd number, then $G(A_{p-1}) \subseteq A_p$ and $F(A_p) \subseteq A_1$, then there exists $x_{p-1} \in Gx_{p-2}$ (when $x_{p-2} \in A_{p-1}$) such that

$$d(x_{p-2}, x_{p-1}) \leq kH(Fx_{p-3}, Gx_{p-2}) \tag{3.6}$$

and there exists $x_p \in Fx_{p-1} \subseteq A_{p+1} = A_1$ such that

$$d(x_{p-1}, x_p) \leq kH(Gx_{p-2}, Fx_{p-1}). \tag{3.7}$$

Since $x_0 \in A_1$ and $x_1 \in A_2$ and from $I_2$ in Definition 3.4 by using inequality (3.1) we obtain that

$$\varphi(H(Fx_0, Gx_1), d(x_0, x_1), d(x_0, Fx_0), d(x_1, Gx_1), d(x_0, Gx_1), d(x_1, Fx_0)) \leq 0 \tag{3.8}$$

for some $\varphi \in \Psi_6$ and by condition $M_1$ in Definition 3.1, we get

$$\varphi(H(Fx_0, Gx_1), d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), d(x_0, x_2), 0) \leq 0. \tag{3.9}$$

By using triangle inequality we can conclude that

$$\varphi(H(Fx_0, Gx_1), d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), d(x_0, x_1) + d(x_1, x_2), 0) \leq 0 \tag{3.10}$$

and by $M_2$ in Definition 3.1, it implies that

$$d(x_1, x_2) \leq h_1 d(x_0, x_1)$$

for some $h_1 \in [0, 1)$.
Similarly, from inequality (3.1) and $I_2$ in Definition 3.4 to obtain that

$$\varphi(H(Gx_1, Fx_2), d(x_1, x_2), d(x_1, Gx_1), d(x_2, Fx_2), d(x_1, Fx_2), d(x_2, Gx_1)) \leq 0 \tag{3.11}$$

for some $\varphi \in \Psi_6$ whenever $x_1 \in A_2$ and $x_2 \in A_3$.
And by condition $M_1$ in Definition 3.1, we get

$$\varphi(H(Gx_1, Fx_2), d(x_1, x_2), d(x_1, x_2), d(x_2, x_3), d(x_1, x_3), 0) \leq 0. \tag{3.12}$$

By using triangle inequality we can conclude that

$$\varphi(H(Gx_1, Fx_2), d(x_1, x_2), d(x_1, x_2), d(x_2, x_3), d(x_1, x_2) + d(x_2, x_3), 0) \leq 0 \tag{3.13}$$

and by using $M_2$, we get

$$d(x_2, x_3) \leq h_2 d(x_1, x_2)$$

for some $h_2 \in [0, 1)$.

Next, we proved in the same process to obtain that

$$d(x_{p-1},x_p) \le h_{p-1} d(x_{p-2},x_{p-1})$$

for some $h_{p-1} \in [0,1)$.

If we choose $h = \max\{h_1,h_2,\dots,h_{p-1}\}$, then we have

$$d(x_{n-1},x_n) \le h d(x_{n-2},x_{n-1}) \le \dots \le h^{n-1} d(x_0,x_1)$$

when $n$ be positive integer. Now we have the sequence $x_n$ in which

$$\lim_{n\to\infty} d(x_{n-1},x_n) = 0.$$

Next we will show that $x_n$ is Cauchy sequence. Without loss of generality, we assume that $n > m$

$$\begin{aligned} d(x_m,x_n) &\le d(x_m,x_{m+1}) + d(x_{m+1},x_{m+2}) + \dots + d(x_{n-2},x_{n-1}) + d(x_{n-1},x_n) \\ &\le h^m d(x_0,x_1) + h^{m+1} d(x_0,x_1) + \dots + h^{n-1} d(x_0,x_1) \\ &\le d(x_0,x_1)\left[h^m + h^{m+1} + \dots + h^{n-1}\right] \\ &\le d(x_0,x_1)\left[h^m\left(\frac{h^{n-m}-1}{h-1}\right)\right]. \end{aligned}$$

By taking limit as $m \to \infty$ for some $h \in [0,1)$, we get $d(x_n,x_m) \to 0$. Hence $x_n$ is a Cauchy sequence. And since $X$ is complete, then $x_n$ converges to $x^*$ in $X$ which implies that every subsequence $x_{n_p}$ of $x_n$ will converge to $x^*$.

Since $x_0 \in A_1$, then there exists subsequence $x_{n_p} \in A_1$ that converge to $x^*$ and since $A_1$ is closed, then $x^* \in A_1$. Again for $x_1 \in A_2, x_2 \in A_3, \dots, x_p \in A_1$ there are $x_{n_{p+1}} \in A_2$, $x_{n_{p+2}} \in A_3, \dots, x_{n_{2p-1}} \in A_p$ are also converge to $x^*$ and $x^* \in A_i$ when $i = 1,2,\dots,p$ because every $A_i$ are closed. Therefore $x^* \in \bigcap_{i=1}^{p} A_i$.

