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Research Article

Configuring Generalized Form of Half Companion Sequence of Special Dio-Triples involving Centered Triacontagonal Number

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Abstract. In this paper, the generalized half companion sequence of a special Dio-3-tuple covering the centred triacontagonal number of different ranks with different properties are produced. Also, we analyze the extendability of 3-tuples to quadruples for some ranks by performing a suitable algebraic algorithm and various enthralling patterns are presented using MATLAB.

Keywords. Extendability, Rank, Centered triacontagonal number, Dio-3-tuple, Perfect square

Mathematics Subject Classification (2020). 11B83

1. Introduction

Number theory acts as the soul of mathematics as it deals with numbers which play an effective role in counting, measuring and labelling objects. It is also a pure area of mathematics where integer solutions are highly concentrated. Diophantine equations are very fascinating topic in the field of number theory. Diophantus of Alexandria proposed that the set of numbers $\omega_1 = \frac{1}{16}$, $\omega_2 = \frac{33}{16}$, $\omega_3 = \frac{68}{16}$, $\omega_4 = \frac{105}{16}$, satisfying the condition

$$\omega_i \omega_j = Q^2 - 1, \quad \text{for all } i,j = 1,2,3,4,$$

where Q is a rational number. In addition to this, many research works have been carried out on forming a sequence of integers.

A Diophantine k-tuple is formed by a sequence of integers $\{z_1, z_2, \dots, z_k\}$ with property n if

$$z_m z_n + n = z^2, \quad \text{for all } m, n = 1, \dots, k \text{ with } m \neq n \text{ and } z \in Z.$$

Further, a special Dio-3-tuple is formed by three distinct set of polynomials $\{\delta_1, \delta_2, \delta_3\}$ whose coefficients are in $\mathbb{Z}$ with the property $D(t)$ such that $\delta_i \delta_j + \delta_i + \delta_j + t$ is a perfect square for all $i, j = 1, 2, 3$, where 't' denotes a non-zero integer or polynomial with coefficients belongs to $\mathbb{Z}$.

For the existence of integer z, Zhang and Grossman [10] proved necessary and sufficient conditions by considering the Diophantine triples $\{a_1, a_2, a_3\}$ so that $a_j a_k + z = c^2$, for all $k \neq j$, where $z \in \mathbb{Z}$. Extensibility of $D(4)$ pairs using Pellian equation is found by Baćić and Filipin in [4] whereas Earp-Lynch *et al.* [5], generalized their result to distinguish the solutions of Pellian equations corresponding to $D(l^2)$ Dio-3-tuples. Half companion sequences formed by Pell numbers are studied by Gopalan and Geetha [6]. Adédji *et al.* [2] dealt with the extensibility of the Diophantine 3-tuple $\{b_1, b_2, b_3\}$ with additional condition that $b_1 < b_2 < b_3$ with $b_1 = 3b_1$ and arrived at a result that such a set is not extended to a quadruple. They also showed that any Diophantine triple which includes the pair $\{b_1, 3b_1\}$ is regular and they obtained the equivalent result for $b_2 = 8b_1$.

Formation of Dio-3-tuples involving Pentatope number are discussed by Janaki and Saranya [7] whereas Vidhya and Gokila [9] studied that a special Dio-3-tuple formed by using nonagonal pyramidal number cannot be extended to a quadruple and the non-extendability of Diophantine triples containing centered square number is proved by Saranya and Janaki [8]. It is speculated that, suppose $\{k, k+1, a, b\}$ denotes a $D(-t)$ quadruple with the condition $a < b$, then $a = 1$, $b = 4t + 1$, in which $3t + 1$ must be a square number (Adžaga *et al.* [3]). On the other hand, Adedji *et al.* [1] postulates for the value of t of the form $t = h^2 - 1$, where $h \in \mathbb{Z}_+$ and $h \geq 3$.

Motivation of all the above works leads to present this paper on generalized half companion sequences of special Dio-3-tuples using centered triacontagonal number of different ranks $(p - \mu, p + \mu)$, $(p, p + \mu)$, $(p - \mu, p - \mu + 1)$, where $\mu \in \mathbb{Z}$ and with different properties which are discussed in Section 2 along with some numerical and graphical illustrations using MATLAB. Some theorems on validating the extendability of special Dio-3-tuples to quadruples for some different ranks are shown in Section 3.

2. Systematic Approach

In this section, three forms of distinct generalized half companion sequence is constructed by using suitable algebraic algorithms.

Definition 2.1. A *Centered Triacontagonal Number* (*CTG*) is a centered figurate number that gives the number of dots arranged in a triacontagon (30-sided polygon), with a dot in the center and all other dots surrounding the central dot in successive triacontagon layers. The general formula is given by $CTG_p = 15p^2 + 15p + 1$, where $p \in \mathbb{W}$ denotes the rank of CTG. First few CTG's are $1, 31, 91, \dots$.

Genre A. $(p, p + \mu)$: Let g_1, g_2 denotes the centered triacontagonal number of rank p and $p + \mu$, respectively,

$$g_1 = CTG_p = 15p^2 + 15p + 1, \tag{2.1}$$

$$g_2 = CTG_{p+\mu} = 15p^2 + (30\mu + 15)p + 15\mu^2 + 15\mu + 1. \tag{2.2}$$

On validating, it is found that $g_1g_2 + g_1 + g_2$ becomes a perfect square δ^2 (say) by adding the property $D(p)$, where $p = 1 - 45p^2\mu - 45p\mu - 45p\mu^2 + 6\mu^2 - 6\mu$.

