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Strong and Δ-Convergence Theorems for Asymptotically k-Strictly Pseudo-Contractive Mappings in CAT(0) Spaces

Research Article

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Abstract. In this paper, we study and prove fixed point and convergence theorems of modified Mann iteration for asymptotically k-strictly pseudo-contracttive mappings in CAT(0) spaces. Our result extend and improve many results in the literature.

Keywords. Fixed point; Asymptotically k-strictly pseudo-contracttive mappings; Convergence theorems; CAT(0) spaces

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1. Introduction

Let C be a nonempty subset of a real Hilbert space X. Let $T : C \to C$ be a self-mapping. Recall that a mapping T is said to be:

(1) contractive if there exists a constant $k < 1$ such that $\|Tx - Ty\| \le k\|x - y\|$ for all $x, y \in C$;

(2) nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$;

(3) k-strictly pseudo-contractive [20] if there exists a constant $k \in [0,1)$ such that $\|Tx - Ty\|^2 \le k\|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2$ for all $x, y \in C$;

(4) asymptotically nonexpansive if there exists a sequence $\{k_n\}$ in $[1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \le k_n\|x - y\|$ for all $x, y \in C$ and $n \ge 1$;

(5) uniformly L-Lipschitzian [6] if there exists a constant $L > 0$ such that $\|T^n x - T^n y\| \le L\|x - y\|$ for all $x, y \in C$ and $n \ge 1$;

(6) asymptotically k-strictly pseudo-contractive [19] if there exist a sequence $\{k_n\}$ in $[1,\infty]$ with $\lim_{n\to\infty} k_n = 1$ and constant $k \in [0,1)$ such that $\|T^n x - T^n y\|^2 \le k_n\|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y)\|^2$ for all $x, y \in C$ and $n \ge 1$.

The class of strictly pseudo-contractive mappings has been studied by several authors (see for example [2, 8, 16, 18]) that an asymptotically strictly pseudo-contractive mapping is an uniformly L-Lipschitzion mapping.

In this paper, we define the concept of an asymptotically k-strictly pseudo-contractive mapping in a CAT(0) as follows: Let C be a nonempty subset of a CAT(0) space X. A mapping $T : C \to C$ is said to be asymptotically k-strictly pseudo-contractive if there exist a constant $k \in [0,1)$ and sequence $\{k_n\} \in [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x, T^n y)^2 \le k_n d(x, y)^2 + k(d(x, T^n x) + d(y, T^n y))^2 \tag{1.1}$$

for all $x, y \in C$. A point $x \in C$ is called a *fixed point* of T if $x = T(x)$. We denote with $F(T)$ the set of fixed points of T. A sequence $\{x_n\}$ is called approximate fixed point sequence for T if

$$\lim_{n\to\infty}(x_n, Tx_n) = 0.$$

Kirk [10–12] first studied the theory of fixed point in CAT(0) spaces[1]. Later on, the fixed point theory for some mappings in CAT(0) spaces has been rapidly developed by many authors (see, e.g., [5–7, 9, 13, 23, 24]).

In 1953, Mann [15] introduced the following iteration for approximating a fixed point of nonexpansive and pseudo-contractive mappings, sequence $\{x_n\}$ is defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n \tag{1.2}$$

for all $n \ge 1$, where $\{\alpha_n\}$ is an appropriate sequence $(0,1)$.

Motivate and inspired, we modify Mann's iteration (1.2) to asymptotically nonexpansive and asymptotically k-strictly pseudo-contractive mappings in CAT(0) spaces is as below

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n)T^n x_n \tag{1.3}$$

for all $n \ge 1$, where $\{\alpha_n\}$ is an appropriate sequence $(0,1)$.

Qihou [19] proved some convergence results for class of an asymptotically k-strictly pseudo-contractive mapping in Hilbert spaces.

The purpose of this paper, we prove strong and Δ-convergence results by using modified Mann iteration process for an asymptotically k-strictly pseudo-contractive mapping in CAT(0) spaces. In section 2 and 3, we present preliminaries and main results, respectively.