Now, we want to show $x^*$ be fixed point of $F$ and $G$. Firstly, we let $x = x^*$, $y = x_n$ and assume that $x^* \notin Fx^*$, which implies that $d(x^*,Fx^*) > 0$ then we obtain that

$$\begin{aligned} 0 &\ge \varphi\left(H\left(Fx^*,Gx_n\right), d(x^*,x_n), d(x^*,Fx^*), d(x_n,Gx_n), d(x^*,Gx_n), d(x_n,Fx^*)\right) \\ &\ge \varphi\left(H\left(Fx^*,Gx_{2n-1}\right), d(x^*,x_{2n-1}), d(x^*,Fx^*), d(x_{2n-1},Gx_{2n-1}), d(x^*,Gx_{2n-1}), d(x_{2n-1},Fx^*)\right) \\ &\ge \varphi\left(d\left(Fx^*,x_{2n}\right), d(x^*,x_{2n-1}), d(x^*,Fx^*), d(x_{2n-1},x_{2n}), d(x^*,x_{2n}), d(x_{2n-1},Fx^*)\right) \end{aligned}$$

and we take $n \to \infty$ to have

$$0 \ge \varphi\left(d\left(Fx^*,x^*\right), d(x^*,x^*), d(x^*,Fx^*), d(x^*,x^*), d(x^*,x^*), d(x^*,Fx^*)\right) \tag{3.14}$$

while the condition $M_3$ in Definition 3.1 implies that

$$0 < \varphi\left(d\left(Fx^*,x^*\right), d(x^*,x^*), d(x^*,Fx^*), d(x^*,x^*), d(x^*,x^*), d(x^*,Fx^*)\right)$$

for any $d(x^*,Fx^*) > 0$, which contradict. Therefore $x^* \in Fx^*$, meanwhile if we let $x = x_n$, $y = x^*$ and assume that $x^* \notin Gx^*$, which implies that $d(x^*,Gx^*) > 0$ then we obtain that

$$\begin{aligned} 0 &\ge \varphi\left(H\left(Fx_n,Gx^*\right), d(x_n,x^*), d(x_n,Fx_n), d(x^*,Gx^*), d(x_n,Gx^*), d(x^*,Fx_n)\right) \\ &\ge \varphi\left(H\left(Fx_{2n},Gx^*\right), d(x_{2n},x^*), d(x_{2n},Fx_{2n}), d(x^*,Gx^*), d(x_{2n},Gx^*), d(x^*,Fx_{2n})\right) \\ &\ge \varphi\left(d\left(x_{2n+1},Gx^*\right), d(x_{2n},x^*), d(x_{2n},x_{2n+1}), d(x^*,Gx^*), d(x_{2n},Gx^*), d(x^*,Fx_{2n})\right) \end{aligned}$$

and also take $n \to \infty$ to have

$$0 \ge \varphi\left(d\left(x^*,Gx^*\right), d(x^*,x^*), d(x^*,x^*), d(x^*,Gx^*), d(x^*,Gx^*), d(x^*,Fx^*)\right) \tag{3.15}$$

that is

$$0 \geq \varphi\left(d\left(x^*, Gx^*\right), 0, 0, d(x^*, Gx^*), d(x^*, Gx^*), 0\right). \tag{3.16}$$

But the condition $M_3$ in Definition 3.1 implies that

$$0 < \varphi\left(d\left(x^*, Gx^*\right), 0, 0, d(x^*, Gx^*), d(x^*, Gx^*), 0\right) \tag{3.17}$$

for any $d(x^*, Gx^*) > 0$, which contradict then $d(x^*, Gx^*) = 0$. Hence $x^*$ is common fixed point of $F$ and $G$. □

**Corollary 3.6** ([13]). *Let $(X,d)$ be a complete metric space, $p \in \mathbb{N}$ and $A_1, A_2, \dots, A_p$ be nonempty closed subsets of $X$ and $Y = \bigcup_{i=1}^{p} A_i$. If $F : Y \to CB(X)$ is an implicit cyclic contractive mapping for some $T \in \tau_6$, then $F$ has at least a fixed point in $\bigcap_{i=1}^{p} A_i$.*

*Proof.* From Theorem 3.5, when $F$ and $G$ are the same mappings then we obtained that result. □

**Definition 3.7.** Let $(X,d)$ be a complete metric space. Let $A_1, A_2, \dots, A_p$ be nonempty closed subsets of $X$ when $p$ be a positive integer. Then $F : Y \to CB(X)$ and $G : Y \to CB(X)$ are called *Kannan-type of generalized cyclic contractive multivalued mappings* if it satisfies the following condition:

$I_1$: $Y = \bigcup_{i=1}^{p} A_i$ is a generalized cyclic representation of $Y$ with respect to $F$ and $G$ ;

$I_2$: For any $x, y \in A_i \times A_{i+1}$ when $i = 1, 2, \dots, p$ and $A_{p+1} = A_1$,

$$H(Fx, Gy) \leq a\,[d(x, Fx) + d(y, Gy)] \quad \text{where } 0 \leq a < \frac{1}{2}.$$

From the Definition 3.7 and Example 3.2, we can obtain the following corollary.

**Corollary 3.8.** *Let $(X,d)$ be a complete metric space, $p \in \mathbb{N}$ and $A_1, A_2, \dots, A_p$ be nonempty closed subsets of $X$ and $Y = \bigcup_{i=1}^{p} A_i$. If $F : Y \to CB(X)$ and $G : Y \to CB(X)$ are called* Kannan-type of generalized cyclic contractive multivalued mappings*, then $F$ and $G$ has a common fixed point in $\bigcap_{i=1}^{p} A_i$.*

*Proof.* The proof of this Corollary will follow by Theorem 3.5 and Example 3.2, where $a_1, a_4, a_5 = 0$, and $a_2 = a_3 = a$. □

**Remark 3.9.** There are many ways to define the implicit function that satisfies the conditions in Definition 3.1, then we can define and proof the generalization of our main theorem as same as the previous corollary.

## 4. Conclusion

In this work, we consider the extension of cyclic representation and implicit function that established in [13]. We have determined cyclic relation on two mappings which are multivalued mappings and more generalized than the literature. Moreover, our results are also generalized and can reduce to cover the results in [13]. It would be an interesting study to consider for the other spaces by using this cyclic representation under implicit function.

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### Competing Interests

The authors declare that they have no competing interests.

### Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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