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Research Article

Fixed Point Theorems for Multiplicative Cyclic (α,β)-Convex Contraction of Type-2 in Dislocated Quasi b-Multiplicative Metric Space and Its Applications

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Abstract. The dislocated quasi b-multiplicative metric space is introduced as an extension of dislocated quasi b-metric space, which is demonstrated some well-known fixed point theorems and shows fixed point results for a contraction of rational type. Specifically, we generalize the convex contraction to encompass the dislocated quasi b-multiplicative setting. The results of this study not only contribute to the theoretical foundations of fixed point theory but also offers a new perspective on the applicability of dislocated quasi b-multiplicative metric spaces.

Keywords. Dislocated quasi b-multiplicative metric space (dqb-mms), Cauchy sequence, Multiplicative cyclic (α,β)-convex contraction of Type-2, Unique fixed point (UFP), Multiplicative metric space (mms)

Mathematics Subject Classification (2020). Primary 47H10; Secondary 54H25

1. Introduction and Preliminaries

Bashirov *et al.* [1] proposed multiplicative calculus, Özavşar and Çevikel [8] initiated the idea of multiplicative metric spaces (multiplicative distance). The idea of dislocated metric spaces was begin by Hitzler [5] in 2001. In that the point self-distance need not be zero, he also commenced Banach's principle of contraction in these areas. Dislocated quasi metric spaces were first mentioned by Zeyada *et al.* [10], who also expanded on Hitzler's result. Dislocated quasi b-metric space was first discussed by Rahman and Sarwar [9], and Istrăţescu [3] introduced a category of convex contraction mappings in metric spaces, thereby extending the widely recognized Banach contraction notion. Nallaselli *et al.* [6] introduced a novel generalized convex contraction concept for b-metric spaces and 2-metric spaces. Kir and Kizitune [4] established fixed point theorems for Kannan and Chatterjea type contractive mappings in b-metric spaces.

Definition 1.1 ([7])**.** Consider a non-empty set P. A mapping $\omega : P \times P \to R^+$ is termed a *multiplicative metric* if, for all $u_1, v_1, w_1 \in P$:

(i) $\omega(u_1,v_1) > 1$ and $\omega(u_1,v_1) = 1$ if and only if $u_1 = v_1$,

(ii) $\omega(u_1,v_1) = \omega(v_1,u_1)$,

(iii) $\omega(u_1,w_1) \leq \omega(u_1,v_1)\omega(v_1,w_1)$ (*multiplicative triangle inequality*).

Definition 1.2 ([8])**.** Let (P,ω) be two *multiplicative metric spaces*. A mapping $T:P\to P$ is called *multiplicative contraction* if there exists a real constant $\varpi \in [0,1)$ such that $\omega(T(u_1),T(u_2)) \leq \omega(u_1,u_2)^{\varpi}$, for all $u_1, v_1 \in P$.

Theorem 1.1 ([8])**.** *Let (P,ω) be a mms. A $T:P\to P$ be a multiplicative contraction. If (P,ω) is complete, then T has UFP.*

Theorem 1.2 ([8])**.** *Let (P,ω) be a complete mms. Suppose the mapping $T:P\to P$ satisfies the following condition:*

$$\omega(Tu_1,Tv_1) \leq (\omega(Tu_1,u_1)\cdot\omega(Tv_1),v_1))^{\varpi}, \quad \text{for all } u_1,v_1 \in P,\ \varpi \in \left[0,\tfrac{1}{2}\right).$$

Then T has UFP in P. For any $u_1 \in P$, iterative sequence $T^n\{u_1\}$ converges to the fixed point.

Definition 1.3 ([9])**.** Let P be a non-empty set and $k \geq 1$ be a real number. A mapping $\omega : P\times P \to [0,\infty)$ is called *dislocated quasi b-metric* if, for all $u_1,v_1,w_1 \in P$:

(i) $\omega(u_1,v_1) = \omega(v_1,u_1) = 0$ implies $u_1 = v_1$,

(ii) $\omega(u_1,w_1) \leq k[\omega(u_1,v_1)+\omega(v_1,w_1)]$.

The pair (P,ω) is called a *dislocated quasi b*-metric or shortly (*dqb-metric*) space.

2. Main Results

Definition 2.1. A mapping $\omega : P\times P \to [1,\infty)$ is termed a *dislocated quasi b-multiplicative metric* if, for all $u_1,v_1,w_1 \in P$:

(i) $\omega(u_1,v_1) = \omega(v_1,u_1) = 1$ implies $u_1 = v_1$,

(ii) $\omega(u_1,w_1) \leq [\omega(u_1,v_1)\omega(v_1,w_1)]^k$.

The pair (P,ω) is referred to as a *dislocated quasi b-multiplicative metric space*, or simply a (*dqb-mms*).

Remark 2.1. For $k=1$ then it becomes *dislocated quasi multiplicative metric space*.

Example 2.1. Consider $P=[0,1]$ and $\omega : P\times P \to [1,\infty)$ be defined as $\omega(\vartheta,\iota)=a^{(\vartheta-\iota)^3}$, where $\vartheta,\iota \in R$ and $a>1$ with $k=2$. Then $\omega(\vartheta,\iota)$ is *dqb-mms*.

Definition 2.2. In a *dislocated quasi b-multiplicative metric space* (P,ω), where $u_1, v_1 \in P$ and $\epsilon > 1$, the set $B_\epsilon(u_1)$, defined as $\{v_1 \in P \mid \omega(u_1,v_1)\} \preceq \epsilon$, is referred to as the *dqb*-multiplicative open ball with radius ϵ centered at u_1. Similarly, the *dqb*-multiplicative closed ball, denoted as $\bar{B}_\epsilon(u_1)$, is defined as $\{v_1 \in P \mid \omega(u_1,v_1) \preceq \epsilon\}$.