Choose $g_3 \in \mathbb{Z}_+$ such that

$$g_1g_3 + g_1 + g_3 + 1 - 45p^2\mu - 45p\mu - 45p\mu^2 + 6\mu^2 - 6\mu = \Phi_1^2, \tag{2.3}$$

$$g_2g_3 + g_2 + g_3 + 1 - 45p^2\mu - 45p\mu - 45p\mu^2 + 6\mu^2 - 6\mu = \Phi_2^2, \tag{2.4}$$

where $\Phi_1 = l_1 + (g_1 + 1)m_1$ and $\Phi_2 = l_1 + (g_2 + 1)m_1$.

On solving eq. (2.3) and eq. (2.4) by elimination method, the value of l_1^2 results as

$$l_1^2 = -45p^2\mu - 45p\mu - 45p\mu^2 + 6\mu^2 - 6\mu + (g_1 + 1)(g_2 + 1)m_1^2. \tag{2.5}$$

Making use of eq. (2.1) and eq. (2.2) in eq. (2.5) leads to the initial solutions

$$l_1 = 15p^2 + 15p + 15p\mu + 6\mu + 2, \quad m_1 = 1.$$

Substitution of the above initial conditions in Φ_1 (or Φ_2) yields

$$\Phi_1 = 30p^2 + 30p + 15p\mu + 6\mu + 4. \tag{2.6}$$

The g_3 is obtained with the help of eqs. (2.1), (2.3) and (2.6) as follows

$$g_3 = 15\mu^2 + 27\mu + 60p^2 + p(60\mu + 60) + 7. \tag{2.7}$$

Equation (2.7) can also be represented as

$$g_3 = 2(CTG_p + CTG_{p+\mu}) - 15\mu^2 - 3\mu + 3.$$

In order to find $g_4 \in \mathbb{Z}_+$, g_2, g_3 are taken into account.

Consider

$$g_2g_4 + g_2 + g_4 + 1 - 45p^2\mu - 45p\mu - 45p\mu^2 + 6\mu^2 - 6\mu = \Psi_1^2, \tag{2.8}$$

$$g_3g_4 + g_3 + g_4 + 1 - 45p^2\mu - 45p\mu - 45p\mu^2 + 6\mu^2 - 6\mu = \Psi_2^2, \tag{2.9}$$

with $\Psi_1 = c_1 + (g_2 + 1)d_1$ and $\Psi_2 = c_1 + (g_3 + 1)d_1$.

Elimination of g_4 from eq. (2.8) and eq. (2.9) provides the value of c_1^2 as

$$c_1^2 = -45p^2\mu - 45p\mu - 45p\mu^2 + 6\mu^2 - 6\mu + (g_2 + 1)(g_3 + 1)d_1^2.$$

The following initial solutions are determined by using eq. (2.2) and eq. (2.7) in the above equation,

$$c_1 = 30p^2 + 15\mu^2 + 45p\mu + 21\mu + 30p + 4, \quad d_1 = 1 \tag{2.10}$$

Accomplishment of validating g_4 is enhanced by eqs. (2.2), (2.8) and (2.10) which is shown below:

$$g_4 = 60\mu^2 + 180p\mu + 84\mu + 135p^2 + 135p + 17. \tag{2.11}$$

Equation (2.11) implies

$$g_4 = 3CTG_p + 6CTG_{p+\mu} - 30\mu^2 - 6\mu + 8.$$

Similarly, select $g_5 \in \mathbb{Z}_+$. Further, g_5 is depicted with the help of g_3, g_4. Take

$$g_3g_5 + g_3 + g_5 + 1 - 45p^2\mu - 45p\mu - 45p\mu^2 + 6\mu^2 - 6\mu = \zeta_1^2, \tag{2.12}$$

$$g_4g_5 + g_4 + g_5 + 1 - 45p^2\mu - 45p\mu - 45p\mu^2 + 6\mu^2 - 6\mu = \zeta_2^2. \tag{2.13}$$

Suppose $\zeta_1 = h_1 + (g_3 + 1)f_1$ and $\zeta_2 = h_1 + (g_4 + 1)f_1$.

Eradicating g_5 from eqs. (2.12) and (2.13) and using eq. (2.7), eq. (2.11) in the resulting equation, the following initial solutions are generated,

$$h_1 = 90p^2 + 30\mu^2 + 105p\mu + 90p + 48\mu + 12, \quad f_1 = 1.$$

Applying the above values of h_1, f_1 and g_3 in ζ_1,

$$\zeta_1 = 150p^2 + 45\mu^2 + 165p\mu + 150p + 75\mu + 20. \tag{2.14}$$

Operating the eqs. (2.12), (2.14) and (2.7), g_5 is produced as follows

$$g_5 = 135\mu^2 + 450p\mu + 207\mu + 375p^2 + 375p + 49 \tag{2.15}$$

$$\Rightarrow \quad g_5 = 10CTG_p + 15CTG_{p+\mu} - 90\mu^2 - 18\mu + 24$$

$\{(CTG_p, CTG_{p+\mu}, 2(CTG_p + CTG_{p+\mu}) - 15\mu^2 - 3\mu + 3), (CTG_{p+\mu}, 2(CTG_p + CTG_{p+\mu}) - 15\mu^2 - 3\mu + 3, 3CTG_p + 6CTG_{p+\mu} - 30\mu^2 - 6\mu + 8), (2(CTG_p + CTG_{p+\mu}) - 15\mu^2 - 3\mu + 3, 3CTG_p + 6CTG_{p+\mu} - 30\mu^2 - 6\mu + 8, 10CTG_p + 15CTG_{p+\mu} - 90\mu^2 - 18\mu + 24), \dots\}$ represents a half companion sequence of special dio-triples that are formed with the help of centered triacontagonal number for the ranks $(p, p+\mu)$.