[1]The initials of term "CAT" are is honour of E. Cartan, A.D. Alexanderov and V.A. Toponogov.

2. Preliminaries

Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from closed interval $[0,r] \subset \mathbf{R}$ to X such that

$$c(0) = x, c(r) = y \quad \text{and} \quad d(c(t),c(s)) = |t-s|$$

for all $s,t \in [0,r]$.

In particular, c is an isometry and $d(x,y) = r$. The image of c is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic is denoted by $[x,y]$. We denote the point $w \in [x,y]$ if and only if there exists $\alpha \in [0,1]$ such that

$$d(x,w) = \alpha d(x,y) \text{ by } w = (1-\alpha)x \oplus \alpha y.$$

The space (X,d) is called a geodesic space if any two points of X are joined by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x,y \in X$. A subset $D \subseteq X$ is called convex if D includes geodesic segment joining every two points of itself. A geodesic triangle $\triangle(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consist of three points (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for geodesic triangle (or $\triangle(x_1,x_2,x_3)$) in (X,d) is a triangle $\overline{\triangle}(x_1,x_2,x_3) = \triangle(\overline{x}_1,\overline{x}_2,\overline{x}_3)$ in the Euclidean plane $\mathbf{R}^2$ such that

$$d_{\mathbf{R}^2}(\overline{x}_i,\overline{x}_j) = d(x_i,x_j) \quad \text{for } i,j \in \{1,2,3\}.$$

A geodesic metric space is said to be a CAT(0) space [1] if all geodesic triangle satisfy the following comparison axiom. Let $\triangle$ be a geodesic triangle in X and $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then, $\triangle$ is said to satisfy the CAT(0) inequality if for all $x,y \in \triangle$ and all comparison points

$$d(x,y) \le d_{\mathbf{R}^2}(\overline{x},\overline{y}).$$

If x,y_1,y_2 are points of a CAT(0) space and y_0 is the midpoint of the segment $[y_1,y_2]$, which we will denote by $\left(\frac{y_1 \oplus y_2}{2}\right)$, then the CAT(0) inequality implies

$$d\left(x,\frac{y_1 \oplus y_2}{2}\right)^2 \le \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2)^2 - \frac{1}{4}d(y_1,y_2)^2. \tag{2.1}$$

The inequality (2.1) is called the (CN) inequality (see more details Bruhat and Titz [3]).

A geodesic metric space is a CAT(0) space if and only if it satisfies the (CN) inequality.

A subset C of a CAT(0) space X is convex if for any $x,y \in C$, then $[x,y] \subset C$.

Lemma 2.1 ([7]). *Let X be a* CAT(0) *space.*

(i) *For any $x,y,z \in X$ and $t \in [0,1]$, has*

$$d((1-t)x \oplus ty,z) \le (1-t)d(x,z) + td(y,z). \tag{2.2}$$

(ii) *For any $x,y,z \in X$ and $t \in [0,1]$, has*

$$d((1-t)x \oplus ty,z)^2 \le (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2. \tag{2.3}$$

Next, we refer some elementary properties about CAT(0) spaces as follows: Let $\{x_n\}$ be a bounded sequence in a CAT(0) space (X,d). For all $x \in X$, we set

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,\{x_n\}).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}.$$

We know that in a complete CAT(0) space, (see [6]) $A(\{x_n\})$ consists of exactly one point.

In 1976, a concept of convergence in a general metric space introduced by Lim [14] setting which is called Δ-convergent. In 2008, Kirk and Panyanak [12] used the concept of Δ-convergent to prove on the CAT(0) space.