Definition 2.3. Consider two *dqb-multiplicative metric spaces* (P,ω_1) and (Υ,ω_2), along with a function $T : P \to \Upsilon$. If, for every $\epsilon > 1$, there exists $\delta > 1$ such that $T(B_\delta(u_1)) \subset B_\epsilon(T(u_1))$, where T maps the ball of radius δ centered at u_1 in P to a ball of radius ϵ centered at $T(u_1)$ in Υ, then T is *multiplicative continuous* at $u_1 \in P$.

Definition 2.4. Consider a *dislocated quasi b-multiplicative metric space* denoted by (P,ω). In this context:

(i) A point $u_1 \in P$ is considered a *multiplicative limit point* of a subset $Z \subset P$ if and only if the intersection of the set $(B_\epsilon(u_1)/(u_1))$ and Z is non-empty for every $\epsilon > 1$.

(ii) A subset $Z \subset P$ is deemed *multiplicative closed* within the space (P,ω) if it contains all of its *multiplicative limit points*.

(iii) A set Z is labeled as *multiplicative bounded* if there exists a point $u_1 \in P$ and a constant $M > 1$ such that Z is entirely contained within the ball $B_M(u_1)$.

Definition 2.5. A sequence $\{u_n\}$ is called *dq-converges* to u_1 iff

$$\lim_{n\to\infty} \omega(u_n,u_1) = \lim_{n\to\infty} \omega(u_1,u_n) = 1.$$

In this case, u_1 is called *dq-limit* of u_n.

Definition 2.6. A sequence $\{u_n\}$ in *dqb-mms* (P,ω) is called *cauchy if for each* $\epsilon > 1$, there exists $n_0 \in N$ such that $n,m \geq n_0$, $\omega(u_n,u_m) \prec \epsilon$ and $\omega(u_m,u_n) \prec \epsilon$.

Definition 2.7. A *dqb-mms* (P,ω) is called *complete if every cauchy sequence* in it is *dq b-multiplicative convergent* in P.

Definition 2.8. In a *dislocated quasi b-multiplicative metric space* (P,ω) with a designated point $Z \in P$, we define $u_1 \in A$ as a *multiplicative interior point* of Z *if there exists* an $\epsilon > 1$ such that the ball $B_\epsilon(u_1)$ is entirely contained within Z. The set comprising all such interior points of Z is referred to as the *multiplicative interior* of Z, denoted by $Int(Z)$.

Definition 2.9. Let (P,ω) be a *dqb-mms* and $Z \subset P$. If every point of Z is a *multiplicative interior point* of Z, this mean, $Z = Int(Z)$, then Z is called a *multiplicative open set*.

Lemma 2.1. *Let (P,ω) be a dqb-mms, every subsequence of any convergent sequence is convergent.*

Lemma 2.2. *Let (P,ω) be a dqb-mms. If a sequence $\{u_n\}$ is dqb-mms convergent, then the dq-limit point is unique.*

Theorem 2.1. *Let (P,ω) be a complete dq-bmms and a continuous function $T:P\to P$ satisfies:*

$$\omega(Tu_1,Tv_1)\preceq\omega(u_1,v_1)^{\varpi},$$

where $u_1,v_1\in P$, $\varpi\in[0,1/k]$ and $0\preceq k\varpi\prec 1$. Then, T has a UFP.

Proof. Initiate $\{u_n\}\subset P$, choose $u_0\in P$ and inductively construct the sequence $\{u_n\}$ of points of T,

$$u_1=Tu_0,\quad u_2=T^2u_0,\quad u_3=T^3u_0.$$

Likewise, $u_n=Tu_{n-1}=T^nu_0$.

Clearly, $\{u_n\}$ is images of u_0 under repeated of application of T,

$$\begin{aligned}\omega(u_n,u_{n+m})&\preceq\omega(u_n,u_{n+1})^{k^n}\omega(u_{n+1},u_{n+2})^{k^{n+1}}\dots\ \omega(u_{n+m-1},u_{n+m})^{k^{n+m-1}}\\&=\omega(T^nu_0,T^nu_1)^{k^n}\omega(T^{n+1}u_0,T^{n+1}u_1)^{k^{n+1}}\dots\ \omega(T^{n+m-1}u_0,T^{n+m-1}u_1)^{k^{n+m-1}}\\&\preceq\omega(u_0,u_1)^{\varpi k^n/1-\varpi k}.\end{aligned}$$

As $n\to\infty$ and $\varpi\prec 1/k$ which implies $k\varpi\prec 1$, then $\{u_n\}$ is a *multiplicative cauchy sequence*.

Since (P,ω) is complete then $\{u_n\}$ is convergent such that $\lim\limits_{n\to\infty}u_n=u$.

To demonstrate u is a *FP* of T,

$$\begin{aligned}\omega(u,Tu)&\preceq(\omega(u,u_n)\omega(u_n,Tu))^k\\&\preceq(\omega(u,u_n)\omega(u_{n-1},u_n)^{\varpi})^k\\&\to 0\ \text{ as }\ n\to\infty\\&=0,\\u&=Tu.\end{aligned}$$

Therefore, u is a *FP* of T.

Let, if possible, u and v be two fixed points of T in P. Then, $Tu=u$, $Tv=v$.

Now,

$$\begin{aligned}&\omega(u,v)=\omega(Tu,Tv)\preceq\omega(u,v)^{\varpi}\\\Longrightarrow\quad&\omega(u,v)=1\\\Longrightarrow\quad&u=v\end{aligned}$$

Then T has a *UFP*. □

Theorem 2.2. *Let (P,ω) be a complete dqb-mms with $k\geq 1$. Let $T:P\to P$ be a mapping for which there exists $\varpi\in[0,1/2)$ such that*

$$\omega(Tu,Tv)\preceq[\omega(u,Tu)\omega(v,Tv)]^{\varpi}, \tag{2.1}$$

for all $u,v\in P$. Then there exists $u\in P$ such that $u_n\to u$ and u is UFP of T.

