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Research Article

On Two-Dimensional Landsberg Space with A Special (α,β)-Metric

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Abstract. The purpose of the present paper is to study a Finsler space with a special (α,β)-metric $L(\alpha,\beta)=\alpha+\epsilon\beta+\kappa\frac{\beta^2}{\alpha}$ (ϵ and $k\neq 0$ are real constants) satisfying some conditions. First we find a condition for a Finsler space with a special (α,β)-metric to be a Berwald space. Then we show that if a two-dimensional Finsler space with a special (α,β)-metric $L(\alpha,\beta)=\alpha+\epsilon\beta+\kappa\frac{\beta^2}{\alpha}$ (ϵ and $k\neq 0$ are real constants) is a Landsberg space, then it is a Berwald space.

Keywords. Berwald space; Cartan connection; Finsler space; Landsberg space; Main scalar

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1. Introduction

The real Landsberg spaces, in particular the real Berwald spaces, have been a major subject of study for many geometers over the years. In 1926, L. Berwald [4] introduced a special class of Finsler spaces which took his name in 1964 [8]. It is known that a real Finsler space is called a Berwald space if the local coeffcients of the Berwald connection depend only on position coordinates. In the Cartan connection $C\Gamma$, a Finsler space is called Landsberg space, if the covariant derivative $C_{hij|k}$ of the C-torsion tensor $C_{hij}=\dot\partial_h\dot\partial_i\dot\partial_j(L^2/4)$ satisfies $C_{hij|k}(x,y)y^k=0$. A Berwald space is characterized by $C_{hij|k}=0$. Berwald spaces are specially interesting and important, because the connection is linear, and many examples of a Berwald space have been known. But any concrete example of a Landsberg space which is not a Berwald space is not known yet. If a Finsler space is a Landsberg space and satisfies some additional conditions, then it is merely a Berwald space [3]. On the other hand, in the two- dimensional case, a general Finsler space is a Landsberg space, if and only if its main scalar $I(x,y)$ satisfies $I_{|i}y^i = 0$ [7].

The purpose of the present paper is to find a two-dimensional Landsberg space with a special (α,β)-metric $L(\alpha,\beta) = \alpha+\epsilon\beta+\kappa\frac{\beta^2}{\alpha}$ satisfying some conditions, where ϵ, $\kappa \neq 0$ are real constants. First we find the condition for a Finsler space with a special (α,β)-metric to be a Berwald space (see Theorem 3.1). Next, we determine the difference vector and the main scalar of F^2 with the aforesaid metric.

Finally, we derive the condition for a two-dimensional Finsler space F^2 with a special (α,β)-metric $L(\alpha,\beta) = \alpha+\epsilon\beta+\kappa\frac{\beta^2}{\alpha}$ (ϵ and $k \neq 0$ are real constants) to be a Landsberg space, and we show that if F^2 with the mentioned metric is a Landsberg space, then it is a Berwald space (see Theorem 4.1).

2. Preliminaries

Let $F^n = (M^n, L(\alpha,\beta))$ be an n-dimensional Finsler space with an (α,β)-metric and $R^n = (M^n,\alpha)$ the associated Riemannian space, where $\alpha^2 = a_{ij}(x)y^iy^j$, Â$^-$ $\beta = b_i(x)y^i$. Since the metric tensor a_{ij} is invertible, we put $a^{ij} = (a_{ij})^{-1}$.

The Riemannian metric α is not supposed to be positive-definite and we shall restrict our discussions to a domain of (x,y) where β does not vanish. The covariant differentiation in the Levi-Civita connection $(\gamma^i_{jk}(x))$of R^n is denoted by the semi-colon. Let us list the symbols here for the late use:

(i) $b^i = a^{ir}b_r$, $b^2 = a^{rs}b_rb_s$,

(ii) $2r_{ij} = b_{i;j} + b_{j;i}$, $2s_{ij} = b_{i;j} - b_{j;i}$,

(iii) $r^i_j = a^{ir}r_{rj}$, $s^i_j = a^{ir}s_{rj}$, $r_i = b_r r^r_i$, $s_i = b_r s^r_i$.

(iv) $L_\alpha = \partial L/\partial\alpha$, $L_\beta = \partial L/\partial\beta$, $L_{\alpha\alpha} = \partial^2 L/\partial\alpha^2$, $L_{\beta\beta} = \partial^2 L/\partial\beta^2$.