Table 1. A few examples of Half companion sequences of special Dio-3-tuples formed by $CTG_p, CTG_{p+\mu}$

Rank p	μ	(g_1, g_2, g_3)	(g_2, g_3, g_4)	(g_3, g_4, g_5)	$D(p)$
1	1	(31,91,229)	(91,229,611)	(229,611,1591)	$D(-134)$
	2	(31,181,361)	(181,361,1055)	(361,1055,2653)	$D(-347)$
	3	(31,301,523)	(301,523,1619)	(523,1619,3985)	$D(-638)$
	4	(31,451,715)	(451,715,2303)	(715,2303,5587)	$D(-1007)$
2	1	(91,181,529)	(181,529,1331)	(529,1331,3541)	$D(-359)$
	2	(91,301,721)	(301,721,1955)	(721,1955,5053)	$D(-887)$
	3	(91,451,943)	(451,943,2699)	(943,2699,6835)	$D(-1583)$
	4	(91,631,1195)	(631,1195,3563)	(1195,3563,8887)	$D(-2447)$

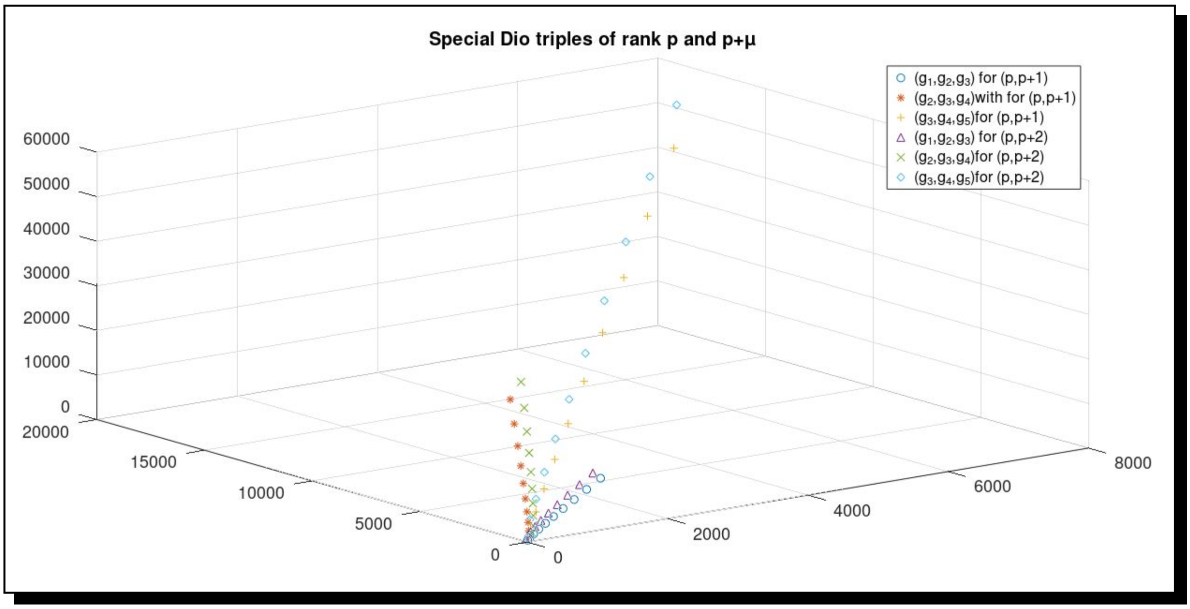

Figure 1. Scattered plot of special Dio-triples for the ranks $(p, p+1)$ and $(p, p+2)$

Genre B. $(p-\mu, p+\mu)$: Let the centered triacontagonal numbers of rank $p-\mu$ and $p+\mu$ be denoted by x_1, x_2 respectively and are given by

$$x_1 = CTG_{p-\mu} = 15p^2 - 30p\mu + 15\mu^2 + 15p - 15\mu + 1, \tag{2.16}$$

$$x_2 = CTG_{p+\mu} = 15p^2 + 30p\mu + 15\mu^2 + 15p + 15\mu + 1. \tag{2.17}$$

On calculating the value of $x_1x_2 + x_1 + x_2$, it is seen that a perfect square is obtained with the help of the property $D(105\mu^2+1)$.