Proof. Initiate $\{u_n\}\subset P$. Fix $u_0\in P$, inductively construct $\{u_n\}$ of points of P,

$$u_1=Tu_0,\ \ u_2=Tu_1=T^2u_0,\ \ u_3=Tu_2=T^3u_0.$$

Similarly, $u_n = Tu_{n-1} = T^n u_0$,

$$\begin{aligned}\omega(u_n,u_{n+1}) &= \omega(Tu_{n-1},Tu_n)\\ &\preceq [\omega(u_{n-1},u_n)\omega(u_n,u_{n+1})]^{\varpi}\\ &\preceq \omega(u_{n-1},u_n)^{\varpi/1-\varpi}.\end{aligned}$$

Likewise,

$$\omega(u_n,u_{n+1}) \preceq \omega(u,u_1)^{(\varpi/1-\varpi)^n}.$$

Note that $\varpi \in [0,1/2)$ then $\varpi/1-\varpi \in [0,1)$. Thus T is a contraction mapping. We deduce, in similar manner to that in the proof of Theorem 2.1 that $\{u_n\}_{n=1}^{\infty}$ is a *cauchy sequence* and hence, a *convergent sequence*, too. We consider that $\{u_n\}_{n=1}^{\infty}$ *convergent* to $u \in P$. The result as,

$$\begin{aligned}\omega(u,Tu) &\preceq [\omega(u,u_n)\omega(u_n,Tu)]^k\\ &\preceq \omega(u,u_n)^k[\omega(u_{n-1},u_n)\omega(u_n,Tu]^{k\varpi},\end{aligned}$$

and we arrive at

$$\omega(u,Tu) \preceq \omega(u,u_n)^{(k/1-k\varpi)}\omega(u_n,Tu)^{(k\varpi/1-k\varpi)}. \tag{2.2}$$

Use the equation (2.2),

$$\omega(u,Tu) \preceq \omega(u,u_n)^{(k/1-k\varpi)}\omega(u_0,u_1)^{(k\varpi/1-kw)(\varpi/1-\varpi)^n}. \tag{2.3}$$

Let $n \to \infty$ in equation (2.3),

$$\lim_{n\to\infty} \omega(u,Tu) = 1$$

Therefore, $u = Tu$ and implies that u is a *FP* of T. □

Lemma 2.3. *Let (P,ω) be a dqb-mms with coefficient $k \geq 1$ and $T : P \to P$ be a mapping. Suppose that $\{u_n\}$ is a sequence in T induced by $u_{n+1} = Tu_n$ such that*

$$\omega(u_n,u_{n+1}) \preceq \omega(u_{n-1},u_n)^{\varpi},$$

for all $n \in N$, where $\varpi \in [0,1)$ is a constant. Then $\{u_n\}$ is a dqb-multiplicative cauchy sequence.

Theorem 2.3. *Consider a dqb-mms (P,ω) with a coefficient $k \geq 1$, and let $T : P \to P$ be a mapping on P. Assume that $\varpi_1,\varpi_2,\varpi_3$ are nonnegative real numbers satisfying $\varpi_1+\varpi_3 < 1$ and $\frac{\varpi_1+\varpi_2}{k-\varpi_3} < 1$. In this context, we have the following inequality:*

$$\omega(Tu,Tv)^k \preceq \omega(u,v)^{\varpi_1}\left[\frac{\omega(u,Tv)\omega(v,Tv)}{1+\omega(Tu,Tv)}\right]^{\varpi_2}\omega(Tu,Tv)^{\varpi_3} \tag{2.4}$$

holds for each $u,v \in P$. Then T has a UFP.

Proof. Let u_0 be arbitrary in P. We define $\{u_n\}$ in P such that

$$u_{n+1} = Tu_n, \quad \text{for all } n \in N. \tag{2.5}$$

Utilizing (2.4) with $u = u_n$ and $v = u_{n-1}$,

$$\begin{aligned}\omega(u_{n+1},u_n)^k &= \omega(Tu_n,Tu_{n-1})^k\\ &\preceq \omega(u_n,u_{n-1})^{\varpi_1}\left(\frac{\omega(u_n,Tu_n)\omega(u_{n-1},Tu_{n-1})}{1+\omega(Tu_n,Tu_{n-1})}\right)^{\varpi_2}\omega(Tu_n,Tu_{n-1})^{\varpi_3}\\ &= \omega(u_n,u_{n-1})^{\varpi_1}\left(\frac{\omega(u_n,u_{n+1})\omega(u_{n-1},u_n)}{1+\omega(u_{n+1},u_n)}\right)^{\varpi^2}\omega(u_{n+1},u_n)^{\varpi_3}\end{aligned}$$

$$\preceq \omega(u_n,u_{n-1})^{\varpi_1}\left(\frac{\omega(u_n,u_{n+1})\omega(u_{n-1},u_n)}{1+\omega(u_{n+1},u_n)}\right)^{\varpi_2}\omega(u_{n+1},u_n)^{\varpi_3}$$
$$\preceq \omega(u_{n-1},u_n)^{\frac{\varpi_1+\varpi_2}{k-\varpi_3}}.$$

Use Lemma 2.3, say $\{u_n\}$ is a *dqb-multiplicative cauchy sequence* in (P,ω). Since (P,ω) is a *complete-dqb-mms*, then $\{u_n\}$ converges to some $u\in P$ as $n\to\infty$.

We show that $Tu=u$.

By dislocated quasi b-multiplicative triangle inequality and (2.4),

$$\begin{aligned}\omega(u,Tu)&\preceq[\omega(u,u_{n+1})\omega(u_{n+1},Tu)]^k\\&=\omega(u,u_{n+1})^k\omega(Tz,Tu_n)^k\\&\preceq\omega(u,u_{n+1})^k\omega(u,u_n)^{k\varpi_1}\left(\frac{\omega(u_n,Tu_n)\omega(u,Tu)}{1+\omega(Tu_n,Tu)}\right)^{k\varpi_2}\omega(Tu,Tu_n)^{k\varpi_3}.\end{aligned}$$

Taking the limit as $n\to\infty$, $\omega(u,Tu)^{1-\varpi_3}\preceq 1$.