In the present paper Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ of F^n plays one of the leading roles. Denote by B^i_{jk} the difference tensor of Matsumoto [7] of G^i_{jk} from (γ^i_{jk}):

$$G^i_{jk}(x,y) = \gamma^i_{jk}(x,y) + B^i_{jk}(x,y). \tag{2.1}$$

With the subscript 0, the transvection by y^i, we have

$$G^i_j = \gamma^i_{0j} + B^i_j,\ 2G^i = \gamma^i_{00} + 2B^i \tag{2.2}$$

and then $B^i_j = \dot\partial_j B^i$ and $B^i_{jk} = \dot\partial_k B^i_j$. On account of Matsumoto [7], the Berwald connection $B\Gamma$ of a Finsler space with (α,β)-metric $L(\alpha,\beta)$ is given by (2.1) and (2.2), where B^i_{jk} are the components of a Finsler tensor of $(1,2)$-type which is determined by

$$L_\alpha B^k_{ji}y^iy_k = \alpha L_\beta(b_{j;i} - B^k_{ji}b_k)y^j. \tag{2.3}$$

According to Matsumoto [7], $B^i(x;y)$ is called the *difference vector*. If

$$\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha} \neq 0,$$

where $\gamma^2 = b^2\alpha^2 - \beta^2$, then B^i is written as follows:

$$B^i = \frac{E^*}{\alpha} y^i + \frac{\alpha L_\beta}{L_\alpha} s^i_0 - \frac{\alpha L_{\alpha\alpha}}{L\alpha} C^* \left(\frac{1}{\alpha} y^i - \frac{\alpha}{\beta} b^i\right), \tag{2.4}$$

where

$$E^* = \left(\frac{\beta L_\beta}{L}\right) C^*,\ C^* = \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})},$$

Furthermore, by means of Hashiguchi, Hojo and Matsumoto [4], we have

$$\alpha_{|i} = -\frac{L_\beta}{L_\alpha} \beta_{|i}, \tag{2.5}$$

$$\beta_{|i} y^i = r_{00} - 2b_r B^r, \tag{2.6}$$

$$b^2_{|i} y^i = 2(r_0 + s_0), \tag{2.7}$$

$$\gamma^2_{|i} y^i = 2(r_0 + s_0)\alpha^2 - 2\left(\frac{L_\beta}{L_\alpha} b^2\alpha + \beta\right)(r_{00} - 2b_r B^r). \tag{2.8}$$

The following lemmas have been shown:

Lemma 2.1 ([2, 6])**.** *If $\alpha^2 \equiv 0 (\mathrm{mod}\,\beta)$, that is, $a_{ij}y^i y^j$ contains $b_i(x)y^i$ as a factor, then the dimension n is equal to 2 and b^2 vanishes. In this case we have $1-$form $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.*

Lemma 2.2 ([5, 6])**.** *We consider the two-dimensional case.*

(i) *If $b^2 \neq 0$, then there exist a sign $\epsilon = \pm 1$ and $\delta = d_i(x)y^i$ such that $\alpha^2 = \frac{\beta^2}{b^2} + \epsilon\delta^2$ and $d_i b^i = 0$.*

(ii) *If $b^2 = 0$, then there exists $\delta = d_i(x)y^i$ such that $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.*

If there are two functions $f(x)$ and $g(x)$ satisfying $f\alpha^2 + g\beta^2 = 0$, then $f = g = 0$ is obvious, because $f \neq 0$ implies a contradiction $\alpha^2 = \frac{-g}{f}\beta^2$.

Throughout the paper, we shall say "homogeneous polynomial (s) in (y^i) of degree r" as $hp(r)$ for brevity. Thus γ^i_{00} are $hp(2)$.

3. Berwald Space

In this section, we find the condition for a Finsler space F^n with a special (α,β)-metric to be a Berwald space.

Let $F^n = (M^n, L(\alpha,\beta))$ be an n-dimensional Finsler space with a special (α,β)-metric given by

$$L(\alpha,\beta) = \alpha + \epsilon\beta + \kappa\frac{\beta^2}{\alpha}, \tag{3.1}$$

where ϵ, $\kappa \neq 0$ are real constants.

