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Polynomial G_L, Yang-Baxter Equation and Quantum Group $SL(2)_q$

Research Article

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Abstract. In this paper, we define the polynomial G_L by way of the braids. We construct the abstract tensor model of the polynomial G_L and we obtain the new solutions relevant with the state model of the polynomial G_L to the Yang-Baxter equation. We also construct the vacuum-vacuum expectation model of the polynomial G_L and we show that the studies performed using the Kaufmann bracket on the quantum group $SL(2)_q$ with $q = A^2$ are valid for the state model of the polynomial G_L without $q = A^2$.

Keywords. Polynomial G_L; Jones polynomial; Regular isotopy; Braid; Abstract tensor; Vacuum-vacuum expectation; Quantum group $SL(2)_q$

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1. Introduction

In [2], we constructed a one-variable Laurent polynomial invariant of oriented knots and links, and denoted it by N_L for an oriented link diagram L. The primary version of this is an invariant of regular isotopy for oriented knot and link diagrams, denoted G_L. We have seen that G_L is a special case of a general polynomial, $[L]$, well defined on equivalence classes of oriented diagrams. Since the polynomial N_L obtained from G_L by multiplying a normalizing factor, it is an invariant for oriented knots and links. In [3] we give some properties of these polynomials. We also calculate the polynomials G_L and N_L of the knots through nine crossings and the two-component links through eight crossing. The polynomial N_L may be compared with the original

Jones polynomial [8], [9], [10] and with the normalized bracket polynomial [11], [12], [13], [14]. In fact, the polynomial N_L yields the Jones polynomial and the normalized bracket polynomial (see [2, Theorem 2.11]). Thus, this gives an oriented state model for the Jones polynomial in principle. In [2], we also used the polynomial G_L to prove that the number of crossings in connected, reduced alternating projection of a link L is a topological invariant of L. This is a remarkable application of the polynomial G_L. It solves some of old conjectures about alternating knots due to Tait, see [15], [16], [17].

This paper is organized as follows. The Section 2 contain brief information about the polynomial G_L, braid and the Jones polynomial. In Section 3, we demonstrate the normalized polynomial N_L is a version of the original Jones polynomial by way of the theory of braids. For this purpose, by discussing generalities about braids and the construction of the original Jones polynomial we look directly at the polynomial G_L on closed braids. We prove that there is a representation of the Artin braid group to Temperley-Lieb algebra. This representation is defined by the formulas obtained by applying the state model of the polynomial G_L to the generators of braid group in Theorem 6. In the process, the structure of the Jones polynomial and its associated representations of the braid groups will naturally emerge.

In Section 4, We construct an abstract tensor model for the polynomial G_L. We prove that this abstract model is an invariant of regular isotopy in Theorem 8 and obtain the polynomial G_L from this abstract model. Therefore we have the new solutions relevant with the state model of the polynomial G_L to the Yang-Baxter equation.

In the last section, by choosing a creation matrix, M^{ab}, and a annihilation matrix, M_{ab}, give us a tensor model for the polynomial G_L we express the polynomial G_L as vacuum-vacuum expectation and show that the relations of the algebra known as the $SL(2)_q$ quantum group in the literature leaves invariant the matrices $\tilde{\varepsilon}^{ab}$ and $\tilde{\varepsilon}_{ab}$ which are the special cases of the matrices M^{ab} and M_{ab}. The relations of the algebra are the same as the relations of the algebra which leaves invariant the $\tilde{\varepsilon}$-matrix obtained the vacuum-vacuum expectation of the Kauffmann bracket with $q = A^2$ in Theorem 13. Thus the studies performed using the Kauffmann bracket in Quantum Group $SL(2)_q$ are valid for the state model of the polynomial G_L without $q = A^2$.

2. Polynomial G_L

We begin by defining a 3-variable polynomial on oriented link diagrams. A link L of k components is a subset of $\mathbb{R}^3 \subset \mathbb{R}^3 \cup \infty = S^3$, consisting of k disjoint piecewise linear simple closed curves; a knot is a link with one component. Although links live in $\mathbb{R}^3$, we usually represent them by link diagrams: the regular projections of links into with over passing curves specified.

Definition 1. Given an oriented link diagram L, let $[L] \in Z[p,q,r]$ denote the corresponding polynomial in commuting algebraic variables p, q and r. The polynomial $[L]$ satisfies the axioms:

(1) $[\bigcirc] = r, \quad [\bigcirc \sqcup L] = r[L]$

(2) $p[L_+] - q[L_-] = (p-q)[L_\circ]$

where L_+, L_- and $L_\circ$ are diagrams in Figure 1, $\bigcirc$ is the oriented diagram with zero-crossing of the unknot and $\sqcup$ denotes disjoint union.

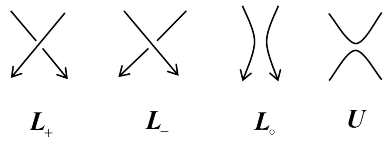


Figure 1. Crossings and splits.

We consider the model:

$$[L_+] = [L_\circ] + q[U]$$
$$[L_-] = [L_\circ] + p[U]. \tag{2.1}$$

The axiom 2 of Definition 1 is obtained from the model (2.1) by multiplying the first equality by p and the second equality by q, and taking the difference. The split U in the model (2.1) acquires orientations outside the category of link diagrams. It is useful to think of these new vertices as abstract Feynman diagrams with a local arrow of time that is coincident with the direction of the diagrammatic arrows. Then $L_\circ$ represents an interaction and U represents creations and annihilations. For details, see [14].

The model (2.1) with $q = p^{-1}$ and $r = -q - q^{-1}$ is an invariant of regular isotopy for oriented link diagrams [2]. If we take $p = q^{-1}$, $r = -q - q^{-1}$ and $G_L = r^{-1}[L]$ so that $G_\circ = 1$ then we have the following definition from the model (2.1) and Definition 1. Henceforth, we call the state model of the polynomial G_L to the model (2.1).

Definition 2. Let L denote an oriented knot or link diagram. Then $G_L \in Z[q, q^{-1}]$ is a Laurent polynomial in the variable q assigned to oriented link diagram L. The polynomial satisfies the properties:

(1) $G_\circ = 1, \quad G_\circ \sqcup L = rG_L, \quad r = -q - q^{-1}$,

(2) $q^{-1}G_{L_+} - qG_{L_-} = (q^{-1} - q)G_{L_\circ}$,

(3) G_L is an invariant of regular isotopy.