Since $0<\varpi_3<1$, then $\omega(u,Tu)\preceq 1$, which is a contradiction, so $\omega(u,Tu)=1$. Hence $Tu=u$, thus u is a *FP* of T. Now we say T have a *UFP*. Suppose u and v are different fixed point of T, then from (2.4) that

$$\begin{aligned}\omega(u,v)^k&=\omega(Tu,Tv)^k\\&\preceq\omega(u,v)^{\varpi_1}\left(\frac{\omega(u,Tu)\omega(v,Tv)}{1+\omega(Tu,Tv)}\right)^{\varpi_2}\omega(Tu,Tv)^{\varpi_3}\\&=\omega(u,v)^{\varpi_1}\left(\frac{\omega(u,u)\omega(v,v)}{1+\omega(Tu,Tv)}\right)^{\varpi_2}\omega(u,v)^{\varpi_3}\\&=\omega(u,v)^{\varpi_1+\varpi_3}.\end{aligned}$$

Since $\varpi_1+\varpi_3$ is non negative reals with $\varpi_1+\varpi_3<1$, then we have $\omega(u,v)=1$.

Thus, T have a *UFP* in P. □

Definition 2.10. In a metric space (P,ω), let T be a self-mapping and consider $\epsilon>0$. We define $u_0\in P$ as an ϵ-fixed point of T if $\omega(u_0,Tu_0)<\epsilon$. This is denoted by $\Upsilon_{\epsilon>0}(T)=\{u\in P\mid\omega(Tu,u)<\epsilon\}$, and the set of all fixed points of T is denoted by $Fix(T)$. We say that T adheres to the *approximate fixed point theory* (AFPP) if, for every $\epsilon>0$, there exists an ϵ-fixed point of T, i.e., $\Upsilon_\epsilon(T)\neq\emptyset$.

Definition 2.11. A self-mapping $T:P\to P$ on a non-empty set P is considered α-admissible if, for any $u,v\in P$ such that $\alpha(u,v)\geq 1$, it follows that $\alpha(Tu,Tv)\geq 1$.

Definition 2.12. Let P possess property (H) if, for every pair u,v in the fixed points of T, there exists an element u in P such that $\alpha(v,u)\geq 1$ and $\alpha(w,u)\geq 1$.

Lemma 2.4. *A self mapping T is asymptotically regular on a dq b-mms (P,ω) at a point $u_0\in P$, i.e., $\omega(T^nu_0,T^{n+1}u_0)\to 1$ then T has the AFPP.*

Definition 2.13. Let (P,ω) be a dislocated quasi b-multiplicative metric space with $k\geq 1$. We say that a self mapping T on P is a multiplicative cyclic (α,β)-convex contraction of Type-2 if there exists a mapping $\alpha,\beta:X\to[1,\infty]$ and $\varpi_i\geq 0$, for all $i=\{1,2,3,\ldots,8\}$ with

$\varpi_1+\varpi_2+\varpi_3+\varpi_4+\varpi_5+\varpi_6+2\varpi_7+2\varpi_8<\frac{1}{h^2}$ such that

$$\alpha(u)\beta(v)\omega(T^2u,T^2v)\le\omega(u,v)^{\varpi_1}\omega(Tu,Tv)^{\varpi_2}\omega(u,Tu)^{\varpi_3}\omega(Tu,T^2u)^{\varpi_4}\omega(v,Tv)^{\varpi_5}\omega(Tv,T^2v)^{\varpi_6}\cdot\left[\frac{\omega(u,Tv)\omega(v,Tu)}{2}\right]^{\varpi_7}\left[\frac{\omega(Tu,T^2v)\omega(Tv,T^2u)}{2}\right]^{\varpi_8},$$

for all $u,v\in P$.

Theorem 2.4. *Consider a (P,ω), which is a dqb-mms with $k\ge1$. Let $T:P\to P$ denote a cyclic (α,β)-convex contraction of Type-2. We assume that T is a multiplicative cyclic (α,β) admissible map, and that there exists an element $u_0\in P$ satisfying $\alpha(u_0)\ge1$ and $\beta(u_0)\ge1$. Under these conditions, T possesses the approximate fixed point property. If T is continuous, it implies that there exists a point that remains unchanged under the action of T. If, in addition, for every pair of elements u and v belonging to the set of fixed points of T, it holds that $\alpha(u)\ge1$ and $\beta(v)\ge1$,then T possesses a unique fixed point within the domain P.*

Proof. Given an initial point $u_0\in P$ satisfying $\alpha(u_0)\ge1$ and $\beta(u_0)\ge1$, we define the sequence $\{u_n\}$ in P as $u_{n+1}=T^{n+1}u_0$, for all $n\in Z^+\cup\{0\}$. If there exists an $n\in Z^+\cup\{0\}$ such that $u_n=u_{n+1}$, then u_n is a fixed point of T. Assuming $u_n\ne u_{n+1}$, for all $n\in Z^+\cup\{0\}$ and given that T is a cyclic mapping that is (α,β) admissible mapping, we have

$$\alpha(u_0)\ge1 \implies \beta(u_1)=\beta(Tu_0)\ge1, \tag{2.6}$$

$$\beta(u_0)\ge1 \implies \alpha(u_1)=\alpha(Tu_0)\ge1. \tag{2.7}$$

Using a comparable approach,

$$\alpha(u_n)\ge1 \text{ and } \beta(u_n)\ge1, \quad \text{for all } n\in N,$$

this implies

$$\alpha(u_{n-1})\beta(u_n)\ge1, \quad \text{for all } n\in Z^+\cup\{0\}.$$

Let's symbolize

$$m=\max\{\omega(u_0,Tu_0),\omega(Tu_0,T^2u_0)\},$$

$$v=\varpi_1+\varpi_2+\varpi_3+\varpi_4+\varpi_5+2\varpi_7+\varpi_8$$

and $\wp=1-\varpi_6-\varpi_8$. Since $\alpha(u_{n-1})\beta(u_n)\ge1$, for all $n\in N$. By Definition 2.13, taking $u=u_0$, $v=Tu_0$,