Then from the above we have

$$L_\alpha = 1-\frac{k\beta^2}{\alpha^2},\ \ L_\beta = \epsilon+\frac{2k\beta}{\alpha},\ \ L_{\alpha\alpha} = \frac{2k\beta^2}{\alpha^3},\ \ L_{\beta\beta} = \frac{2k}{\alpha}. \tag{3.2}$$

Substituting (3.2) into (2.3), we obtain

$$(\alpha^2-k\beta^2)B^k_{ji}y^iy_k+\alpha(-2k\alpha\beta-\epsilon\alpha^2)(b_j;i-B^k_{ji}b_k)y^j = 0. \tag{3.3}$$

Assume that the Finsler space with metric (3.1) be a Berwald space, that is, $G^i_{jk} = G^i_{jk}(x)$. Then we have $B^k_{ji} = B^k_{ji}(x)$, so the left-hand side of (3.3) has a form

$$P(x,y)+\alpha Q(x,y) = 0, \tag{3.4}$$

where P, Q are polynomials in (y^i) while α is irrational in (y^i). Hence the equation (3.3) shows $P = Q = 0$.

Thus we have

$$B^k_{ji}a_{kh}y^jy^h = 0,\quad (b_{j;i}-B^k_{ji}b_k)y^j = 0. \tag{3.5}$$

The former yields $B^k_{ji}a_{kh}+B^k_{hi}a_{kj} = 0$, so we have $B^k_{ji} = 0$. Then the latter leads to $b_{j;i=0}$ directly.

Conversely, if $b_{i;j} = 0$, then $(\gamma^i_{jk},\gamma^i_{0j},0)$ becomes the Berwald connection of F^n due to the well-known Okada's axioms. Thus F^n is a Berwald space. Therefore, we have

Theorem 3.1. *The Finsler space F^n with special metric (3.1) satisfying $b^2 \neq 0$ is a Berwald space if and only if $b_{j;i} = 0$, and then the Berwald connection is essentially Riemannian $(\gamma^i_{jk},\gamma^i_{0j},0)$.*

4. Two-dimensional Landsberg Space

Let the Finsler space $F^n = (M^n, L(\alpha,\beta))$ with an (α,β)-metric given by (3.1) be a Landsberg space.

The difference vector B^i of the Finsler space has been first given in [7]. Here, by means of (2.4) and (3.2), we have

$$2B^i = \frac{A}{(\alpha^2-k\beta^2)\Omega}\left(2k\alpha^2b^i+\frac{By^i}{\alpha L}\right)+\frac{2\alpha^2(\epsilon\alpha+2k\beta)}{(\alpha^2-k\beta)}s^i_0, \tag{4.1}$$

where

$$A = r_{00}(\alpha^2-k\beta^2)-2\alpha^2s_0(\epsilon\alpha+2k\beta),$$
$$B = (\epsilon\alpha^3-3\epsilon k\alpha\beta^2-4k^2\beta^3),$$
$$\Omega = (\alpha^2+2kb^2\alpha^2-3k\beta^2).$$

It is trivial that $(\alpha^2-\beta^2) \neq 0$ and $\Omega \neq 0$, because α is irrational in (y^i).

From (4.1) it follows that

$$r_{00}-2b_rB^r = \frac{A(\alpha^2-k\beta^2)}{\alpha L\Omega}. \tag{4.2}$$

Now we deal with the condition for a two-dimensional Finsler space F^2 with (3.1) to be a Landsberg space. It is known that in the two-dimensional case, a general Finsler space is a Landsberg space, if and only if its main scalar $I(x,y)$ satisfies $I_{|i}y^i = 0$ ([1], [6]).

The main scalar of F^2 is obtained as follows:

$$\epsilon I^2 = \frac{9\gamma^2 M^2}{4\alpha L\Omega^3}, \tag{4.3}$$

where

$$M = \epsilon(1+2kb^2)\alpha^3 - 8k\beta^3 + 4b^2k^2\alpha^2\beta - 5\epsilon k\alpha\beta^2,$$
$$\Omega = (1+2b^2k)\alpha^2 - 3k\beta^2.$$

The covariant differentiation of (4.3) leads to

$$4\alpha^2 L\Omega^4\epsilon I^2_{|i} = 9M(\alpha\Omega M\gamma^2_{|i} + 2\alpha\Omega\gamma^2 M_{|i} - \Omega\gamma^2 M\alpha_{|i} - 3\alpha\gamma^2 M\Omega_{|i}). \tag{4.4}$$