It is possible to create an invariant of ambient isotopy associated with the polynomial G_L for oriented diagram L. For this we use the writhe, $\omega(L)$, of the oriented link diagram L ($\omega(L)$ is the sum of all crossing signs). Recall that $\omega(L)$ is also a regular isotopy invariant. Thus we may give the following definition.

Definition 3. We define a polynomial $N_L \in Z[q, q^{-1}]$ for an oriented link diagram L by the formula

$$N_L = (-q^{-1})^{-\omega(L)} N_L(q).$$

Call N_L a normalized polynomial of the polynomial G_L by the writhe.

The normalized polynomial N_L is an invariant of ambient isotopy and it satisfies the following skein relation [2]:

$$q^{-2}N_{L_+} - q^{-2}N_{L_-} = (q^{-1} - q)N_{L_\circ}.$$

In fact, the polynomial N_L is an oriented state model for the Jones polynomial $V_L(t)$ [8], [9], [10] and it is also a version of the normalized bracket polynomial $f_L(A)$ [11], [12], [13]. That is,

$$N_L(t^{-2}) = V_L(t) \quad \text{and} \quad N_L(A^{-2}) = f_L(A).$$

3. Polynomial G_L for Braids

A braid is formed when n points on a horizontal line are connected by n strings to n points on another horizontal line directly below the first n points so that parallel planes intersect with each strand at one point.

A general *n-braid* is constructed from the trivial *n-braid* by successive applications of operations b_i, $i = 1,2,\ldots,n-1$. The operations b_i and its inverse b_i^{-1} are best understood by the graphs. Every braid can be written as a product of the generators b_i and their inverses b_i^{-1}, (Figure 2). Hence a set of generators $b_1, b_2, \ldots, b_{n-1}$ defines the braid group B_n. By regarding the trivial *n-braid* as the identity operation in B_n, we can identity any element in B_n as an *n-braid*.

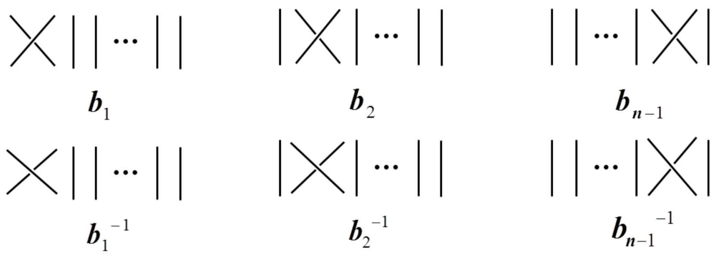


Figure 2. The generators of the braid group B_n.

To guarantee the topological equivalence between different expressions of a braid in terms of braid group elements, Artin prove that the following conditions are necessary and sufficient [4].

$$\begin{aligned} b_i b_i^{-1} &= 1, & i &= 1,2,\ldots,n-1 \\ b_i b_j &= b_j b_i, & |i-j| &> 1 \\ b_i b_{i+1} b_i &= b_{i+1} b_i b_{i+1}. \end{aligned}$$

We call them defining relations of the braid group B_n. Then each topologically equivalent class (isotopy class) of the braids is identified with an element in B_n. The braid group B_n has significance for knot theory in following sense [5].

For any braid, one may form a link by tying by opposite ends. According to Alexander's theorem any link is represented by a closed braid [1]. This fact gives the braid group a fundamental role in the knot theory. However, infinitely many braids given the same link when they closed. Therefore we need another important theorem due to Markov [5]. The equivalent braids expressing the same link are mutually transformed by successive applications of two types of operation, the Markov moves of type I and type II (Figure 3).

$$\begin{aligned} AB &\leftrightarrow BA, \quad (A,B \in B_n), \\ B &\leftrightarrow Bb_n \quad (\text{or } B \leftrightarrow Bb_n^{-1}), \quad (B \in B_n, b_n \in B_{n+1}). \end{aligned}$$

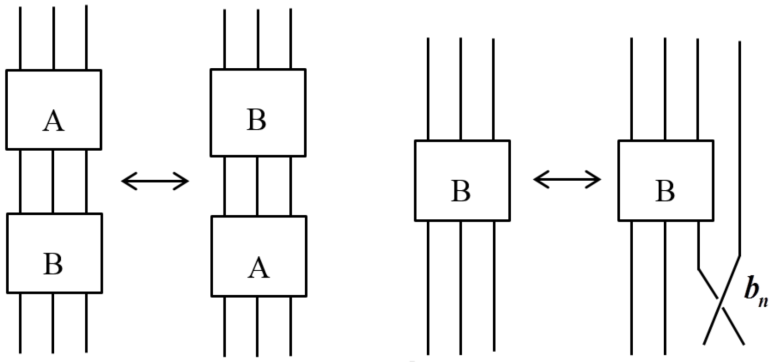


Figure 3. Markov moves of type I (left) and type II (right).

Thus the closed braid $\overline{Bb_n^{\pm 1}}$ is obtained from the closed braid $\overline{B}$ by a type I Reidemesiter moves. Another way to make a braid with the same closure is to choose any braid A in B_n and take the conjugate braid ABA^{-1}. When we close ABA^{-1} to $\overline{ABA^{-1}}$, the braid A and its inverse A^{-1} can be each other out by interacting through the closure strands. The fundamental theorem that relates the theory of knots and the theory of braids is the following theorem.

Theorem 4 (Markov)**.** *Let $A \in B_m$ and $B \in B_n$ be two braids in the braid groups B_m and B_n respectively. Then the links* (*closures of the braids A, B*) *$L = \overline{A}$ and $K = \overline{B}$ are ambient isotopic if only if A can be obtained from B by a series of*

(1) *Equivalence in a given braid group,*
(2) *Conjugation in a given braid group,*
(3) *Markov moves.*

For a proof of the Markov theorem the reader may wish to consult [5].

Jones constructed the invariant by a route involving braid groups and von Neumann algebras [7]. Although there is much more to say about von Neumann algebras, it is sufficient here a sequence of algebras $\mathscr{A}_n$ $(n = 2,3,\ldots)$ with multiplicative generators $e_1, e_2, \ldots, e_{n-1}$ and relations:

$$\begin{aligned} e_i^2 &= e_i \\ e_i e_{i\pm1} e_i &= t e_i \\ e_i e_j &= e_j e_i, \quad |i-j| > 1. \end{aligned}$$

Here t is a scalar, commuting with all the other elements. For our purposes we can let $\mathscr{A}_n$ be the free additive algebra on these generators viewed as a module over the ring $\mathbb{C}[t,t^{-1}]$, $\mathbb{C}$ denotes the complex numbers.