Let x_3 be a non-zero integer such that the following equations hold with the assumptions $\omega_1 = i_1 + (x_1+1)j_1$ and $\omega_2 = i_i + (x_2+1)j_1$,

$$x_1x_3 + x_1 + x_3 + 105\mu^2 + 1 = \omega_1^2, \tag{2.18}$$

$$x_2x_3 + x_2 + x_3 + 105\mu^2 + 1 = \omega_2^2. \tag{2.19}$$

By eliminating x_3, the value of i_1^2 can be accomplished as

$$i_1^2 = 105\mu^2 + (x_1+1)(x_2+1)j_1^2. \tag{2.20}$$

Utilizing eqs. (2.16), (2.17) in (2.20), it leads to the initial solution

$$i_1 = 15p^2 - 15\mu^2 + 15p + 2, \quad j_1 = 1. \tag{2.21}$$

Equation (2.21) helps in finding ω_1 as

$$\omega_1 = 30p^2 + 30p - 30p\mu - 15\mu + 4. \tag{2.22}$$

On applying the known values in eq. (2.18),

$$x_3 = 60p^2 + 60p + 7 \tag{2.23}$$

$$\Rightarrow \quad x_3 = 2(CTG_{p-\mu} + CTG_{p+\mu}) - 60\mu^2 + 3$$

To find a non-zero positive integer x_4, both x_2, x_3 will be considered with the conditions $\Omega_1 = a_1 + (x_2+1)b_1$ and $\Omega_2 = a_1 + (x_3+1)b_1$ such that the following equations hold

$$x_2x_4 + x_2 + x_4 + 105\mu^2 + 1 = \Omega_1^2, \tag{2.24}$$

$$x_3x_4 + x_3 + x_4 + 105\mu^2 + 1 = \Omega_2^2. \tag{2.25}$$

The initial solution $a_1 = 30p^2 + 30p + 30p\mu + 15\mu + 4$, $b_1 = 1$ is obtained by elimination of x_4 from eqs. (2.24) and (2.25) and this helps in finding Ω_1 as

$$\Omega_1 = 45p^2 + 45p + 60p\mu + 15\mu^2 + 30\mu + 6. \tag{2.26}$$

With the help of eqs. (2.24) and (2.26), x_4 is generated as

$$x_4 = 15\mu^2 + 90p\mu + 45\mu + 135p^2 + 135p + 17 \tag{2.27}$$

$$\Rightarrow \quad x_4 = 3CTG_{p-\mu} + 6CTG_{p+\mu} - 120\mu^2 + 8$$

x_3, x_4 are used to detect some $x_5 \in \mathbb{Z}_+$ where

$$x_3x_5 + x_3 + x_5 + 105\mu^2 + 1 = \Delta_1^2, \tag{2.28}$$

$$x_4x_5 + x_4 + x_5 + 105\mu^2 + 1 = \Delta_2^2. \tag{2.29}$$

Assume that $\Delta_1 = q_1 + (x_3+1)r_1$ and $\Delta_2 = q_1 + (x_4+1)r_1$.

Removing x_5 from eqs. (2.28) and (2.29) leads to the foremost solution $q_1 = 90p^2 + 90p + 30p\mu + 15\mu + 12$ and $r_1 = 1$ and this solution gives the value of Δ_1 as $\Delta_1 = 150p^2 + 150p + 30p\mu + 15\mu + 20$. On using the value of Δ_1 and x_3 in eq. (2.28),

$$x_5 = 15\mu^2 + 75\mu + 375p^2 + 150p\mu + 375p + 49 \tag{2.30}$$

$$\Rightarrow \quad x_5 = 10CTG_{p-\mu} + 15CTG_{p+\mu} - 360\mu^2 + 20$$

Thus, the half companion sequence formed by the special Diophantine 3-tuples involving centered triacontagonal number of the ranks $(p-\mu, p+\mu)$ is given by $\{(CTG_{p-\mu}, CTG_{p+\mu}, 2(CTG_{p-\mu}+CTG_{p+\mu})-60\mu^2+3), (CTG_{p+\mu}, 2(CTG_{p-\mu}+CTG_{p+\mu})-60\mu^2+3, 3CTG_{p-\mu}+6CTG_{p+\mu}-120\mu^2+8), (2(CTG_{p-\mu}+CTG_{p+\mu})-60\mu^2+3, 3CTG_{p-\mu}+6CTG_{p+\mu}-120\mu^2+8, 10CTG_{p-\mu}+15CTG_{p+\mu}-360\mu^2+20), \dots\}$.

Table 2. Examples of half companion sequences of special Dio-3-tuples formed by $CTG_{p-\mu}$, $CTG_{p+\mu}$

Rank p	μ	$p-\mu$	$p+\mu$	(x_1,x_2,x_3)	(x_2,x_3,x_4)	(x_3,x_4,x_5)	$D(105\mu^2+1)$
1	1	0	2	(1,91,127)	(91,127,437)	(127,437,1039)	$D(106)$
2	1	1	3	(31,181,367)	(181,367,1067)	(367,1067,2689)	$D(106)$
	2	0	4	(1,301,367)	(301,367,1337)	(367,1337,3109)	$D(421)$
3	1	2	4	(91,301,727)	(301,727,1967)	(727,1967,5089)	$D(106)$
	2	1	5	(31,451,727)	(451,727,2327)	(727,2327,5659)	$D(421)$
	3	0	6	(1,631,727)	(631,727,2717)	(727,2717,6259)	$D(946)$
4	1	3	5	(181,451,1207)	(451,1207,3137)	(1207,3137,8239)	$D(106)$
	2	2	6	(91,631,1207)	(631,1207,3587)	(1207,3587,8959)	$D(421)$
	3	1	7	(31,841,1207)	(841,1207,4067)	(1207,4067,9709)	$D(946)$
	4	0	8	(1,1081,1207)	(1081,1207,4577)	(1207,4577,10489)	$D(1681)$