$$\begin{aligned}\omega(T^2u_0,T^3u_0)&\le\omega(u_0,Tu_0)^{\varpi_1}\omega(Tu_0,T^2u_0)^{\varpi_2}\omega(u_0,Tu_0)^{\varpi_3}\omega(Tu_0,T^2u_0)^{\varpi_4}\\&\quad\cdot\omega(Tu_0,T^2u_0)^{\varpi_5}\omega(T^2u_0,T^3u_0)^{\varpi_6}\left[\frac{\omega(u_0,T^2u_0)\omega(Tu_0,Tu_0)}{2}\right]^{\varpi_7}\\&\quad\cdot\left[\frac{\omega(Tu_0,T^2u_0)\omega(T^2u_0,T^2u_0)}{2}\right]^{\varpi_8}\\&=\omega(u_0,Tu_0)^{\varpi_1+\varpi_3}\omega(Tu_0,T^2u_0)^{\varpi_2+\varpi_4+\varpi_5}\\&\quad\cdot\omega(T^2u_0,T^3u_0)^{\varpi_6}\left[\frac{\omega(u_0,T^2u_0)}{2}\right]^{\varpi_7}\left[\frac{\omega(Tu_0,T^2u_0)}{2}\right]^{\varpi_8}\\&\le\omega(u_0,Tu_0)^{\varpi_1+\varpi_3}\omega(Tu_0,T^2u_0)^{\varpi_2+\varpi_4+\varpi_5}\omega(T^2u_0,T^3u_0)^{\varpi_6}\\&\quad\cdot[\omega(u_0,Tu_0)\omega(Tu_0,T^2u_0)]^{\varpi_7}[\omega(Tu_0,T^2u_0)\omega(T^2u_0,T^3u_0)]^{\varpi_8}\end{aligned}$$

$$
\begin{aligned}
&= \omega(u_0,Tu_0)^{\varpi_1+\varpi_3+\varpi_7}\omega(Tu_0,T^2u_0)^{\varpi_2+\varpi_4+\varpi_5+\varpi_7+\varpi_8}\\
&\quad\cdot\omega(T^2u_0,T^3u_0)^{\varpi_6+\varpi_8},\\
\omega(T^2u_0,T^3u_0)^{1-\varpi_6-\varpi_8} &\le m^{\varpi_1+\varpi_3+\varpi_7}m^{\varpi_2+\varpi_4+\varpi_5+\varpi_7+\varpi_8}\\
&= m^{\varpi_1+\varpi_2+\varpi_3+\varpi_4+\varpi_5+2\varpi_7+\varpi_8},\\
\omega(T^2u_0,T^3u_0) &\le m^{\frac{v}{\wp}},
\end{aligned}
$$

where $\frac{v}{\wp}<1$ as $\varpi_1+\varpi_2+\varpi_3+\varpi_4+\varpi_5+2\varpi_7+2\varpi_8<\frac{1}{h^2}$.

Since T is cyclic (α,β) admissible. Taking $u=Tu_0$, $v=T^2u_0$,

$$
\begin{aligned}
\omega(T^3u_0,T^4u_0) &\le \omega(Tu_0,T^2u_0)^{\varpi_1}\omega(T^2u_0,T^3u_0)^{\varpi_2}\omega(Tu_0,T^2u_0)^{\varpi_3}\omega(T^2u_0,T^3u_0)^{\varpi_4}\\
&\quad\cdot\omega(T^2u_0,T^3u_0)^{\varpi_5}\omega(T^3u_0,T^4u_0)^{\varpi_6}\\
&\quad\cdot\left[\frac{\omega(Tu_0,T^3u_0)\omega(T^2u_0,T^2u_0)}{2}\right]^{\varpi_7}\left[\frac{\omega(T^2u_0,T^4u_0)\omega(T^3u_0,T^3u_0)}{2}\right]^{\varpi_8}\\
&= \omega(Tu_0,T^2u_0)^{\varpi_1+\varpi_3+\varpi_7}\omega(T^2u_0,T^3u_0)^{\varpi_2+\varpi_4+\varpi_5+\varpi_7+\varpi_8}\omega(T^3u_0,T^4u_0)^{\varpi_6+\varpi_8}\\
&= m^{\varpi_1+\varpi_2+\varpi_3+\varpi_4+\varpi_5+2\varpi_7+\varpi_8}\\
&\le m^{\frac{v}{\wp}},\\
\omega(T^4u_0,T^5u_0) &\le \omega(T^2u_0,T^3u_0)^{\varpi_1}\omega(T^3u_0,T^4u_0)^{\varpi_2}\omega(T^2u_0,T^3u_0)^{\varpi_3}\omega(T^3u_0,T^4u_0)^{\varpi_4}\\
&\quad\cdot\omega(T^3u_0,T^4u_0)^{\varpi_5}\omega(T^4u_0,T^5u_0)^{\varpi_6}\\
&\quad\cdot\left[\frac{\omega(T^2u_0,T^4u_0)\omega(T^3u_0,T^3u_0)}{2}\right]^{\varpi_7}\left[\frac{\omega(T^3u_0,T^5u_0)\omega(T^4u_0,T^4u_0)}{2}\right]^{\varpi_8}\\
&= \omega(T^2u_0,T^3u_0)^{\varpi_1+\varpi_3+\varpi_7}\omega(T^3u_0,T^4u_0)^{\varpi_2+\varpi_4+\varpi_5+\varpi_7+\varpi_8}\omega(T^4u_0,T^5u_0)^{\varpi_6+\varpi_8}\\
&= m^{\left(\frac{v}{\wp}\right)^{\varpi_1+\varpi_3+\varpi_7}}m^{\left(\frac{v}{\wp}\right)^{\varpi_2+\varpi_4+\varpi_5+\varpi_7+\varpi_8}}\omega(T^4u_0,T^5u_0)^{\varpi_6+\varpi_8}\\
&\le m^{\left(\frac{v}{\wp}\right)^2}.
\end{aligned}
$$

Continuing in this way

$$
\omega(T^nu_0,T^{n+1}u_0)\le m^{\left(\frac{v}{\wp}\right)^n},
$$

whenever $n=2\delta$ or $n=2\delta-1$, for $\delta>2$.