He constructed a representation $J_n : B_n \to \mathscr{A}_n$ of the Artin braid group to the algebra $\mathscr{A}_n$. The representation has the form $J_n(b_i) = ae_i + b$ with a and b chosen appropriately. Since $\mathscr{A}_n$ has a trace $tr : \mathscr{A}_n \to \mathbb{C}[t,t^{-1}]$, one can obtained a mapping $tr \circ J_n : B_n \to \mathbb{C}[t,t^{-1}]$. Upon appropriate normalization, this mapping is the Jones polynomial $V_L(t)$.

We have the information needed to use the presentations of the braid groups to extract topological information about knots and links. In particular, it is now possible to explain how the Jones polynomial works in relation to braids. Let R a commutative ring. We define functions $V_n : B_n \to R$, for $n = 2,3,\ldots$, from the *n-strand braid* group to the ring R. Then the Markov theorem assures us that the family of functions $\{V_n\}$ can be used to construct link invariants if the following conditions are satisfied:

- If A and B are equivalent braids, then $V_n(A) = V_n(B)$.
- If $A, B \in B_n$, then $V_n(B) = V_n(ABA^{-1})$.
- If $B \in B_n$, then there is a constant $\alpha \in \mathbb{R}$ such that

$$V_{n+1}(Bb_n) = \alpha V_n(B)$$
$$V_{n+1}(Bb_n^{-1}) = \alpha^{-1} V_n(B).$$

We see that for the closed braid $\overline{A} = \overline{Bb_n}$, the result of the Markov move $B \to A$ is to perform a type I on $\overline{B}$. Furthermore, $\overline{Bb_n}$ corresponds to a type I move of positive sign, $\overline{Bb_n^{-1}}$ corresponds to a type move I of negative sign. For this reason, we can choose the convention for α and α^{-1} as above.

Bearing in mind these remarks, we define the writhe of a braid B, $\omega(B)$, to be its exponent sum. That is,

$$\omega(B) = \sum_{l=1}^{k} a_l$$

in any braid word $b_{i_1}^{a_1} b_{i_2}^{a_2}, \ldots, b_{i_k}^{a_k}$ representing B. It is clear that $\omega(B) = \omega(\overline{B})$ where $\overline{B}$ is the oriented link obtained by closing the braid B.

Definition 5. Let $V_n : B_n \to R$ be given with properties as listed above, call $\{V_n\}$ a Markov trace on $\{B_n\}$. For any link L, let $L \sim \overline{B}$, $B \in B_n$ via Alexander's theorem ($\sim$ denotes ambient isotopy). $V_L \in R$ is defined with the formula

$$V_L = \alpha^{-\omega(B)} V_n(B).$$

and called V_L the link invariant for the Markov trace $\{V_n\}$.

Let V be the link invariant corresponding to the Markov trace $\{V_n\}$. Then V is an invariant of ambient isotopy for oriented links. That is, if $L \sim L'$ then $V_L = V_{L'}$, see [14].

After discussing the above generalizations about braids, we will now demonstrate that the normalized polynomial $N_L(q)$ is a version of the original Jones polynomial $V_L(t)$ by way of the theory of braids. For this, we can look directly at the polynomial G_L on closed braids. In the process, the structure of the Jones polynomial and its associated representations of the braid groups will naturally emerge.

In order to design this discussion, let's define $[\ ] : B_n \to Z[q, q^{-1}]$ via $[B] = [\overline{B}]$, the evaluation of the state model of the polynomial G_L on the closed braid B. In terms of the Markov trace formalism, we are letting $V_n : B_n \to Z[q, q^{-1}] = R$ via $V_n(B) = [\overline{B}]$. In fact, from the polynomial G_L, it is obvious that $\{V_n\}$ is a Markov trace with $\alpha = -q^{-1}$.

We now consider the states of a braid. That is, we consider the states determined by the state model of the polynomial G_L on oriented link diagrams (by forgetting the orientation).

Diagrammatic version of this model is as in Figure 4. So we have the following model for the braids:

$$[b_i] = [1_n] + q[U_i]$$
$$[b_i^{-1}] = [1_n] + q^{-1}[U_i]$$

where 1_n denotes the identity element in B_n (henceforth denoted by 1), and U_i is a new element written in braid input-output form, but with a cup-$\bigcup$ cap-$\bigcap$ combination $\overset{\cup}{\cap}$ at i-th and $i+1$-th strands.

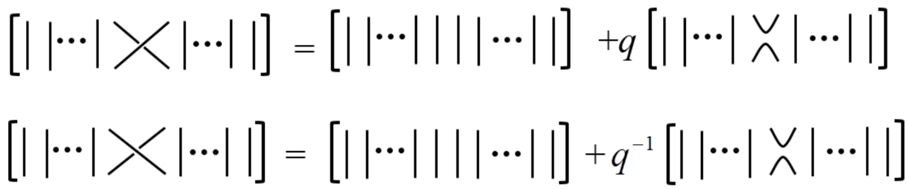


Figure 4. Diagrammatic version of the state model of the polynomial G_L for braids.

Since a state for $\overline{B}$ is obtained by choosing splice direction for each crossing of B, we see that each state of $\overline{B}$ can be written as the closure of an (input-output) product of the elements U_i. For convenience, we write the above model as follows:

$$b_i = 1 + qU_i$$
$$b_i^{-1} = 1 + q^{-1}U_i.$$

In this context, we write $B \equiv \bigcup(B)$ for a braid B, where $\bigcup(B)$ is a sum of products of the U_i's. Each product of the U_i's, when closed, gives a collection of loops. Thus, if U is such a product, then $[U] = [\overline{U}] = r^{\|U\|}$ where $\|U\| = \#(\overline{U}) - 1$ (#: the number of loops in $\overline{U}$) and $r = -(q^{-1} + q)$. Finally if $\bigcup(B)$ is given by

$$\bigcup(B) = \sum_s [B|s]U_s$$

where s indexes all the terms in the product, and $[B|s]$ is the product of p's and q^{-1}'s in each U_s, then

$$[B] = \left[\bigcup(B)\right] = \sum_s [B|s]U_s$$
$$[B] = \sum_s [U|s]r^{\|U\|}.$$

This is braid-analog of the state expansion for the polynomial G_L. The upshot of these observations is in calculating the polynomial G_L for braids in B_n. It is the free additive algebra $\mathscr{C}_n$ with generators $U_1, U_2, \ldots, U_{n-1}$ and multiplicative relations coming from the interpretation the generators U_i's as cup-cap combinations. This algebra $\mathscr{C}_n$ will be regarded as a module over the ring $Z[q,q^{-1}]$ with $r = -q - q^{-1} \in Z[q,q^{-1}]$ the designated loop value. We shall call $\mathscr{C}_n$ the Temperley-Lieb Algebra [5], [11]. By considering Figure 5, the relations of $\mathscr{C}_n$ are as follows:

$$U_iU_{i+1}U_i = U_i,$$
$$U_i^2 = rU_i, \tag{3.1}$$

$$U_iU_j = U_jU_i, \quad |i-j| > 1.$$

Note that the Temperley-Lieb algebra and the Jones algebra are closely related.