Trasvevting (4.4) by y^i, we have

$$4\alpha^2 L\Omega^4\epsilon I^2_{|i} = 9M(U\gamma^2_{|i}y^i + QM_{|i}y^i - R\alpha_{|i}y^i - S\Omega_{|i}y^i), \tag{4.5}$$

where

$$U = \epsilon\alpha^6 - 8\epsilon k\alpha^4\beta^2 + 4\epsilon kb^2\alpha^6 - 8k^2\alpha^3\beta^3 + 4b^2k^2\alpha^5\beta + 15\epsilon k^2\alpha^2\beta^4 - 16\epsilon k^2b^2\alpha^4\beta^2$$
$$+ 24k^3\alpha\beta^5 - 28k^3b^2\alpha^3\beta^3 + 4\epsilon k^2b^4\alpha^6 + 8k^3b^4\alpha^5\beta,$$
$$Q = 2b^2\alpha^5 - 10kb^2\alpha^3\beta^2 + 4kb^4\alpha^5 - 2\alpha^3\beta^2 + 6k\alpha\beta^4,$$
$$R = \epsilon b^2\alpha^7 - 12\epsilon kb^2\alpha^5\beta^2 + 4\epsilon kb^4\alpha^7 - \epsilon\alpha^5\beta^2 + 8\epsilon k\alpha^3\beta^4 + 25\epsilon k^2b^2\alpha^3\beta^4 - 20\epsilon k^2b^4\alpha^5\beta^2$$
$$- 15\epsilon k^2\alpha\beta^6 + 4\epsilon k^2b^6\alpha^7 + 6\epsilon k^2b^2\alpha^3\beta^4 - 12k^2b^2\alpha^4\beta^3 + 52k^3b^2\alpha^2\beta^5 - 36k^3b^4\alpha^4\beta^3$$
$$+ 8k^2\alpha^2\beta^5 - 24k^3\beta^7 + 4k^2b^4\alpha^6\beta + 8k^3b^6\alpha^6\beta,$$
$$S = 3\epsilon b^2\alpha^6 - 21\epsilon kb^2\alpha^4\beta^2 + 6\epsilon kb^4\alpha^6 - 36k^2b^2\alpha^3\beta^3 + 12k^2b^4\alpha^5\beta - 3\epsilon\alpha^4\beta^2$$
$$+ 15\epsilon k\alpha^2\beta^4 + 24k^2\alpha\beta^5.$$

Thus the equation (4.5) is written in the form

$$4\alpha^2 L\Omega^4\epsilon I^2_{|i} = 9M(U\gamma^2_{|i}y^i + V\alpha_{|i}y^i + W\beta_{|i}y^i + Xb^2_{|i}y^i), \tag{4.6}$$

where

$$V = 14\epsilon kb^2\alpha^5\beta^2 - 12k^2b^4\alpha^6\beta - 5\epsilon k^2b^2\alpha^3\beta^4 + 24\epsilon k^2b^4\alpha^5\beta^2 + 100k^3b^4\alpha^4\beta^3 - 24k^3b^6\alpha^6\beta$$
$$- 10\epsilon k\alpha^3\beta^4 + 68k^2b^2\alpha^4\beta^3 - 15\epsilon k^2\alpha\beta^6 - 100k^3b^2\alpha^2\beta^5 - \epsilon b^2\alpha^7 - 4\epsilon kb^4\alpha^7$$
$$+ \epsilon\alpha^5\beta^2 - 4\epsilon k^2b^6\alpha^7 - 56k^2\alpha^2\beta^5 + 24k^3\beta^7,$$
$$W = -2\epsilon kb^2\alpha^6\beta - 56k^2b^2\alpha^5\beta^2 + 8k^2b^4\alpha^7 - 26\epsilon k^2b^2\alpha^4\beta^3 + 48k^3b^2\alpha^3\beta^4 - 64k^3b^4\alpha^5\beta^2$$
$$- 4\epsilon k^2b^4\alpha^6\beta + 16k^3b^6\alpha^7 + 2\epsilon k\alpha^4\beta^3 + 48k^2\alpha^3\beta^4 + 30\epsilon k^2\alpha^2\beta^5,$$
$$X = (-2\epsilon kb^2\alpha^8 + 8k^2b^2\alpha^7\beta + 22\epsilon k^2b^2\alpha^6\beta^2 + 32k^3b^2\alpha^5\beta^3 - 4\epsilon k^2b^4\alpha^8 - 8k^3b^4\alpha^7\beta$$
$$+ 2\epsilon k\alpha^6\beta^2 - 8k^2\alpha^5\beta^3 - 18\epsilon k^2\alpha^4\beta^4 - 24k^3\alpha^3\beta^5).$$

Consequently, the two-dimensional Finsler space F^2 with (3.1) is a Landsberg space, if and only if

$$U\gamma^2_{|i}y^i + V\alpha_{|i}y^i + W\beta_{|i}y^i + Xb^2_{|i}y^i = 0, \tag{4.7}$$

since $M \neq 0$. If $M = 0$, then $b^2 = 0$, namely, it is a contradiction.