Therefore, $\omega(T^nu_0,T^{n+1}u_0)\le m^{\left(\frac{v}{\wp}\right)^{j-1}}$, whenever $n=2\delta$ or $2\delta-1$, for $\delta>1$. Therefore, $\omega(T^nu_0,T^{n+1}u_0)\to 1$ as $n\to\infty$. So, T is asymptotically regular at u_0. By Lemma 2.4, conclude that the AFPP.

Assuming that (P,ω) is a complete *dqb-mms* with T being continuous, in order to prove that $\{u_n\}$ is a Cauchy sequence in P, consider two distinct non-zero positive integers, denoted as σ and ς, where $\sigma<\varsigma$. This gives rise to two cases:

Case 1: For $n=2\delta$ with δ, $\chi>1$, then

$$
\begin{aligned}
\omega(T^nu_0,T^{n+\chi}u_0) &= \omega(T^{2\delta}u_0,T^{2\delta+\chi}u_0)\\
&\le \omega(T^{2\delta}u_0,T^{2\delta+1}u_0)^k\omega(T^{2\delta+1}u_0,T^{2\delta+2}u_0)^{k^2}\omega(T^{2\delta+2}u_0,T^{2\delta+3}u_0)^{k^3}\dots\\
&\le m^{\left(\frac{v}{\wp}\right)^{\delta}k}m^{\left(\frac{v}{\wp}\right)^{\delta}k^2}m^{\left(\frac{v}{\wp}\right)^{\delta+1}k^3}m^{\left(\frac{v}{\wp}\right)^{\delta+1}k^4}\dots
\end{aligned}
$$

$$\le m^{(k+k^2)\left(\frac{v}{\wp}\right)^{\delta}\left(1+k^2\left(\frac{v}{\wp}\right)+\dots\right)}$$
$$\le m^{(k+k^2)\left(\frac{v}{\wp}\right)^{\delta}\frac{1}{1-k^2\left(\frac{v}{\wp}\right)}}.$$

Case 2: For $n = 2\delta+1$ with δ, $\chi > 1$,

$$\begin{aligned}\omega(T^n u_0, T^{n+\chi}u_0) &= \omega(T^{2\delta+1}u_0, T^{2\delta+\chi+1}u_0)\\ &\le \omega(T^{2\delta+1}u_0, T^{2\delta+2}u_0)^k \omega(T^{2\delta+2}u_0, T^{2\delta+3}u_0)^{k^2}\\ &\quad\cdot \omega(T^{2\delta+3}u_0, T^{2\delta+4}u_0)^{k^3} \omega(T^{2\delta+4}u_0, T^{2\delta+5}u_0)^{k^4}\dots\\ &\le m^{\left(\frac{v}{\wp}\right)^{\delta}k} m^{\left(\frac{v}{\wp}\right)^{\delta}k^2} m^{\left(\frac{v}{\wp}\right)^{\delta+1}k^3} m^{\left(\frac{v}{\wp}\right)^{\delta+1}k^4}\dots\\ &\le m^{(k+k^2)\left(\frac{v}{\wp}\right)^{\delta}\left(1+k^2\left(\frac{v}{\wp}\right)+\dots\right)}\\ &\le m^{(k+k^2)\left(\frac{v}{\wp}\right)^{\delta}\frac{1}{1-k^2\left(\frac{v}{\wp}\right)}}.\end{aligned}$$

As $\delta\to\infty$ in all cases, we have $k^2\left(\frac{v}{\wp}\right) < 1$, this implies that $\omega(T^n u_0, T^n u_0) \to 1$. Consequently, $\{u_n\}$ forms a Cauchy sequence in P. Given that P is complete, there exists a point $\sigma \in P$ such that $u_n = T^n u_0 \to \sigma \in P$ as $n \to \infty$. Utilizing the continuity of T, we deduce $\sigma = \lim_{n\to\infty} T\sigma$. This establishes σ as a fixed point of T. Now, let's demonstrate the uniqueness of this fixed point. Let $\sigma,\sigma^* \in Fix(T)$ with $\sigma \ne \sigma^*$. By being cyclic (α,β) admissible, we have $\alpha(\sigma) > 1$ and $\beta(\sigma^*) > 1$ and from Definition 2.13, taking $u = \sigma$ and $v = \sigma^*$, we obtain

$$\begin{aligned}\omega(\sigma,\sigma^*) &= \omega(T^2\sigma, T^2\sigma^*)\\ &\le \omega(\sigma,\sigma^*)^{\varpi_1}\omega(T\sigma,T\sigma^*)^{\varpi_2}\omega(\sigma,T\sigma)^{\varpi_3}\omega(T\sigma,T^2\sigma)^{\varpi_4}\omega(\sigma^*,T\sigma^*)^{\varpi_5}\omega(T\sigma^*,T^2\sigma^*)^{\varpi_6}\\ &\quad\cdot\left[\frac{\omega(\sigma,T\sigma^*)\omega(\sigma^*,T\sigma)}{2}\right]^{\varpi_7}\left[\frac{\omega(T\sigma,T^2\sigma^*)\omega(T\sigma,T^2\sigma)}{2}\right]^{\varpi_8}\\ &\le \omega(\sigma,\sigma^*)^{\varpi_1}\omega(\sigma,\sigma^*)^{\varpi_2}\omega(\sigma,\sigma^*)^{\varpi_7}\omega(\sigma,\sigma^*)^{\varpi_8}\\ &\le \omega(\sigma,\sigma^*)^{\varpi_1+\varpi_2+\varpi_7+\varpi_8}.\end{aligned}$$

It follows that $\omega(\sigma,\sigma^*)^{1-\varpi_1-\varpi_2-\varpi_7-\varpi_8} \le 1$, which is contradiction. Therefore, $\omega(\sigma,\sigma^*) = 1$. Hence T has a unique fixed point in P. □