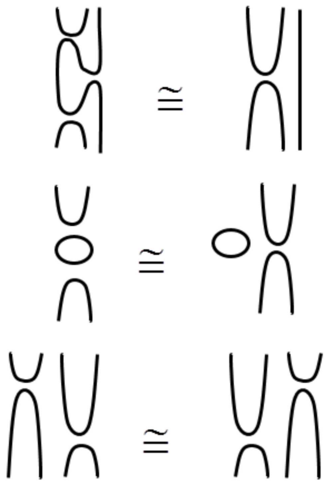


Figure 5. Relations of Temperley-Lieb algebra.

Theorem 6. *Let $\sigma : B_n \to \mathscr{C}_n$ be a mapping from the n-strand Artin braid to the n-strand the Temperley-Lieb Algebra by defining it on generators of the braid group by the formulas*

$$\sigma(b_i) = 1 + qU_i$$
$$\sigma(b_i^{-1}) = 1 + q^{-1}U_i.$$

Then $\sigma : B_n \to \mathscr{C}_n$ is a representation of the Artin braid group.

Proof. It will suffice to show that

$$\sigma(b_i)\sigma(b_i^{-1}) = 1$$
$$\sigma(b_i)\sigma(b_{i+1})\sigma(b_i) = \sigma(b_{i+1})\sigma(b_i)\sigma(b_{i+1})$$
$$\sigma(b_i)\sigma(b_j) = \sigma(b_j)\sigma(b_i) \quad \text{for } |i-j| > 1,$$

since these equations are the image of the relations in B_n under σ. Considering the second relation of (3.1), we reach

$$\begin{aligned}\sigma(b_i)\sigma(b_i^{-1}) &= (1+qU_i)(1+q^{-1}U_i)\\ &= 1+(q^{-1}+q)U_i + U_i^2\\ &= 1+(q^{-1}+q)U_i + rU_i\\ &= 1+(q^{-1}+q)U_i - (q^{-1}+q)U_i = 1\end{aligned}$$

and the first and second relations of (3.1)

$$\begin{aligned}\sigma(b_i)\sigma(b_{i+1})\sigma(b_i) &= (1+qU_i)(1+q^{-1}U_{i+1})(1+qU_i)\\ &= 1+2qU_i+qU_{i+1}+q^2U_i^2+q^2(U_{i+1}U_i+U_iU_{i+1})+q^3U_iU_{i+1}U_i\\ &= 1+q(U_i+U_{i+1})+q^2(U_{i+1}U_i+U_iU_{i+1}).\end{aligned}$$

Since this last expression is invariant under the interchange of i and $i+1$, we conclude that

$$\sigma(b_i)\sigma(b_{i+1})\sigma(b_i) = \sigma(b_{i+1})\sigma(b_i)\sigma(b_{i+1}).$$

Finally considering the third relation of (3.1)

$$\begin{aligned}\sigma(b_i)\sigma(b_j) &= (1+qU_i)(1+qU_j)\\ &= (1+qU_j)(1+qU_i)\\ &= \sigma(b_j)\sigma(b_i).\end{aligned}$$

This completes the proof of theorem. □

4. Abstract Tensor Diagrams and the Polynomial G_L

A abstract (diagrammatic) tensor is a diagrammatic version of matrix algebra where the matrices have many indices. Figure 6 contains some diagrammatic notations of tensor algebra. Here (**a**), (**b**), (**c**), (**d**) and (**e**) denote a matrix $M = (M^a_b)$ with entries M^a_b for a and b in an index set, a tensor-like object has some upper and lower indices, a Kronecker delta, the trace of a matrix and a simple labeling oriented scheme with inputs $\{a,b\}$ and outputs $\{c,d\}$, respectively. (**f**) and (**g**) denote two examples of matrix multiplication. The diagram in (**h**) is a crossover occurring from the matrix multiplication in (**g**). The crossed lines are independent Kronecker delta. See [14] for details.

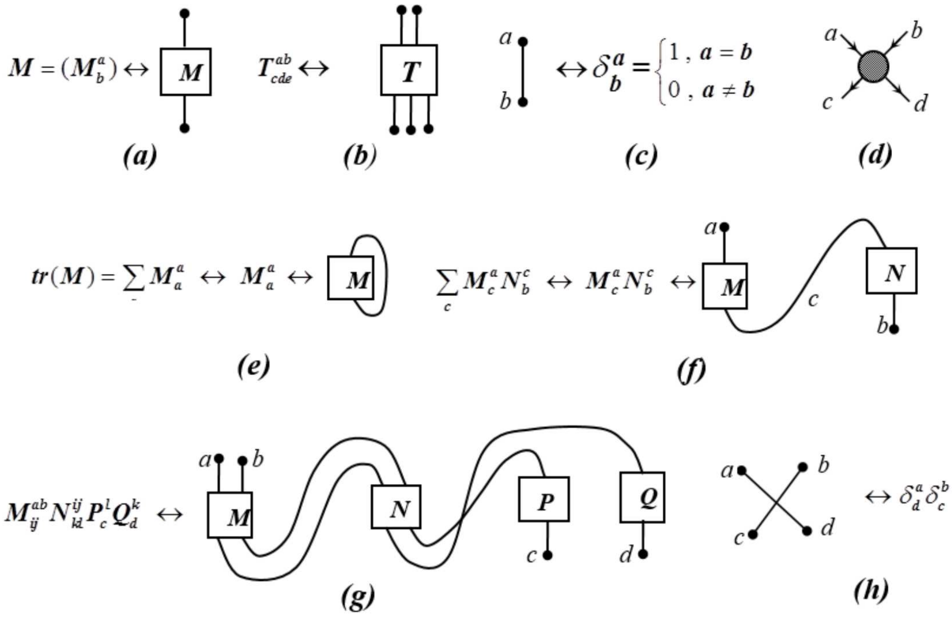


Figure 6. Some diagrammatic representation of tensor algebra.