Definition 2.2 ([4,6,14,17,21,22]). A sequence $\{x_n\}$ in X is said to be Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Lemma 2.3 ([7]). *If $\{x_n\}$ is a bounded sequence in a complete* CAT(0) *space with $A(\{x_n\}) = \{x\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence of $\{d(x_n,u)\}$ is convergent then $x = u$.*

Let C be a closed convex subset of a CAT(0) space X and $\{x_n\}$ be a bounded sequence in C. We given the notation as follows:

$$x_n \rightharpoonup \omega \Leftrightarrow \Phi(\omega) = \inf_{x\in C} \Phi(x).$$

Proposition 2.4 ([17]). *Let C be a closed convex subset of a* CAT(0) *space X and $\{x_n\}$ be a bounded sequence in C. Then $\Delta - \lim_{n\to\infty} x_n = p$ implies that $\{x_n\} \rightharpoonup p$.*

Lemma 2.5. (i) (*see* [11]) *Every bounded sequences in a complete* CAT(0) *space always has an Δ-convergent subsequence.*

(ii) (*see* [5]) *Let C be a nonempty closed convex subset of a complete* CAT(0) *space and let $\{x_n\}$ be a bounded sequence in C. Then the asymptotic center of $\{x_n\}$ is in C.*

Definition 2.6 ([22]). Let (X,d) be a metric space and C be its nonempty subset. Then $T : C \to C$ is said to be semi-compact if for a sequence x_n in C with $\lim_{n\to\infty} d(x_n,Tx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in C$.

3. Main Results

Next, we state our results of my work.

Theorem 3.1. *Let C be a nonempty bounded closed convex subset of a complete* CAT(0) *space X and let $T : C \to C$ be an asymptotically k-strictly pseudo-contractive mapping. Then T has a fixed point. Moreover fixed point set $F(T)$ is closed and convex.*

Proof. $F(T)$ closed is evident. Since T is continuous. We will show that the fixed point set of T is convex. Let $p,q \in F(T)$ and $t \in (0,1)$. Setting $z = (1-t)p \oplus tq$ we get,

$$d(z,p)^2 \le t^2 d(p,q)^2 \quad \text{and} \quad d(z,q)^2 \le (1-t)^2 d(p,q)^2.$$

Since T is an asymptotically k-strictly pseudo-contractive mapping, from Lemma 2.1, we obtain

$$\begin{aligned} d(z,T^n z)^2 &= d((1-t)p \oplus tq, T^n z)^2 \\ &\le (1-t)d(p,T^n z)^2 + td(q,T^n z)^2 - t(1-t)d(p,q)^2 \\ &\le (1-t)\{k_n d(z,p)^2 + k(d(z,T^n z)+d(p,p))^2\} \\ &\quad + t\{k_n d(z,q)^2 + k(d(z,T^n z)+d(q,q))^2\} - t(1-t)d(p,q)^2 \\ &= (1-t)\{k_n t^2 d(p,q)^2 + kd(z,T^n z)^2\} \\ &\quad + t\{k_n(1-t)^2 d(p,q)^2 + kd(z,T^n z)^2\} - t(1-t)d(p,q)^2 \\ &= t(1-t)(tk_n + (1-t)k_n - 1)d(p,q)^2 + (1-t+t)kd(z,T^n z)^2 \\ &= t(1-t)(k_n-1)d(p,q)^2 + kd(z,T^n z)^2, \end{aligned}$$

where $k \in [0,1)$ and a sequence $\{k_n\}$ in $[1,\infty)$ such that $\lim_{n\to\infty} k_n = 1$ for any $n \ge 1$. It follow that

$$d(z,T^n z)^2 \le \frac{t(1-t)d(p,q)^2}{1-k}(k_n - 1).$$

Hence, taking limit as $n \to \infty$ on the both side the above inequality and by using the fact of $k_n \to 1$ as $n \to \infty$, we have $\lim_{n\to\infty} d(z,T^n z) = 0$. From the continuity of T, we obtain

$$Tz = T(\lim_{n\to\infty} T^n z) = \lim_{n\to\infty} T^{n+1} z = z.$$

Therefore $z \in F(T)$, that is, $F(T)$ is convex. This complete proof. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a complete* CAT(0) *space X and let $T: C \to C$ be an asymptotically k-strictly pseudo-contractive mappings such that $k \in [0,\frac{1}{2})$ there exist sequence $\{k_n\} \subset [1,\infty)$ $\lim_{n\to\infty} k_n = 1$ and $F(T) \neq \emptyset$. Let $\{x_n\}$ be a bounded sequence in C such that $\Delta - \lim_{n\to\infty} x_n = \omega$ and $\limsup_{m\to\infty}\limsup_{n\to\infty} d(x_n,T^m x_n) = 0$. Then $T\omega = \omega$.*

Proof. By the hypothesis, $\Delta - \lim_{n\to\infty} x_n = \omega$. By Proposition 2.4 we have $x_n \rightharpoonup \omega$.