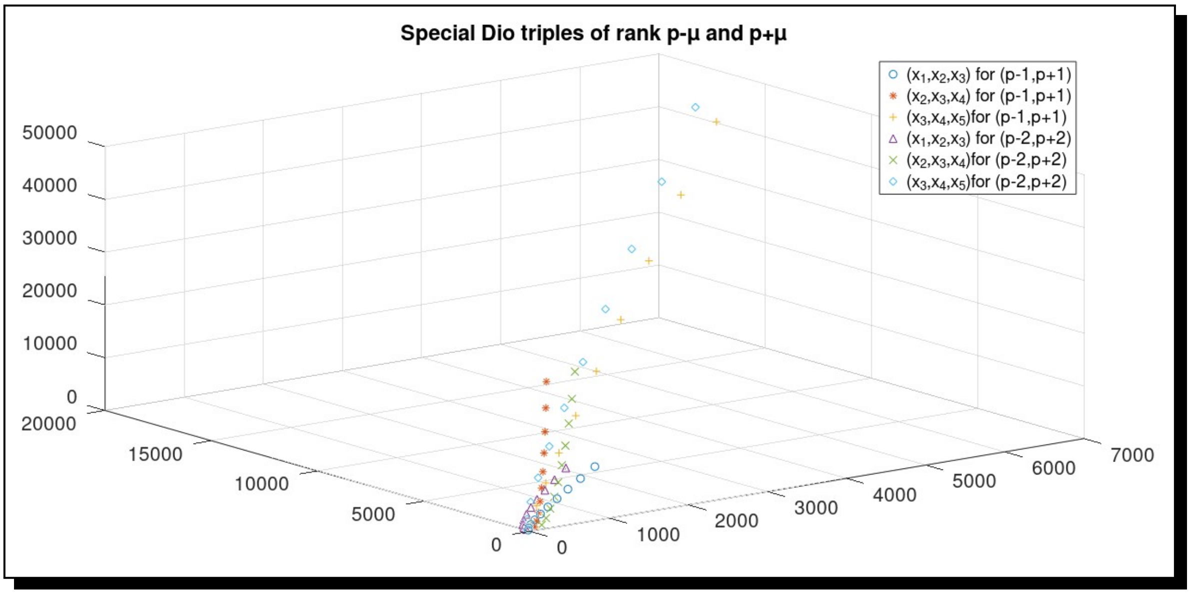

Figure 2. Scattered plot of special Dio-triples for the ranks $(p-1, p+1)$ and $(p-2, p+2)$

Genre C. $(p-\mu, p-\mu+1)$: Take the centered triacontagonal numbers of the ranks $p-\mu$ and $p-\mu+1$,

$$k_1 = CTG_{p-\mu} = 15p^2 - 30p\mu + 15p + 15\mu^2 - 15\mu + 1, \tag{2.31}$$

$$k_2 = CTG_{p-\mu+1} = 15p^2 - 30p\mu + 45p + 15\mu^2 - 45\mu + 31. \tag{2.32}$$

It is found that $k_1k_2 + k_1 + k_2$ becomes a perfect square on adding the property $D(a)$, where $a = 1 - 45p^2 - 45\mu^2 + 90p\mu - 90p + 90\mu)$.

Table 3. Examples of half companion sequences of special Dio-3-tuples formed by $CTG_{p-\mu}$, $CTG_{p-\mu+1}$

Rank p	μ	$p-\mu$	$p-\mu+1$	(k_1,k_2,k_3)	(k_2,k_3,k_4)	(k_3,k_4,k_5)	$D(a)$
1	-1	2	3	(91,181,529)	(181,529,1331)	(529,1331,3541)	$D(-359)$
	0	1	2	(31,91,229)	(91,229,611)	(229,611,1591)	$D(-134)$
	1	0	1	(1,31,49)	(31,49,161)	(49,161,391)	$D(1)$
2	-2	4	5	(301,451,1489)	(451,1489,3581)	(1481,3581,9691)	$D(-1079)$
	-1	3	4	(181,301,949)	(301,949,2321)	(949,2321,6241)	$D(-674)$
	0	2	3	(91,181,529)	(181,529,1331)	(529,1331,3541)	$D(-359)$
	1	1	2	(31,91,229)	(91,229,611)	(229,611,1591)	$D(-134)$
	2	0	1	(1,31,49)	(31,49,161)	(49,161,391)	$D(1)$
3	1	2	3	(91,181,529)	(181,529,1331)	(529,1331,3541)	$D(-359)$
	2	1	2	(31,91,229)	(91,229,611)	(229,611,1591)	$D(-134)$
	3	0	1	(1,31,49)	(31,49,161)	(49,161,391)	$D(1)$
4	1	3	4	(181,301,949)	(301,949,2321)	(949,2321,6241)	$D(-674)$
	2	2	3	(91,181,529)	(181,529,1331)	(529,1331,3541)	$D(-359)$
	3	1	2	(31,91,229)	(91,229,611)	(229,611,1591)	$D(-134)$
	4	0	1	(1,31,49)	(31,49,161)	(49,161,391)	$D(1)$
5	1	4	5	(301,451,1489)	(451,1489,3581)	(1481,3581,9691)	$D(-1079)$
	2	3	4	(181,301,949)	(301,949,2321)	(949,2321,6241)	$D(-674)$
	3	2	3	(91,181,529)	(181,529,1331)	(529,1331,3541)	$D(-359)$
	4	1	2	(31,91,229)	(91,229,611)	(229,611,1591)	$D(-134)$
	5	0	1	(1,31,49)	(31,49,161)	(49,161,391)	D(1)