By means of (2.5), (2.6), (2.7) and (2.8), the above equation is written as

$$\begin{aligned}&2(\alpha^2-k\beta^2)(\alpha^2U+X)(r_0+s_0)+[(\alpha^2-k\beta^2)W-V\alpha(\epsilon\alpha+2k\beta)\\&\quad-\{\alpha^2b^2(\epsilon\alpha+2k\beta)+\beta(\alpha^2-k\beta^2)\}U](r_{00}-2b_rB^r)=0.\end{aligned}\tag{4.8}$$

Substituting the values of U,V,W,X and $(r_{00}-2b_rB^r)$ in (4.8), we obtain

$$\begin{aligned}&\alpha^4\Big[2\epsilon\alpha^{10}+8\epsilon kb^2\alpha^{10}-18\epsilon k\alpha^8\beta^2+164\epsilon k^3\alpha^4\beta^6-4\epsilon k^2\alpha^6\beta^4-208\epsilon k^3b^2\alpha^6\beta^4\\&-126\epsilon k^4\alpha^2\beta^8+128\epsilon k^4b^2\alpha^4\beta^6+8\epsilon k^2b^4\alpha^{10}+72\epsilon k^3b^4\alpha^8\beta^2-40\epsilon k^4b^4\alpha^6\beta^4\\&+72\epsilon k^5b^2\alpha^2\beta^8-40\epsilon k^5b^4\alpha^4\beta^6-18\epsilon k^5\beta^{10}\Big](r_0+s_0)\\&+\alpha^5\beta\Big[-52\epsilon^2k^2\alpha^6\beta^2+32\epsilon^2k^2b^2\alpha^8-160\epsilon^2k^3b^2\alpha^6\beta^2+144\epsilon^2k^3\alpha^4\beta^4+20\epsilon^2k^4\alpha^2\beta^6\\&+176\epsilon^2k^5b^2\alpha^2\beta^6+56\epsilon^2k^3b^4\alpha^8-32\epsilon^2k^4\alpha^6\beta^2-72\epsilon^2k^5b^4\alpha^4\beta^4+2\epsilon^2\alpha^8\\&-114\epsilon^2k^5\beta^8-72\epsilon^2k^4b^2\alpha^4\beta^4+24k^6b^2\beta^8-16k^6b^4\alpha^2\beta^6\Big](r_0+s_0)\\&+\alpha^2\beta\Big[-64b^2\alpha^8\beta^2+62\epsilon k^2\alpha^6\beta^4+218\epsilon k^3b^2\alpha^6\beta^4-80\epsilon k^3b^4\alpha^8\beta^2-162\epsilon k^3\alpha^4\beta^6\\&+16\epsilon k^3b^6\alpha^{10}-6\epsilon kb^2\alpha^{10}-\epsilon\alpha^{10}+11\epsilon k\alpha^8\beta^2-208\epsilon k^4b^2\alpha^4\beta^6+99\epsilon k^4\alpha^2\beta^8\\&+66\epsilon k^5b^2\alpha^2\beta^8-80\epsilon k^5b^4\alpha^4\beta^6-9\epsilon k^5\beta^{10}+16\epsilon k^5b^6\alpha^6\beta^4+160\epsilon k^4b^4\alpha^6\beta^4\\&-32\epsilon k^4b^6\alpha^8\beta^2\Big]r_{00}\\&+\alpha\Big[-98k^2b^2\alpha^{10}\beta^2+154k^5b^2\alpha^8\beta^4-88k^3b^4\alpha^{10}\beta^2+16k^3b^2\alpha^{12}+68k^2\alpha^8\beta^4\\&+80k^4b^4\alpha^8\beta^4-86k^3\alpha^6\beta^6+8k^2b^4\alpha^{12}-\epsilon^2\alpha^{10}\beta^2-2k^4b^2\alpha^6\beta^6+16b^6k^4\alpha^{10}\beta^2\\&+42k^4\alpha^4\beta^8-146b^2k^5\alpha^4\beta^8+72k^5b^4\alpha^6\beta^6-72k^6b^4\alpha^4\beta^8+95k^5\alpha^2\beta^{10}\\&+76k^6b^2\alpha^2\beta^{10}+16k^6b^6\alpha^6\beta^6-24k^6\beta^{12}-94k^4\alpha^4\beta^8-32k^5b^6\alpha^8\beta^4\Big]r_{00}\\&+2\alpha^4\Big[94\epsilon k^2b^2\alpha^8\beta^2+68\epsilon k^3b^2\alpha^6\beta^4+80\epsilon k^3b^4\alpha^8\beta^2-16\epsilon k^3b^2\alpha^{10}-87\epsilon k^2\alpha^6\beta^4\\&+160\epsilon k^4b^4\alpha^6\beta^4-125\epsilon k^3\alpha^4\beta^6-8\epsilon k^2b^4\alpha^{10}+3\epsilon k\alpha^8\beta^2-354\epsilon k^4b^2\alpha^4\beta^6\\&-48\epsilon k^4b^6\alpha^8\beta^2+251\epsilon k^4\alpha^2\beta^8-232\epsilon k^5b^4\alpha^4\beta^6+208\epsilon k^5b^2\alpha^2\beta^8+48\epsilon k^5b^6\alpha^6\beta^4\\&-42\epsilon k^5\beta^{10}\Big]s_0\\&+2\alpha^3\beta\Big[234\epsilon^2k^5b^2\alpha^8\beta^2-206\epsilon^2k^2\alpha^6\beta^4-286\epsilon k^3b^2\alpha^6\beta^4+240\epsilon^2k^3b^4\alpha^8\beta^2\\&+128\epsilon^2k^3\alpha^4\beta^6-31\epsilon^2k^3b^6\alpha^{10}-26\epsilon^2k^4b^2\alpha^{10}+\epsilon^2\alpha^{10}-8\epsilon^2k\alpha^8\beta^2\\&-80\epsilon^2k^4b^4\alpha^6\beta^4-16k^3b^4\alpha^{10}-74\epsilon^2k^4b^2\alpha^4\beta^6-\epsilon^2k^4b^6\alpha^8\beta^2+133\epsilon^2k^5\alpha^2\beta^8\\&-144k^6b^4\alpha^4\beta^6+152k^6b^2\alpha^2\beta^8+32k^6b^6\alpha^6\beta^4-48k^6\beta^{10}\Big]s_0=0.\end{aligned}\tag{4.9}$$