Then, we have $A(x_n) = \omega$ by Lemma 2.5(ii). Since $\limsup_{m\to\infty}\limsup_{n\to\infty} d(x_n,T^m x_n) = 0$. Then, we have

$$\Phi(x) = \limsup_{n\to\infty} d(T^m x_n, x) = \limsup_{n\to\infty} d(Tx_n, x) \tag{3.1}$$

for all $x \in C$. From (3.1) taking $x = T^m\omega$, we obtain

$$\begin{aligned} \Phi(T^m\omega)^2 &= \limsup_{n\to\infty} d(T^m x_n, T^m\omega)^2 \\ &\le \limsup_{n\to\infty} \{k_m d(x_n,\omega)^2 + k(d(x_n,T^m x_n) + d(\omega,T^m\omega))^2\}. \end{aligned}$$

Since $\lim_{n\to\infty} k_n = 1$ and $\limsup_{m\to\infty}\limsup_{n\to\infty} d(x_n,T^m x_n) = 0$. Taking $\limsup_{m\to\infty}$ of the both sides the above inequality, we have

$$\begin{aligned} \limsup_{m\to\infty} \Phi(T^m\omega)^2 &\le \limsup_{m\to\infty} k_m \limsup_{n\to\infty} d(x_n,\omega)^2 + k \limsup_{m\to\infty}\limsup_{n\to\infty} d(\omega,T^m\omega)^2 \\ &= \Phi(\omega)^2 + k \limsup_{m\to\infty} d(\omega,T^m\omega)^2. \end{aligned} \tag{3.2}$$

By the (CN) inequality implies that

$$d\left(x_n, \frac{\omega \oplus T\omega}{2}\right)^2 \le \frac{1}{2}d(x_n,\omega)^2 + \frac{1}{2}d(x_n,T^m\omega)^2 - \frac{1}{4}d(\omega,T^m\omega)^2.$$

for any $n, m \geq 1$. Letting $n \to \infty$ and taking limit supremum on the both sides, we get

$$\Phi\left(\frac{\omega \oplus T\omega}{2}\right)^2 \leq \frac{1}{2}\Phi(\omega)^2 + \frac{1}{2}\Phi(T^m\omega)^2 - \frac{1}{4}d(\omega, T^m\omega)^2$$

for any $m \geq 1$. Since $A(\{x_n\}) = \{\omega\}$, we get

$$\begin{aligned}\Phi(\omega)^2 &\leq \Phi\left(\frac{\omega \oplus T\omega}{2}\right)^2 \\ &\leq \frac{1}{2}\Phi(\omega)^2 + \frac{1}{2}\Phi(T^m\omega)^2 - \frac{1}{4}d(\omega, T^m\omega)^2\end{aligned}$$

which implies that

$$d(\omega, T^m\omega)^2 \leq 2\Phi(T^m\omega)^2 - 2\Phi(\omega)^2.$$

Taking limit supremum on the both sides, we obtain

$$\limsup_{m\to\infty} d(\omega, T^m\omega)^2 \leq 2\limsup_{m\to\infty} \Phi(T^m\omega)^2 - 2\Phi(\omega)^2. \tag{3.3}$$

From inequalities (3.2) and (3.3), we get

$$\begin{aligned}\limsup_{m\to\infty} d(\omega, T^m\omega)^2 &\leq 2(\Phi(\omega)^2 + k\limsup_{m\to\infty} d(\omega, T^m\omega)^2) - 2\Phi(\omega)^2 \\ &\leq 2k\limsup_{m\to\infty} d(\omega, T^m\omega).\end{aligned}$$