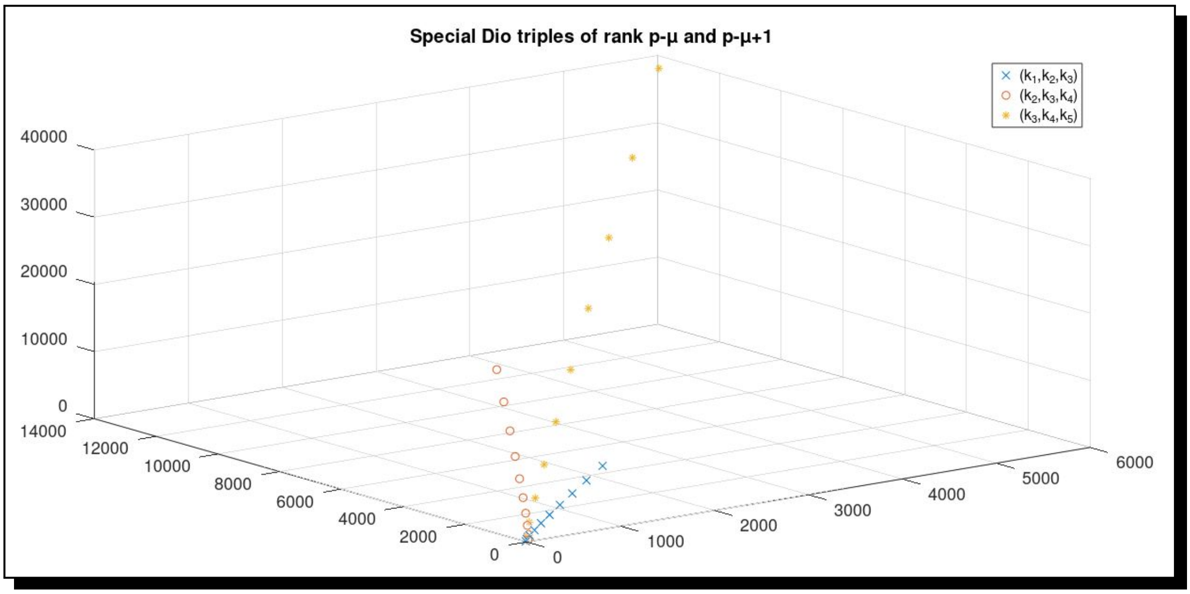

Figure 3. Displays the scattered plot of special Dio-triples for the rank $(p-1,p)$

The values k_3, k_4, k_5 which are given below are obtained by applying a similar procedure as in the previous cases,

$$k_3 = 2(CTG_{p-\mu} + CTG_{p-\mu+1}) - 15,$$

$$k_4 = 3CTG_{p-\mu} + 6CTG_{p-\mu+1} - 28,$$

$$k_5 = 10CTG_{p-\mu} + 15CTG_{p-\mu+1} - 84.$$

$\{(CTG_{p-\mu}, CTG_{p-\mu+1}, 2(CTG_{p-\mu} + CTG_{p-\mu+1}) - 15), (CTG_{p-\mu+1}, 2(CTG_{p-\mu} + CTG_{p-\mu+1}) - 15, 3CTG_{p-\mu} + 6CTG_{p-\mu+1} - 28), (2(CTG_{p-\mu} + CTG_{p-\mu+1}) - 15, 3CTG_{p-\mu} + 6CTG_{p-\mu+1} - 28, 10CTG_{p-\mu} + 15CTG_{p-\mu+1} - 84), \dots\}$ configures a half companion sequence. Some numerical illustrations are available in Table 3 and special Dio-3-tuples for the ranks $(p-1, p)$ are plotted in Figure 3.

3. Theorems on Extendibility

Theorem 3.1. *Special Diophantine triples formed by centered triacontagonal numbers of ranks p and $p+\mu$ cannot be extended to quadruples for all μ, p with both p, $\mu \neq 0$, simultaneously.*

Proof. Consider g_1, g_2, g_3 as in eqs. (2.1), (2.2), (2.7) and choose the property $D(p)$, where $p = 1 - 45p^2\mu - 45p\mu - 45p\mu^2 + 6\mu^2 - 6\mu$ as in *Genre* A.

Let $s \in \mathbb{Z}_+$ such that

$$g_1 s + g_1 + s + p = \Phi_1^2, \tag{3.1}$$

$$g_2 s + g_2 + s + p = \Phi_2^2, \tag{3.2}$$

$$g_3 s + g_3 + s + p = \Phi_3^2, \tag{3.3}$$

where $\Phi_1 = l_1 + (g_1+1)m_1$, $\Phi_2 = l_1 + (g_2+1)m_1$ and $\Phi_3 = l_1 + (g_3+1)m_1$.