The simplest way to interpret a diagram of a link as a abstract tensor diagram is to use an oriented diagram and to associate two matrices to the types of crossing. Thus, by convention an oriented link diagram L is mapped to specific contracted abstract tensor $T(L)$ as in Figure 7. By using the concept of a state s, we rewrite

$$\sum_s [L|s]$$

where s runs over all states of L and $[L|s]$ denotes the product of the vertex weights R^{ab}_{cd} (or $\overline{R}^{ab}_{cd}$) assigned to the crossings of L by the given state.

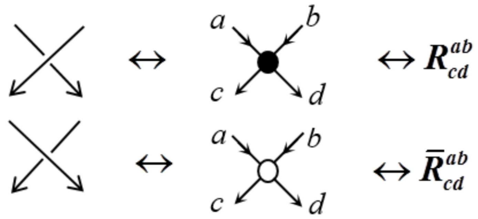


Figure 7. Abstract tensor diagrams of crossing.

To see that the model $T(L)$ is a invariant of regular isotopy for oriented link diagrams, we examine the behavior of its under the Reidemeister moves of types II and III for oriented diagrams. Abstract tensor diagrams of these moves are drawn in Figure 8.

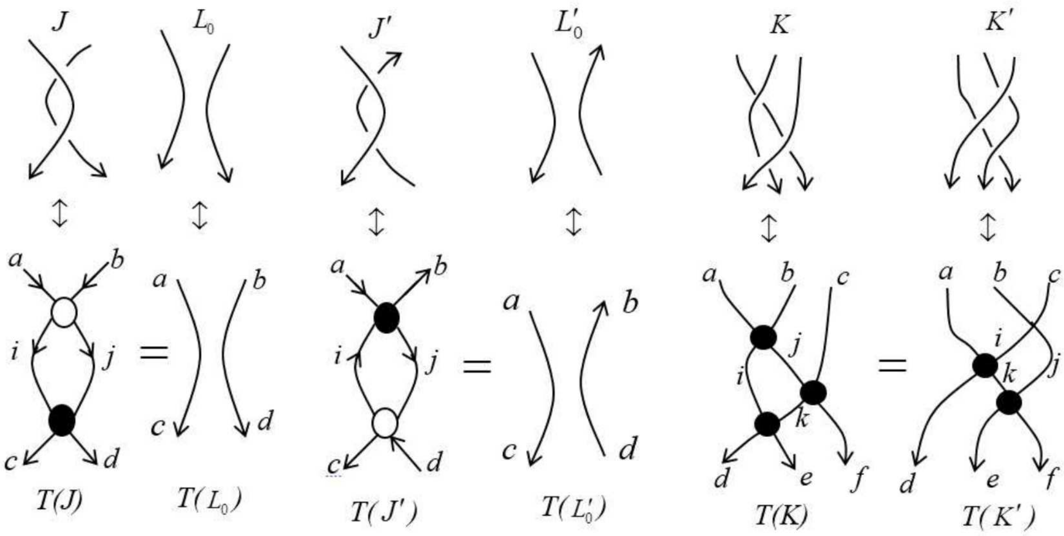


Figure 8. Abstract tensor diagrams of the Reidemeister moves of types II and III: $J \leftrightarrow L_\circ$ (channel unitarity); $J' \leftrightarrow L'_\circ$ (cross- channel unitarity); $K \leftrightarrow K'$ (triangle invariance).

Thus, from channel unitarity, cross-channel unitarity and triangle invariance, we have

$$\overline{R}^{ab}_{ij} R^{ij}_{cd} = \delta^a_c \delta^b_d, R^{ia}_{jb} \overline{R}^{jd}_{ic} = \delta^a_c \delta^d_b \quad \text{and} \quad R^{ab}_{ij} R^{jc}_{kf} R^{ik}_{de} = R^{bc}_{ij} R^{ai}_{dk} R^{kj}_{ef},$$

respectively. The channel unitarity will be satisfied if R and $\overline{R}$ are inverse matrices. If the diagrams K and K' in the triangle invariance have crossing of negative sings, then we have equation:

$$\overline{R}^{ab}_{ij} \overline{R}^{jc}_{kf} \overline{R}^{ik}_{de} = \overline{R}^{bc}_{ij} \overline{R}^{ai}_{dk} \overline{R}^{kj}_{ef}.$$

As seen the triangle invariance, there is a matrix condition that will guarantee invariance of $T(L)$ under the third Reidemeister moves of positive sings and negative sings. These equations are:

$$(\text{III},+): \qquad \sum_{i,j,k \in I} R^{ab}_{ij} R^{jc}_{kf} R^{ik}_{de} = \sum_{i,j,k \in I} R^{bc}_{ij} R^{ai}_{dk} R^{kj}_{ef}$$

$$(\text{III},-): \qquad \sum_{i,j,k \in I} \overline{R}^{ab}_{ij} \overline{R}^{jc}_{kf} \overline{R}^{ik}_{de} = \sum_{i,j,k \in I} \overline{R}^{bc}_{ij} \overline{R}^{ai}_{dk} \overline{R}^{kj}_{ef} \ .$$

The equations $(\mathrm{III},+)$ and $(\mathrm{III},-)$ are known as the Yang-Baxter equations for R and $\overline{R}$. Hence, we have the following theorem.

Theorem 7. *If the matrices R and $\overline{R}$ satisfy the channel unitarity, the cross- channel unitarity and the Yang-Baxter equations, then $T(L)$ is a invariant of regular isotopy for oriented diagrams L.*

Now, we connote the polynomial G_L as abstract tensor model. From Figure 7 we can write the abstract tensor model of the state model of the polynomial G_L as

$$T(L_+) = T(L_\circ) + qT(U)$$
$$T(L_-) = T(L_\circ) + pT(U).$$

In such a model we have matrices $R^{ab}_{cd} = \delta^a_c\delta^b_d + q\delta^{ab}\delta_{cd}$, $\overline{R}^{ab}_{cd} = \delta^a_c\delta^b_d + p\delta^{ab}\delta_{cd}$ and the loop value, $T(\bigcirc) = \delta^a_a = r$. Here

$$\delta^a_b = \begin{cases} 1, & a = b \\ 0, & a \neq b \end{cases} \quad \text{and} \quad \delta^{ab} = \begin{cases} 1, & a = b \\ 0, & a \neq b \end{cases}$$

are Kronecker deltas and δ^a_a is the trace of a Kronecker delta. Thus, if this model is an invariant of regular isotopy, then $G_L = r^{-1}T(L)$ by Definition 2. Thus, we can express the following theorem.