So, we obtain

$$(1-2k)\limsup_{m\to\infty} d(\omega, T^m\omega)^2 \leq 0. \tag{3.4}$$

Since $k \in [0, \frac{1}{2})$, we get $\limsup_{m\to\infty} d(\omega, T^m\omega) = 0$, which implies $\lim_{m\to\infty} T^m(\omega) = \omega$ therefore $T\omega = \omega$. The proof is completed. □

Theorem 3.3. *Let C be a nonempty closed convex subset of a complete* CAT(0) *space X and let $T: C \to C$ be an asymptotically k-strictly pseudo-contractive mapping with $k \in [0, \frac{1}{2})$ and a sequence $\{k_n\}$ in $[1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$. Let $\{x_n\}$ be a sequence in C defined by* (1.3) *and $\{\alpha_n\}$ is a sequence in $(0,1)$. Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\lim_{n\to\infty} d(x_n, p)$ exist for all $p \in F(T)$.*

Proof. First, we will prove that $\lim_{n\to\infty} d(x_n, p)$ exist. It follow from Theorem 3.1 such that $F(T) \neq \emptyset$ and $p \in F(T)$. Since T is asymptotically k-strictly pseudo-contractive mapping and using Lemma 2.1 and Mann iteration (1.3), we have

$$\begin{aligned}d(x_{n+1}, p)^2 &= d(\alpha x_n \oplus (1-\alpha_n)T^n x_n, p)^2 \\ &\leq \alpha_n d(x_n, p)^2 + (1-\alpha_n)d(T^n x_n, p)^2 - \alpha_n(1-\alpha_n)d(x_n, T^n x_n)^2 \\ &\leq \alpha_n d(x_n, p)^2 + (1-\alpha_n)\{k_n d(x_n, p)^2 + k(d(x_n, T^n x_n)^2\} - \alpha_n(1-\alpha_n)d(x_n, T^n x_n)^2 \\ &= \alpha_n d(x_n, p)^2 + k_n d(x_n, p)^2 - k_n\alpha_n d(x_n, p)^2 + (1-\alpha_n)k d(x_n, T^n x_n)^2 \\ &\quad - \alpha_n(1-\alpha_n)d(x_n, T^n x_n)^2.\end{aligned}$$

From $\lim_{n\to\infty} k_n = 1$, taking limit as $n \to \infty$, we obtain

$$\begin{aligned}d(n_{n+1}, p)^2 &\leq d(x_n, p)^2 - \left[(1-\alpha_n)(\alpha_n - k)d(x_n, T^n x_n)^2\right] &(3.5)\\ &\leq d(x_n, p)^2. &(3.6)\end{aligned}$$

It follow that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, p)$ exist for all $p \in F(T)$.

Assume that $\lim_{n\to\infty} d(x_n,p)=r$. From (3.6), we get

$$\lim_{n\to\infty} d(x_{n+1},p)=r. \tag{3.7}$$

By (3.5), we also have

$$d(x_n,T^n x_n)^2 \le \frac{1}{(1-\alpha_n)(\alpha_n-k)}\left[d(x_n,p)^2-d(n_{n+1},p)^2\right]. \tag{3.8}$$

Since $\lim_{n\to\infty} d(x_n,p)$ exist, we obtain

$$\lim_{n\to\infty} d(x_n,T^n x_n)=0. \tag{3.9}$$

Next step, we prove that $\lim_{n\to\infty} d(x_n,Tx_n)=0$. It follow from (1.3) and (3.9), we obtain

$$\begin{aligned} d(x_{n+1},x_n) &= d(\alpha x_n \oplus (1-\alpha_n)T^n x_n, x_n)\\ &\le (1-\alpha_n)d(x_n,T^n x_n) \quad \to 0 \quad \text{as } n\to\infty. \end{aligned} \tag{3.10}$$

Since T is an uniformly L-lipschitzian mapping, from (3.9) and (3.10) for any $n\ge 1$, we have

$$\begin{aligned} d(x_n,Tx_n) &\le d(x_n,x_{n+1})+d(x_{n+1},T^{n+1}x_{n+1})+d(T^{n+1}x_{n+1},T^{n+1}x_n)+d(T^{n+1}x_n,Tx_n)\\ &\le d(x_n,x_{n+1})+d(x_{n+1},T^{n+1}x_{n+1})+Ld(x_{n+1},x_n)+Ld(T^n x_n,x_n)\\ &=(1+L)d(x_n,x_{n+1})+d(x_{n+1},T^{n+1}x_{n+1})+Ld(x_n,T^n x_n)\\ &\to 0 \ \text{ as } n\to\infty. \end{aligned}$$

The proof is completed. □

Next, we are ready to prove our results of an Δ-convergence theorem.