3. Application to System of Integral Equation

Let $P = C([\breve{e},\breve{c}], R_+)$ be a *set of all continuous function* on $[\breve{e},\breve{c}]$ is a *closed* and *bounded interval*. For a *real number* $\tau > 0$, define $\omega : P \times P \to [1,\infty]$ by

$$\omega(\zeta,\eta) = \sup\left|\frac{\zeta(t)}{\eta(t)}\right|^{\tau}, \tag{3.1}$$

for all $\zeta,\eta \in C([\breve{e},\breve{c}], R_+)$ with these settings (ω,P) becomes a complete *dqb-mms* with $h = 2^{\tau-1}$. We utilize Theorem 2.3 to demonstrate the *existence* of a solution for the *Fredholm integral* of a specified type, as defined by

$$u(t) = \int_{\breve{e}}^{\breve{c}} k(t,s,u(s))^{ds}, \tag{3.2}$$

for all $t,s \in [\breve{e},\breve{c}]$,

(i) k is the function from $[\breve{e},\breve{c}]\times[\breve{e},\breve{c}]\times P\to R$ are *continuous functions* on $[\breve{e},\breve{c}]$.

(ii) $|\lambda|\le 1$.

(iii) For every $u,v\in P$ with $u\neq v$ and $t,s\in[\breve{e},\breve{c}]$ meeting the subsequent inequality:

$$\left|\frac{k(t,s,Tu(s))}{k(t,s,Tv(s))}\right|^{\tau}\le D(t,s)\max\Big\{|u(s)-v(s)|,|Tu(s)-Tv(s)|,|u(s)-Tu(s)|,\\ \cdot|Tu(s)-T^2u(s)|,|v(s)-Tv(s)|,|Tv(s)-T^2v(s)|,\\ \cdot\left(\frac{|u(s)-Tv(s)||v(s)-Tu(s)}{2}\right),\\ \cdot\left(\frac{|Tu(s)-T^2v(s)|\,|Tv(s)-T^2v(s)}{2}\right)\Big\}^{\tau}. \tag{3.3}$$

(iv) $\max\limits_{t\in[\breve{e},\breve{c}]}\int_{\breve{e}}^{\breve{c}}D(t,s)^{ds}\le\frac{1}{(e\,\check{-}\,c)^{\tau-1}}$, where $h=2^{\tau-1}$.

Theorem 3.1. *Given assumptions* (i)-(iv)*, the integral equation* (3.2) *possesses a solution within the set P.*

Define $T:P\to P$ by

$$Tu(t)=\int_{\breve{e}}^{\breve{c}}k(t,s,u(s))^{ds}. \tag{3.4}$$

Observe that u is a solution for (3.2) *iff u is a $fixed\ point$ of T. Let $q\in R$ such that $\frac{1}{\tau}+\frac{1}{q}=1$. We establish the novelty of the generalized convex contraction operator T on the space $C[\breve{e},\breve{c}]$ by employing the Holder inequality and satisfying the conditions* (i)-(iv)*. By equations* (3.3) *and* (3.4)*, we obtain*

$$\begin{aligned} d(T^2u,T^2v) &\le \sup_{t\in[\breve{e},\breve{c}]}|T^2u(t)-T^2v(t)|^{\tau}\\ &\le \sup_{t\in[\breve{e},\breve{c}]}|\lambda|^{\tau}\left(\int_{\breve{e}}^{\breve{c}}\left|\frac{k(t,s,Tu(s))}{k(t,s,Tv(s))}\right|^{ds}\right)^{\tau}\\ &\le\left[\sup_{t\in[\breve{e},\breve{c}]}\left(\int_{\breve{e}}^{\breve{c}}1^{\tau ds}\right)^{\frac{1}{q}}\left(\int_{\breve{e}}^{\breve{c}}\left|\left(\frac{k(t,s,Tu(s))}{k(t,s,Tv(s))}\right)^{\tau}\right|^{ds}\right)^{\frac{1}{\tau}}\right]^{\tau}\\ &=(\breve{e}-\breve{c})^{\tau-1}\frac{1}{(\breve{e}-\breve{c})^{\tau-1}}\max\Big\{d(u,v),d(Tu,Tv),d(u,Tu),(Tu,T^2u),d(v,Tv),\\ &\qquad d(Tv,T^2v),\frac{d(u,Tv)d(v,Tu)}{2},\frac{d(Tv,T^2u)d(Tv,T^2u)}{2}\Big\}\\ &=\breve{S}(u,v). \end{aligned}$$

Therefore, $\alpha(u,v)d(Tu,Tv)\le\breve{S}(u,v)$.

Define $\alpha:P\times P\to R^{+}$ by $\alpha(u,v)=1$, for all $u,v\in P$. Therefore, T is α-admissible. Let u_0 and $\{u_n\}$ in P defined by $u_{n+1}=Tu_n=T^{n+1}u_0$, for all $n\ge 0$. Equation (3.4)*, we have*

$$u_{n+1}=Tu_n(t)=\frac{1}{t-s}\int_{\breve{e}}^{\breve{c}}k(t,s,u_n(s))^{ds}.$$

All conditions stated in Theorem 2.3 are met, and hence T has a unique fixed point.

4. An Application to Dynamic Programming

This section establishes the existence of a solution for a certain class of functional equations using rational-type contraction in a dislocated quasi b-multiplicative metric space

$$\dot{u}(\breve{x}) = \sup_{\breve{y}\in D}\{\mu(\breve{x},\breve{y}) + E(\breve{x},\breve{y},\dot{u}(\Gamma(\breve{x},\breve{y}))\} \tag{4.1}$$

(cf., Bellman and Lee [2]). Here, S represents the state space, D is the decision space, and $\Gamma: S\times D$, $\mu: S\times D \to \mathcal{R}$, and $E: S\times D\times\mathcal{R}\to\mathcal{R}$ are mappings provided for the interactions within the system.