Theorem 8. *The matrices*

$$R^{ab}_{cd} = \delta^a_c\delta^b_d + q\delta^{ab}\delta_{cd}$$
$$\overline{R}^{ab}_{cd} = \delta^a_c\delta^b_d + p\delta^{ab}\delta_{cd}$$

satisfy the conditions of the Theorem 7.

Proof. Firstly, let's check the Yang-Baxter equations. It is sufficient to check the equation $(\mathrm{III},+)$. The equation $(\mathrm{III},-)$ can be controlled in a similar manner.

$$(\mathrm{III},+): \quad \sum_{i,j,k\in I} R^{ab}_{ij}R^{jc}_{kf}R^{ik}_{de} = \sum_{i,j,k\in I} R^{bc}_{ij}R^{ai}_{dk}R^{kj}_{ef}$$

$$\sum_{i,j,k\in I} R^{ab}_{ij}R^{jc}_{kf}R^{ik}_{de} - \sum_{i,j,k\in I} R^{bc}_{ij}R^{ai}_{dk}R^{kj}_{ef} = 0$$

$$\begin{aligned}\sum_{i,j,k\in I} R^{ab}_{ij}R^{jc}_{kf}R^{ik}_{de} - \sum_{i,j,k\in I} R^{bc}_{ij}R^{ai}_{dk}R^{kj}_{ef} &= (\delta^a_i\delta^b_j + q\delta^{ab}\delta_{ij})(\delta^j_k\delta^i_k + q\delta^{jc}\delta_{kf})(\delta^i_d\delta^k_e + q\delta^{ik}\delta_{de}) \\ &\quad -(\delta^b_i\delta^c_j + q\delta^{bc}\delta_{ij})(\delta^a_d\delta^i_k + q\delta^{ai}\delta_{dk})(\delta^k_e\delta^j_f + q\delta^{ik}\delta_{de}) \\ &= (q + rq^2 + q^3)[(\delta^a_d\delta^{bc}\delta_{ef}) - (\delta^{ab}\delta_{de}\delta^c_f)].\end{aligned}$$

The matrix R to satisfy the equation $(\mathrm{III},+)$ it is sufficient that

$$(q + rq^2 + q^3) = q(1 + rq + q^2) = 0.$$

Thus $q = 0$ gives a trivial solution to the equation $(\mathrm{III},+)$. Otherwise, we require that $1 + rq + q^2$ and so (assuming $q \neq 0$) $r = -q - q^{-1}$ for the loop value. As similarly, the matrix $\overline{R}$ to satisfy the equation $(\mathrm{III},-)$, it is necessary that

$$(p + rp^2 + p^3) = p(1 + rp + p^2) = 0.$$

Here $p = 0$ gives a trivial solution to equation (III, −). Also, the equation $1 + rp + p^2 = 0$ with the loop value $r = -p - p^{-1}$ is a solution to equation (III, −).

Cross-channel unitarity:

$$\begin{aligned} R^{ia}_{jb}\overline{R}^{jd}_{ic} &= (\delta^i_j\delta^a_b + q\delta^{ia}\delta_{jb})(\delta^j_i\delta^d_c + p\delta^{jd}\delta_{ic} \\ &= (r+p+q)\delta^a_b\delta^d_c + qp\delta^a_c\delta^d_b. \end{aligned}$$

For $R^{ia}_{jb}\overline{R}^{jd}_{ic} = \delta^a_c\delta^d_b$, cross-channel unitarity requires

$$r + p + q = 0 \quad \text{and} \quad qp = 1. \tag{4.1}$$

Channel unitarity:

$$\begin{aligned} \overline{R}^{ab}_{ij}R^{ij}_{cd} &= (\delta^a_i\delta^b_j + p\delta^{ab}\delta_{ij})(\delta^i_c\delta^j_d + q\delta^{ij}\delta_{cd} \\ &= (pqr + p + q)\delta^{ab}\delta_{cd} + \delta^a_c\delta^b_d. \end{aligned}$$

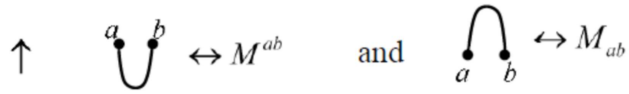
For $\overline{R}^{ab}_{ij}R^{ij}_{cd} = \delta^a_c\delta^b_d$, channel unitarity requires

$$pqr + p + q = 0. \tag{4.2}$$

From (4.1) and (4.2), we have $p = q^{-1}$ and $r = -q - q^{-1}$. Thus, by setting $p = q^{-1}$, $r = -q - q^{-1}$ the matrices R and $\overline{R}$ becomes inverse of each other and the abstract tensor model of the model of the state model of the polynomial G_L (2.1) becomes an invariant of regular isotopy. □

5. Polynomial G_L as Vacuum-vacuum Expectation and Quantum Group $SL(2)_q$

In the previous section, we construct the solutions relevant with state model of the polynomial G_L to the Yang-Baxter equation. Each solution gives the polynomial G_L at special values corresponding to solutions of $r + q + q^{-1} = 0$ for a given positive integer r. We end up Yang-Baxter state models for infinitely many specializations of the state model of the polynomial G_L. In fact there is a way to construct a solution to the Yang-Baxter equation and a corresponding state model for the polynomial G_L. To do this we use the picture of creations, annihilations and interactions of spins from the vacuum. Then the fragments



with the time's arrow, means a creation and a annihilation of spins a and b from the vacuum, respectively. In any case, we allow matrices M_{ab} and M^{ab} corresponding to caps and cups. The matrix values M_{ab} and M^{ab} represent abstract amplitudes for these processes to take place. Together with caps and cups, we have the matrices R corresponding to our knot-theoretic interactions. Thus, a given link diagram L may be represented with respect time's arrow so that it is naturally decomposed into caps, cups and interactions. If the tensors are numerically valued, then the corresponding expression $\tau(L)$ of the link diagram L in the language of creation, annihilation and interaction represents a vacuum-vacuum expectation for the processes indicated by the diagram and this arrow time. The expression $\tau(L)$ can be considered as the basic form of vacuum-vacuum expectation in a highly simplified quantum field theory of link diagrams. We would like this to be a topological quantum field theory in the sense that $\tau(L)$ should be an invariant of regular isotopy of L.