Theorem 3.4. *Let C be a nonempty closed convex subset of a complete* CAT(0) *space X and let $T: C\to C$ be an asymptotically k-strictly pseudo-contractive mapping with $k\in[0,\frac{1}{2})$ and a sequence $\{k_n\}$ in $[1,\infty)$ such that $\lim_{n\to\infty} k_n=1$. Let $\{x_n\}$ be a sequence in C defined by* (1.3) *and $\{\alpha_n\}$ is a sequence in $(0,1)$. Then the sequence $\{x_n\}$ is Δ-convergent to a fixed point of T.*

Proof. The first, we prove that

$$W_\Delta(x_n)=\bigcup_{\{u_n\}\subseteq\{x_n\}} A(\{u_n\})\subseteq F(T).$$

Let $u\in W_\Delta(x_n)$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\})=\{u\}$. By Lemma 2.5, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta-\lim_n v_n=v\in C$. By Theorem 3.3 and Theorem 3.2, we have $v\in F(T)$. Since $\lim_{n\to\infty} d(x_n,v)$ exists, so $u=v$ by Lemma 2.3. This show that $W_\Delta(x_n)\subseteq F(T)$.

Next, we prove that Δ-converges to a point in $F(T)$, it is sufficient to show that $W_\Delta(\{x_n\})$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\})=\{u\}$ and let $A(\{x_n\})=\{x\}$. We have already seen that $u=v$ and $v\in F(T)$. Since $u\in W_\Delta(x_n)\subseteq F(T)$, by Theorem 3.3, $\lim_{n\to\infty} d(x_n,u)$ exists. Hence, we obtain $x=u$ by Lemma 2.3. This shows $W_\Delta(x_n)=\{x\}$. This completes the proof. □

Theorem 3.5. *Let C be a nonempty closed convex subset of a complete* CAT(0) *space X. and let $T: C\to C$ be an uniformly continuous asymptotically k-strictly pseudo-contractive mapping with $k\in[0,\frac{1}{2})$ and a sequence $\{k_n\}$ in $[1,\infty)$ such that $\lim_{n\to\infty} k_n=1$. Let $\{x_n\}$ be a sequence in C defined* *by* (1.3) *and* $\{\alpha_n\}$ *is a sequence in* $(0,1)$. *Assume that* T^s *is semi-compact for some* $s \in N$. *Then the sequence* $\{x_n\}$ *is converges strongly to a fixed point of* T.

Proof. By Theorem 3.3, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since T is an uniformly continuous, then

$$d(x_n, T^s(x_n)) \le d(x_n, T(x_n)) + d(T(x_n), T^2(x_n)) + \ldots + d(T^{s-1}(x_n), T^s(x_n)) \to 0 \text{ as } n \to \infty.$$

That is, $\{x_n\}$ is an approximate fixed point sequence for T^s. By Definition 2.6, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $p \in C$ such that $\lim_{k\to\infty} x_{n_k} = p$. Again, by the uniform continuity of T, we obtain

$$d(T(p), p) \le d(T(p), T(x_{n_k})) + d(T(x_{n_k}), x_{n_k}) + d(x_{n_k}, p) \to 0 \text{ as } k \to \infty.$$

That is, $p \in F(T)$. From again Theorem 3.3, we get $\lim_{n\to\infty} d(x_n, p)$ exist, therefore p is the strong limit of the sequence $\{x_n\}$ itself. The proof is completed. □

4. Conclusion

In this work, we studied and proved strong and Δ-convergence theorems by using modified Mann iteration process for an asymptotically k-strictly pseudo-contractive mapping in a CAT(0) spaces.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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