Separating (4.9) in the rational and irrational terms with respect to (y^i), we have

$$\{\alpha^4D_1(r_0+s_0)+\alpha^2\beta E_1r_{00}+2\alpha^4F_1s_0\}+\alpha\{\alpha^4\beta D_2(r_0+s_0)+E_2r_{00}+2\alpha^2\beta F_2s_0\}=0,\tag{4.10}$$

where

$$D_1 = 2\epsilon\alpha^{10} + 8\epsilon k b^2\alpha^{10} - 18\epsilon k\alpha^8\beta^2 + 164\epsilon k^3\alpha^4\beta^6 - 4\epsilon k^2\alpha^6\beta^4 - 208\epsilon k^3 b^2\alpha^6\beta^4$$
$$- 126\epsilon k^4\alpha^2\beta^8 + 128\epsilon k^4 b^2\alpha^4\beta^6 + 8\epsilon k^2 b^4\alpha^{10} + 72\epsilon k^3 b^4\alpha^8\beta^2 - 40\epsilon k^4 b^4\alpha^6\beta^4$$
$$+ 72\epsilon k^5 b^2\alpha^2\beta^8 - 40\epsilon k^5 b^4\alpha^4\beta^6 - 18\epsilon k^5\beta^{10},$$

$$D_2 = -52\epsilon^2 k^2\alpha^6\beta^2 + 32\epsilon^2 k^2 b^2\alpha^8 - 160\epsilon^2 k^3 b^2\alpha^6\beta^2 + 144\epsilon^2 k^3\alpha^4\beta^4 + 20\epsilon^2 k^4\alpha^2\beta^6$$
$$+ 176\epsilon^2 k^5 b^2\alpha^2\beta^6 + 56\epsilon^2 k^3 b^4\alpha^8 - 32\epsilon^2 k^4\alpha^6\beta^4 - 72\epsilon^2 k^5 b^4\alpha^4\beta^4 + 2\epsilon^2\alpha^8$$
$$- 114\epsilon^2 k^5\beta^8 - 72\epsilon^2 k^4 b^2\alpha^4\beta^4 + 24k^6 b^2\beta^8 - 16k^6 b^4\alpha^2\beta^6,$$

$$E_1 = -64b^2\alpha^8\beta^2 + 62\epsilon k^2\alpha^6\beta^4 + 218\epsilon k^3 b^2\alpha^6\beta^4 - 80\epsilon k^3 b^4\alpha^8\beta^2 - 162\epsilon k^3\alpha^4\beta^6$$
$$+ 16\epsilon k^3 b^6\alpha^{10} - 6\epsilon k b^2\alpha^{10} - \epsilon\alpha^{10} + 11\epsilon k\alpha^8\beta^2 - 208\epsilon k^4 b^2\alpha^4\beta^6 + 99\epsilon k^4\alpha^2\beta^8$$
$$+ 66\epsilon k^5 b^2\alpha^2\beta^8 - 80\epsilon k^5 b^4\alpha^4\beta^6 - 9\epsilon k^5\beta^{10} + 16\epsilon k^5 b^6\alpha^6\beta^4 + 160\epsilon k^4 b^4\alpha^6\beta^4$$
$$- 32\epsilon k^4 b^6\alpha^8\beta^2,$$