Two link diagrams arranged transversal to a given time direction are regularly isotopic if and only if can be obtained from the other by a sequence of moves of types: topological moves, twist, vertical type II move illustrated in Figure 9 and vertical type III move with all crossings of the same type relative to time's arrow which is corresponding to the Yang- Baxter equation for R and $\overline{R}$. Thus we can express the following theorem.

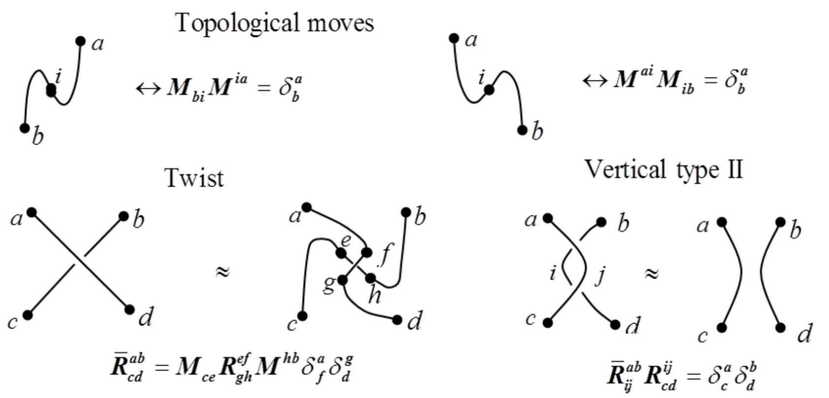


Figure 9. The expression $\tau(L)$ of the Reidemeister moves of types 0, II and the move twist.

Theorem 9. *If the interaction matrix R and its inverse $\overline{R}$ satisfy the Yang-Baxter relation, the interrelation with M_{ab} and M^{ab} are also specified by the twist and if M_{ab} and M^{ab} are inverse matrices then $\tau(L)$ will be an invariant of regular isotopy.*

If B is a braid and $L = \overline{B}$, then each braid strand contributes a matrix of the form $\eta^b_a = M_{ai}M^{bi}$ because each closure strand of L has one maximum and one minimum. If $\sigma(B)$ denotes the interaction tensor coming from the braid, then we have that $\tau(L) = Trace(\eta^{\otimes n}\sigma(B))$ where there are n braid strands, and $\sigma(B)$ is regarded as living in a tensor product of matrices as in previous section (see [14] for details). Now let's obtain the polynomial G_L and its R-matrix. If we wish $\tau(L)$ to satisfy the state model of the polynomial G_L (2.1) as

$$\tau(L_+) = \tau(L_\circ) + q\tau(U), \qquad \tau(L_-) = \tau(L_\circ) + q^{-1}\tau(U)$$

with $p = q^{-1}$, then choices for the matrices R and $\overline{R}$ are as in Figure 10. Hence

$$R^{ab}_{cd} = \delta^a_c\delta^b_d + qM^{ab}M_{cd}, \; \overline{R}^{ab}_{cd} = \delta^a_c\delta^b_d + q^{-1}M^{ab}M_{cd}.$$

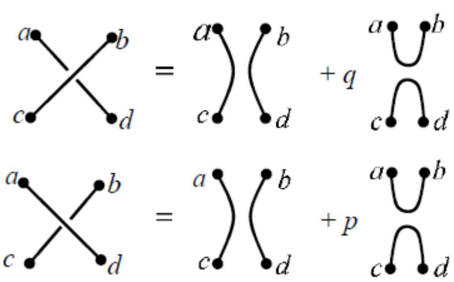
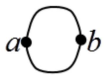


Figure 10. The expression $\tau(L)$ of the state model of the polynomial G_L.

For the moment, we suppose that the matrices M give the correct loop value. That is, we suppose that



$$= \sum_{a,b} M_{ab} M^{ab} = r = -q - q^{-1}.$$

Remark 10. Letting $U = M^{ab} M_{cd}$, we have $U^2 = rU$ and $R = I + qU$, $\overline{R} = I + q^{-1}U$. Then the proof that the matrices R and $\overline{R}$ satisfy the Yang-Baxter equation is identical to the proof given for the corresponding braiding relations in Theorem 6.

Thus, to have a model for the polynomial G_L we need a matrix pair M^{ab} and M_{ab} of inverse matrices whose loop value, $r = -q - q^{-1}$.

Now let's get matrices

$$M^{ab} = \begin{bmatrix} 0 & iq \\ -i & 0 \end{bmatrix}, \quad M_{ab} = \begin{bmatrix} 0 & i \\ -iq^{-1} & 0 \end{bmatrix}, \quad i = \sqrt{-1}.$$

These choices of the creation matrix M^{ab} and the annihilation matrix M_{ab} give us a tensor model for the polynomial G_L. It can be easily seen that all conditions of Theorem 9 are met. Indeed, in the language of matrix we find that

$$M^{ai} M_{ib} = \begin{bmatrix} 0 & iq \\ -i & 0 \end{bmatrix} \begin{bmatrix} 0 & i \\ -iq^{-1} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and

$$M_{ib} M^{ai} = \begin{bmatrix} 0 & i \\ -iq^{-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & iq \\ -i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

$$M^{ai} M_{ib} = I = M_{ib} M^{ai} \quad \text{(topological move is satisfied)},$$

$$U = M^{ab} \otimes M_{ab} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & 1 & 0 \\ 0 & 1 & -q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$r = Trace(U) = \sum_{a,b} M^{ab} M_{ab} = -q - q^{-1}, \quad R = I + qU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-q^2 & q & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\overline{R} = I + q^{-1}U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & q^{-1} & 1-q^{-2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad R\overline{R} = \overline{R}R = I.$$

If we get

$$M_{ab} = M^{ab} = M = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad i = \sqrt{-1},$$

then $M^2 = I$ and $r = -1 - 1 = -2$. This is a special model for the polynomial G_L with $q = \mp 1$.

The special case $M = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ is the same the special case $A = \pm 1$ of the chosen matrix $\begin{bmatrix} 0 & iA \\ -iA^{-1} & 0 \end{bmatrix}$ for the bracket belongs to Kauffman. In quantum theory, there are much works about this special case by Kauffman [14]. Let's write this matrix as

$$M = i\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = i\varepsilon$$

where ε_{ab} denotes the alternating symbol:

$$\varepsilon_{ab} = \begin{cases} 1 & \text{if } a < b \\ -1 & \text{if } a > b\,, \\ 0 & \text{if } a = b \end{cases} \quad a, b \in I = \{1, 2\},\ i = \sqrt{-1}.$$

Lemma 11. *Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix of commuting scalars. Then*

$$P\varepsilon P^T = \det(P)\varepsilon$$

where P^T denotes the transpose of P.