Removal of 's' from eqs. (3.1) and (3.3) leads to the initial solutions

$$l_1 = 30p^2 + 30p + 15p\mu + 6\mu + 4, \quad m_1 = 1.$$

Φ_1 can be detected with the help of the above initial solutions and eq. (3.1),

$$\Phi_1 = 45p^2 + 15p\mu + 45p + 6\mu + 6. \tag{3.4}$$

Equation (3.1) generates the value of s by using eqs. (2.1) and (3.4) as

$$s = 15\mu^2 + 90p\mu + 39\mu + 135p^2 + 135p + 17.$$

Substituting all the known values in eq. (3.2) gives

$$\begin{aligned}\Phi_2^2 &= 2025p^4 + 225\mu^4 + 4050p^3 + 810\mu^3 + 2925p^2\mu^2 + 2565p^2 + 891\mu^2 + 7965p^2\mu \\ &\quad + 5400p^3\mu + 1800\mu^3 p + 4770p\mu^2 + 2025p^2\mu^2 + 342\mu + 3240p\mu + 540p + 36.\end{aligned} \tag{3.5}$$

If ω_2^2 is a perfect square, then the triples can be extended to quadruples. To check, the following cases are monitored.

Case 1: $\mu < p$

Suppose $mu = p-1$, then eq. (3.5) becomes

$$\Phi_2^2 = 14400p^4 - 4005p^3 - 1539p^2 + 360p \neq \text{perfect square}.$$

Case 2: $\mu > p$

Take $\mu = p+1$ in eq. (3.5),

$$\Phi_2^2 = 14400p^4 + 39195p^3 + 38331p^2 + 15804p + 2304 \neq \text{perfect square}.$$

Case 3: $\mu = p$
Put $\mu = p$ in eq. (3.5),

$$\Phi_2^2 = 14400p^4 + 17595p^3 + 6696p^2 + 882p + 36 \neq \text{perfect square.}$$

For all values of μ, p with $\mu = p \neq 0$, it is found that a special Dio-3-tuples formed by centered triacontagonal number cannot be extendable to quadruples.
Hence the theorem. □

Corollary 3.1. *If $\mu = p = 0$, then the special Dio-3-tuples that are generated by CTG_p and $CTG_{p+\mu}$ are extendable to quadruples.*

Theorem 3.2. *For all values of p, μ with $\mu = p \neq 0$, the special Dio-3-tuples generated by $CTG_{p-\mu}$ and $CTG_{p+\mu}$ are not extendable to quadruples.*

Proof. Consider x_1, x_2, x_3 as in *Genre* B. Choose $z \in \mathbb{Z}_+$ so that the following equations hold,

$$x_1 z + x_1 + z + t = \omega_1^2, \tag{3.6}$$

$$x_2 z + x_2 + z + t = \omega_2^2, \tag{3.7}$$

$$x_3 z + x_3 + z + t = \omega_3^2. \tag{3.8}$$

Here $D(t)$ denotes the property as in *Genre* B with $t = 105\mu^2 + 1$. Assume $\omega_1 = i_1 + (x_1 + 1)j_1$, $\omega_2 = i_1 + (x_2 + 1)j_1$ and $\omega_3 = i_1 + (x_3 + 1)j_1$.
By the similar procedure in Theorem 3.1, z and ω_2^2 are obtained as

$$z = 15\mu^2 - 90p\mu - 45\mu + 135p^2 + 135p + 17,$$

$$\begin{aligned} \omega_2^2 = {} & 2025p^4 + 225\mu^4 - 2475p^2\mu^2 + 2700p^3\mu + 4050p^3 + 4050p^2\mu + 2565p^2 - 900p\mu^3 \\ & - 450p\mu^2 + 1710p\mu + 540p - 450\mu^3 + 2025p^2\mu^2 - 270\mu^2 + 180\mu + 36. \end{aligned} \tag{3.9}$$

In order to check whether ω_2^2 is a perfect square, consider the following cases.

Case 1: $\mu < p$
If $\mu = p - 1$, then eq. (3.9) turns to

$$\omega_2^2 = 3600p^4 + 7650p^3 - 1395p^2 - 639 \neq \text{perfect square.}$$

Case 2: $\mu > p$
Assume $\mu = p + 1$, then eq. (3.9),

$$\omega_2^2 = 3600p^4 + 7650p^3 + 4905p^2 + 540p - 279 \neq \text{perfect square.}$$

Case 3: $\mu = p \neq 0$

$$\omega_2^2 = 3600p^4 + 7200p^3 + 4005p^2 + 720p + 36 \neq \text{perfect square.}$$

In all the above cases, it is found that special dio-triples produced by $CTG_{p-\mu}$ and $CTG_{p+\mu}$ are not extendable for all μ, p with $\mu = p \neq 0$. This proves the theorem. □

Corollary 3.2. *Suppose that $\mu = p = 0$, then the special Dio-3-tuples that are formed by $CTG_{p-\mu}$ and $CTG_{p+\mu}$ are extendable to quadruples.*

Theorem 3.3. *Let (k_1, k_2, k_3) be a special Diophantine triple formed by centered triacontagonal number of ranks $p - \mu$ and $p - \mu + 1$, then (k_1, k_2, k_3) cannot be extendable to a quadruple for all $p < \mu$ and $p > \mu$ but it is extendable for all $\mu = p$.*