$$E_2 = -98k^2 b^2\alpha^{10}\beta^2 + 154k^5 b^2\alpha^8\beta^4 - 88k^3 b^4\alpha^{10}\beta^2 + 16k^3 b^2\alpha^{12} + 68k^2\alpha^8\beta^4$$
$$+ 80k^4 b^4\alpha^8\beta^4 - 86k^3\alpha^6\beta^6 + 8k^2 b^4\alpha^{12} - \epsilon^2\alpha^{10}\beta^2 - 2k^4 b^2\alpha^6\beta^6 + 16b^6 k^4\alpha^{10}\beta^2$$
$$+ 42k^4\alpha^4\beta^8 - 146b^2 k^5\alpha^4\beta^8 + 72k^5 b^4\alpha^6\beta^6 - 72k^6 b^4\alpha^4\beta^8 + 95k^5\alpha^2\beta^{10}$$
$$+ 76k^6 b^2\alpha^2\beta^{10} + 16k^6 b^6\alpha^6\beta^6 - 24k^6\beta^{12} - 94k^4\alpha^4\beta^8 - 32k^5 b^6\alpha^8\beta^4,$$

$$F_1 = 94\epsilon k^2 b^2\alpha^8\beta^2 + 68\epsilon k^3 b^2\alpha^6\beta^4 + 80\epsilon k^3 b^4\alpha^8\beta^2 - 16\epsilon k^3 b^2\alpha^{10} - 87\epsilon k^2\alpha^6\beta^4$$
$$+ 160\epsilon k^4 b^4\alpha^6\beta^4 - 125\epsilon k^3\alpha^4\beta^6 - 8\epsilon k^2 b^4\alpha^{10} + 3\epsilon k\alpha^8\beta^2 - 354\epsilon k^4 b^2\alpha^4\beta^6 - 48\epsilon k^4 b^6\alpha^8\beta^2$$
$$+ 251\epsilon k^4\alpha^2\beta^8 - 232\epsilon k^5 b^4\alpha^4\beta^6 + 208\epsilon k^5 b^2\alpha^2\beta^8 + 48\epsilon k^5 b^6\alpha^6\beta^4 - 42\epsilon k^5\beta^{10},$$

$$F_2 = 234\epsilon^2 k^5 b^2\alpha^8\beta^2 - 206\epsilon^2 k^2\alpha^6\beta^4 - 286\epsilon k^3 b^2\alpha^6\beta^4 + 240\epsilon^2 k^3 b^4\alpha^8\beta^2 + 128\epsilon^2 k^3\alpha^4\beta^6$$
$$- 31\epsilon^2 k^3 b^6\alpha^{10} - 26\epsilon^2 k^4 b^2\alpha^{10} + \epsilon^2\alpha^{10} - 8\epsilon^2 k\alpha^8\beta^2 - 80\epsilon^2 k^4 b^4\alpha^6\beta^4 - 16k^3 b^4\alpha^{10}$$
$$- 74\epsilon^2 k^4 b^2\alpha^4\beta^6 - \epsilon^2 k^4 b^6\alpha^8\beta^2 + 133\epsilon^2 k^5\alpha^2\beta^8 - 144k^6 b^4\alpha^4\beta^6 + 152k^6 b^2\alpha^2\beta^8$$
$$+ 32k^6 b^6\alpha^6\beta^4 - 48k^6\beta^{10}.$$