Proof.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = (ad - bc)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$ □

Thus we have the following definition.

Definition 12. Let $\Re$ a commutative, associative ring with unit. Then

$$SL(2) = SL(2, \Re)$$

is defined to be the set 2×2 matrices P with entries in $\Re$ such that $P\varepsilon P^T = \varepsilon$.

In the discussion surrounding Lemma 11, Kauffman saw solutions to the Yang-Baxter equation and models for the Jones polynomial emerge from the deformed epsilon,

$$\widetilde{\varepsilon} = \begin{bmatrix} 0 & A \\ -A^{-1} & 0 \end{bmatrix}.$$

Since ε is the fundamental defining invariant for $SL(2)$ by Lemma 11, he built an algebraic structure which leaves $\widetilde{\varepsilon}$ basic invariant. The relations of this algebra are given in the Proposition 9.5 in [14]. This algebra is denoted by $SL(2)_q$ with $q = A^2$. It is called "the quantum group $SL(2)_q$" in the literature [6].

Since we have

$$M^{ab} = \begin{bmatrix} 0 & iq \\ -i & 0 \end{bmatrix} = i\begin{bmatrix} 0 & q \\ -1 & 0 \end{bmatrix} = i\widetilde{\varepsilon}^{ab},$$

and

$$M_{ab} = \begin{bmatrix} 0 & i \\ -iq^{-1} & 0 \end{bmatrix} = i\begin{bmatrix} 0 & 1 \\ -q^{-1} & 0 \end{bmatrix} = i\widetilde{\varepsilon}_{ab},$$

the algebra defined by relations in the Proposition 9.5 in [14] will leave invariant the matrices $\widetilde{\varepsilon}^{ab}$ and $\widetilde{\varepsilon}_{ab}$ without $q = A^2$. Thus we have the following therem.

Theorem 13. *Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix that the entries of P belong to an associative but not necessarily commutative algebra. Then, the equations*

$$\left.\begin{array}{l} P\widetilde{\varepsilon}^{ab}P^T = \widetilde{\varepsilon}^{ab} \\ P^T\widetilde{\varepsilon}^{ab}P = \widetilde{\varepsilon}^{ab} \end{array}\right\} \quad \text{and} \quad \left.\begin{array}{l} P\widetilde{\varepsilon}_{ab}P^T = \widetilde{\varepsilon}_{ab} \\ P^T\widetilde{\varepsilon}_{ab}P = \widetilde{\varepsilon}_{ab} \end{array}\right\}$$

are equivalent to the set of relations (5.1) *given bellow:*

$$\left\{\begin{array}{c} ba = qab,\ ca = qac,\ dc = qcd,\ db = qbd,\ bc = cb, \\ ad - bc = (q^{-1}-q)bc,\ ad - q^{-1}bc = 1 \end{array}\right\}. \tag{5.1}$$

Proof.

$$\begin{aligned} P\widetilde{\varepsilon}^{ab}P^T &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & q \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} \\ &= \begin{bmatrix} -ba+qab & -bc+qad \\ -da+qcb & -dc+qcd \end{bmatrix} \end{aligned}$$

and

$$P^T\widetilde{\varepsilon}^{ab}P = \begin{bmatrix} -ca+qac & -bc+qad \\ -da+qbc & -db+qbd \end{bmatrix}.$$

As similarly,

$$\begin{aligned} P\widetilde{\varepsilon}_{ab}P^T &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -q^{-1} & 0 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} \\ &= \begin{bmatrix} -q^{-1}ba+ab & -q^{-1}bc+ad \\ -q^{-1}da+cb & -q^{-1}dc+cd \end{bmatrix} \end{aligned}$$

and

$$P^T\widetilde{\varepsilon}_{ab}P = \begin{bmatrix} -q^{-1}ca+ac & -q^{-1}bc+ad \\ -q^{-1}da+bc & -q^{-1}db+bd \end{bmatrix}.$$

Therefore, we obtain directly from the equations $P\widetilde{\varepsilon}^{ab}P^T = \widetilde{\varepsilon}^{ab}$ and $P^T\widetilde{\varepsilon}^{ab}P = \widetilde{\varepsilon}^{ab}$ that the relations are as follows:

$$\begin{array}{cccc} ba = qab & dc = qcd & ca = qac & db = abd \\ -bc+qad = q & -cb+qad = q & -da+qcb = -1 & -da+qbc = -1. \end{array}$$

It follows that $bc = cb$, and that an equivalent set of relations is given by:

$$\left\{\begin{array}{c} ba = qab,\ ca = qac,\ dc = qcd,\ db = qbd,\ bc = cb, \\ ad - bc = (q^{-1}-q)bc,\ ad - q^{-1}bc = 1 \end{array}\right\}.$$

Note that the last relation, $ad - q^{-1}bc = 1$, becomes the condition $ad - da = 1 = \det(P)$ when $q = 1$. Also, these relations tell us that the elements of the matrix P commute among themselves when $q = 1$. As similarly, the set of relations (5.1) can also be obtained from the equations $P\widetilde{\varepsilon}_{ab}P^T = \widetilde{\varepsilon}_{ab}$ and $P^T\widetilde{\varepsilon}_{ab}P = \widetilde{\varepsilon}_{ab}$. Thus the proof is complete. □

Remark 14. The relations of the Theorem 13 are the same the relations given in the Proposition 9.5 of [14] with $q = A^2$. As seen in the proof of the Theorem 13, q is not necessarily equal to A^2 for keeping invariant the matrices $\widetilde{\varepsilon}^{ab}$ and $\widetilde{\varepsilon}_{ab}$. Thus the studies performed using the Kauffman bracket in the quantum group $SL(2)_q$ are valid for the state model of the polynomial G_L without $q = A^2$.

6. Conclusion

Here, we define the polynomial G_L for braids and construct abstract tensor model of it. We also associate the polynomial G_L with the concepts of quantum theory such as Yang-Baxter equation, vacuum-vacuum expectation model and quantum group $SL(2)_q$. The polynomial G_L can be applied other concepts of quantum theory which are associated with Kauffman bracket polynomial.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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