Lemma 4.1. *If $T,G: S\to\mathcal{R}$ are bounded function, where $\breve{x}\in S$ then for each $\sigma>1$,*

$$\sigma^{|\sup_{\breve{x}\in S} T(\breve{x}) - \sup_{\breve{x}\in S} G(\breve{x})|} \le \sigma^{\sup_{\breve{x}\in S}|T(\breve{x})-G(\breve{x})|}.$$

Consider a non-empty set S and we work in the space $B(S)$ representing the collection of all bounded real functions defined on S. Utilizing standard function addition and scalar multiplication, the norm $\|\cdot\|_\infty$ defined as

$$\|\dot{u}\|_\infty = \sup_{\breve{x}\in S}|\dot{u}(\breve{x})|, \quad \text{for all } \dot{u}\in B(S),$$

then $(B(S),\|\cdot\|)$ is Banach space. Therefore, the distance of dqb-mms in $B(S)$ is

$$\omega_\infty(\dot{u},\dot{v}) = \sigma^{\sup_{\mathbf{x}\in S}|\dot{u}(\breve{x})-\dot{v}(\breve{x})|}, \quad \dot{u},\dot{v}\in B(S).$$

Lemma 4.2. *Assuming that:*

(i) *$\mu: S\times D\to\mathcal{R}$ and $E(\cdot,\cdot,0): S\times D\to\mathcal{R}$ are bounded functions.*

(ii) *There exists $\breve{M}\ge 0$ such that, for all $\breve{x}\in S$, $\breve{y}\in D$, $a,b\in\mathcal{R}$,*

$$\sigma^{|E(\breve{x},\breve{y},a)E(\breve{x},\breve{y},b)|} \le \sigma^{\breve{M}|a-b|}.$$

Then the operator $\mathcal{R}: B(S)\to B(S)$ given, for all $\dot{u}\in B(S)$ and all $\breve{x}\in S$ by

$$(\mathcal{R}\dot{u})(\breve{x}) = \sup_{\breve{y}\in D}\{\mu(\breve{x},\breve{y}) + E(\breve{x},\breve{y},\dot{u}(\Gamma(\breve{x},\breve{y})))\}. \tag{4.2}$$

Theorem 4.1. *Suppose the following assumptions:*

(i) *$\mu: S\times D\to\mathcal{R}$ and $E(\cdot,\cdot,0): S\times D\to\mathcal{R}$ are bounded functions.*

(ii) *There exists $\breve{M}\ge 0$ such that, for all $\breve{x}\in S$, $\breve{y}\in D$ and $a,b\in\mathcal{R}$,*

$$\sigma^{|E(\breve{x},\breve{y},a)-E(\breve{x},\breve{y},b)|^2} \le \sigma^{\breve{M}|a-b|^2}.$$

(iii) *There exists a continuous comparison function $\varphi\in F_{com}$ such that, for all $\breve{x}\in S$, for all $\breve{y}\in D$, for all $\dot{u},\dot{v}\in B(S)$ and for each $\sigma>1$*

$$\sigma^{|E(\breve{x},\breve{y}),\dot{u}(\Gamma(\breve{x},\breve{y}))-E(\breve{x},\breve{y}),\dot{v}(\Gamma(\breve{x},\breve{y}))|^2} \le \sigma^{\varphi(\breve{M}(\dot{u},\dot{v}))^2},$$

where

$$\breve{M}(\breve{x},\breve{y}) = \omega_\infty(\dot{u},\dot{v})^{\varpi_1}\left[\frac{\omega_\infty(\dot{u},\mathcal{R}\dot{u})\omega_\infty(\dot{v},\mathcal{R}\dot{v})}{1+\omega_\infty(\mathcal{R}\dot{u},\mathcal{R}\dot{v})}\right]^{\varpi_2}\omega_\infty(\mathcal{R}\dot{u},\mathcal{R}\dot{v})^{\varpi_3}.$$

Then, equation (4.1) has a unique solution $\dot{u}_0$ in $B(S)$.

Proof. Let $\mathcal{R}: B(S) \to B(S)$ in (4.2), by Lemma 4.2 and the non-decreasing character of φ, we deduce that, for all $\dot{u},\dot{v} \in B(S)$ and all $\check{x} \in S$,

$$\begin{aligned}
\sigma^{|(\mathcal{R}\dot{u})(\check{x})-(\mathcal{R}\dot{v})(\check{x})|^2} &= \sigma^{|\sup_{y\check{\in}D}\{\mu(\check{x},\check{y})+E(\check{x},\check{y},u(\Gamma(\check{x},\check{y})\}-\sup_{y\check{\in}D}\{\mu(\check{x},\check{y})+E(\check{x},\check{y},\dot{u}(\Gamma(\check{x},\check{y})\}|^2} \\
&\le \sigma^{|\sup_{\check{y}\in D}(\mu(\check{x},\check{y})+E(\check{x},\check{y},\dot{u}(\Gamma(\check{x},\check{y})))-(\mu(\check{x},\check{y})+E(\check{x},\check{y},\dot{v}(\Gamma(\check{x},\check{y}))))|^2} \\
&= \sigma^{\sup_{y\in D}|E(\check{x},\check{y}),\dot{u}(\Gamma(\check{x},\check{y}))-E(\check{x},\check{y}),\dot{v}(\Gamma(\check{x},\check{y}))|^2} \\
&\le \sigma^{\varphi(\check{M}(\dot{u},\dot{v}))^2}, \\
\omega_\infty(\mathcal{R}\dot{u},\mathcal{R}\dot{v}) &= \sigma^{\sup_{\check{x}\in S}|(\mathcal{R}u((x)-(\mathcal{R}v)(x)))|^2} \\
&\le \sigma^{\varphi(\check{M}(\dot{u},\dot{v}))^2}, \quad \text{for all } \dot{u},\dot{v} \in B(S),
\end{aligned}$$

which means that $\mathcal{R}$ satisfies all hypothesis of theorem. Thus, there exists a unique $\dot{u}_0 \in B(S)$ such that $\mathcal{R}\dot{u}_0 = \dot{u}_0$. Hence, for all $\check{x} \in S$, $\dot{u}_0(\check{x}) = (\mathcal{R}\dot{u}_0)(x) = \sup_{\check{y}\in D}\{\mu(\check{x},\check{y}) + E(\check{x},\check{y},\dot{u}0(\Gamma(\check{x},\check{y}))\}$. This complete the proof. □

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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