The equation (4.10) yields two equations as follows:

$$\alpha^2 D_1(r_0 + s_0) + \beta E_1 r_{00} + 2\alpha^2 F_1 s_0 = 0, \tag{4.11}$$

$$\alpha^4\beta D_2(r_0 + s_0) + E_2 r_{00} + 2\alpha^2\beta F_2 s_0 = 0. \tag{4.12}$$

From (4.12), we obtain

$$-24k^6\beta^{12} r_{00} \equiv 0 (\operatorname{mod} \alpha^2). \tag{4.13}$$

Therefore, there exists a function $f(x)$ such that $r_{00} = \alpha^2 f(x)$. Thus, we have

$$r_{ij} = a_{ij} f(x). \tag{4.14}$$

Transvection by $b^i y^i$ leads to

$$r_0 = \beta f(x); \quad r_j = b_j f(x). \tag{4.15}$$

Eliminating (r_0+s_0) from (4.11) and (4.12), from (4.13), we have

$$\alpha^2 f(x)(\alpha^2\beta^2 D_2E_1 - D_1E_2) + 2\alpha^2\beta s_0(\alpha^2 D_2F_1 - D_1F_2) = 0. \tag{4.16}$$

From $\alpha^2 \neq 0 (\mathrm{mod}\,\beta)$ it follows that there exists a function $g(x)$ satisfying $s_0 = g(x)\beta$.

Hence (4.16) is reduced to

$$\alpha^2\beta^2(f(x)D_2E_1 + 2g(x)D_2F_1) - (f(x)D_1E_2 + 2\beta^2 g(x)D_1F_2) = 0. \tag{4.17}$$

Since only the term $-432\epsilon k^{11}(f(x)+4g(x))\beta^{22}$ of $(f(x)D_1E_2 + 2\beta^2 g(x)D_1F_2)$ seemingly does not contain α^2, we must have $hp(20)V_{20}$ such that $\beta^{22} = \alpha^2 V_{20}$. But it is a contradiction because of $\alpha^2 \neq 0(\mathrm{mod}\,\beta)$, that is, $(f(x)D_1E_2 + 2\beta^2 g(x)D_1F_2)$ does not contain α^2 as a factor. Hence $(f(x)D_1E_2 + 2\beta^2 g(x)D_1F_2)$ must be zero, which implies $f(x) = g(x) = 0$, which leads to $s_0 = 0$ and $s_i = 0$. From (4.14), we get $r_{ij} = 0$.

Summarizing up, we obtain $r_{ij} = 0$ and $s_i = 0$, that is,

$$b_{i;j} + b_{j;i} = 0, \quad b^r b_{r;i} = 0. \tag{4.18}$$

Therefore $b_i(x)$ is the so-called killing vector field with a constant length.

According to Hashiguchi, Hojo and Matsumoto [4], the condition (4.18) is equivalent to $b_{i;j} = 0$. So, we have

Theorem 4.1. *Let F^2 be a two-dimensional Finsler space with a special (α,β)-metric* (3.1) *satisfying $b^2 \neq 0$. If F^2 is a Landsberg space, then F^2 is a Berwald space.*

5. Conclusion

The present paper is devoted to finding a Landsberg space in a two-dimensional Finsler space F^2 with a special (α,β)-metric $L(\alpha,\beta) = \alpha + \epsilon\beta + \kappa\frac{\beta^2}{\alpha}$ satisfying some conditions, where ϵ, $\kappa \neq 0$ are real constants. First we find the condition for a Finsler space with a special (α,β)-metric (3.1) to be a Berwald space (see Theorem (3.1). Next, we determine the difference vector and the main scalar of F^2 with the aforesaid metric. Finally, we show that if the Finsler space F^2 with the metric (3.1) is a Landsberg space, then it becomes a Berwald space under some conditions.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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