Proof. Assume k_1, k_2, k_3 as in *Genre* C. Suppose r is a positive integer such that

$$k_1 r + k_1 + r + a = \sigma_1^2, \tag{3.10}$$

$$k_2 r + k_2 + r + a = \sigma_2^2, \tag{3.11}$$

$$k_3 r + k_3 + r + a = \sigma_3^3, \tag{3.12}$$

where $a = 1 - 45p^2 - 45\mu^2 + 90p\mu - 90p + 90\mu$ is taken as in *Genre* C and $\sigma_1 = \rho_1 + (k_1 + 1)t_1$, $\sigma_2 = \rho_1 + (k_2 + 1)t_1$ and $\sigma_3 = \rho_1 + (k_3 + 1)t_1$.

Eliminating 'r' from eqs. (3.10) and (3.12) provides the initial solution $\rho_1 = 30p^2 + 30\mu^2 - 60p\mu - 45\mu + 45p + 10$, $t_1 = 1$.

This gives the value of σ_1 as

$$45p^2 - 90p\mu + 45\mu^2 - 60\mu + 60p + 12.$$

Applying all the values which are obtained before in eq. (3.10), r can be deduced as

$$r = 135\mu^2 - 270p\mu - 225\mu + 135p^2 + 225p + 71.$$

Substitution of all known values in eq. (3.11) gives

$$\begin{aligned} \sigma_2^2 = {} & 2025\mu^4 - 8100\mu^3 p - 9450\mu^3 + 12150\mu^2 p^2 + 28350\mu^2 p \\ & + 15480\mu^2 - 8100\mu p^3 - 28350\mu p^2 - 30960\mu p - 10350\mu \\ & + 2025p^4 + 9450p^3 + 15480p^2 + 10350p + 2304. \end{aligned} \tag{3.13}$$

Let $p > \mu$. Take $\mu = p - 1$ in eq. (3.13). This shows that σ_2^2 is not a perfect square. Similarly for $p < \mu$, substitution of $\mu = p + 1$ in eq. (3.13) leads to the result that $\sigma_2^2 \neq$ perfect square. Suppose that $p = \mu$), then eq. (3.13) becomes $\sigma_2^2 = 2304$ which is a perfect square. Thus, it is clear that for all values of μ, p satisfying the condition $\mu = p$, the special Dio-triple (k_1, k_2, k_3) is extended to a quadruple (k_1, k_2, k_3, r).

Hence the theorem. □

4. Conclusion

An effort is made to find the generalized version of half companion sequence of special Diophantine triples using centered triacontagonal number of different ranks and their extendability to quadruples are detected for some ranks. It is analysed from the table that all the numbers in the sequence are odd. Also, it is found that the path traced by all the scattered plot of triples are similar on observing all the figures.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] K. N. Adédji, A. Filipin and A. Togbé, The extension of the $D(-k)$-triple $\{1, k, k+1\}$ to a quadruple, *Acta Mathematica Hungarica* **166** (2022), 407 – 422, DOI: 10.1007/s10474-022-01216-3.

[2] K. N. Adédji, B. He, Á. Pintér and A. Togbé, On the diophantine pair $\{a,3a\}$, *Journal of Number Theory* **227** (2021), 330 – 351, DOI: 10.1016/j.jnt.2021.03.011.

[3] N. Adžaga, A. Filipin and Y. Fujita, The extension of the $D(-k)$-pair k, k +1 to a quadruple, *Periodica Mathematica Hungarica* **85** (2021), 148 – 163, DOI: 10.1007/s10998-021-00424-8.

[4] L. Baćić and A. Filipin, The extensibility of $D(4)$-pairs, *Mathematical Communications* **18** (2013), 447 – 456, URL: https://hrcak.srce.hr/file/163349.

[5] B. Earp-Lynch, S. Earp-Lynch and O. Kihel, On certain $D(9)$ and $D(64)$ diophantine triples, *Acta Mathematica Hungarica* **162** (2020), 483 – 517, DOI: 10.1007/s10474-020-01061-2.

[6] M. A. Gopalan and V. Geetha, Observations on half-companion pell-numbers, *International Journal of Scientific and Research Publications* **4**(2) (2014), 6 pages, URL: https://www.ijsrp.org/research-paper-0214.php?rp=P262257.

[7] G. Janaki and C. Saranya, Construction of the diophantine triple involving pentatope number, *International Journal for Research in Applied Science & Engineering Technology* **6** (2018), 2317 – 2319, DOI: 10.22214/ijraset.2018.3535.

[8] C. Saranya and G. Janaki, Some non-extendable diophantine triples involving centered square numbers, *International Journal of Scientific Research in Mathematical and Statistical Sciences* **6** (2019), 105 – 107.

[9] S. Vidhya and T. Gokila, Non-extendability of special Dio-3-tuples involving nonaganal pyramidal number, *International Journal for Research in Applied Science & Engineering Technology* **11** (2023), 889 – 894, DOI: 10.22214/ijraset.2023.49096.

[10] Y. Zhang and G. Grossman, On diophantine triples and quadruples, *Notes on Number Theory and Discrete Mathematics* **21**(4) (2015), 6 – 16, URL: https://nntdm.net/papers/nntdm-21/NNTDM-21-4-06-16.pdf.