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Research Article

Results on Invariant Submanifolds of Hyperbolic Kenmotsu Manifolds

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Abstract. In this paper, our main focus was on studying the geodesic property of invariant submanifolds of Hyperbolic Kenmotsu manifolds. We also investigated different conditions regarding the second fundamental form π and established its equivalence. These conditions included being 2-semiparallel, pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel and 2-Ricci-generalized pseudoparallel.

Keywords. Hyperbolic Kenmotsu manifold, Invariant submanifold, Totally geodesic submanifold

Mathematics Subject Classification (2020). 53C22, 53C17, 53C15

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1. Introduction

The notion of invariant submanifolds plays a crucial role in differential geometry, as it allows us to explore and understand the geometric properties of a manifold under certain transformations or symmetries. Hyperbolic Kenmotsu manifolds are a particular class of Riemannian manifolds that generalize the concept of Kenmotsu manifolds. They possess a special type of metric tensor known as a Kenmotsu metric, which is closely related to the hyperbolic metric. Understanding the existence and properties of invariant submanifolds in hyperbolic Kenmotsu manifolds has significant implications in various fields of mathematics and physics. Specifically, in mathematical physics, invariant submanifolds often arise in the context of gauge theories, where they represent physically meaningful states or configurations.

The concept of Hyperbolic Kenmotsu manifolds has been extensively studied by various authors in different contexts. In a study conducted in 2021 by Pankaj *et al.* [10], the properties of Yamabe metric on 3-dimensional hyperbolic Kenmotsu manifolds were examined. Recently, in 2022, Chaubey *et al.* [5] conducted a study examining the characteristics of invariant submanifolds within hyperbolic Sasakian manifolds. They established that a three-dimensional submanifold is completely geodesic if and only if it is invariant.

In addition, Gill and Dube [8] studied the basic properties, integrability, and totally geodesicity of generalized CR-submanifolds of the trans hyperbolic Sasakian manifold were studied. In a separate investigation, Siddiqi and Akyol [12] studied the concept of anti-invariant $\xi^{\perp}$-Riemannian submersions from hyperbolic β-Kenmotsu manifolds onto Riemannian manifolds was introduced and examined. The study established necessary and sufficient conditions for a specific anti-invariant $\xi^{\perp}$-Riemannian submersion deemed to be totally geodesic.

Invariant submanifolds can be used in digital design to create smooth and deformable shapes, such as character animation in video games or computer-generated images. By defining invariant submanifolds, designers can ensure that specific parts of the shape remain unchanged during deformation, resulting in more realistic and visually appealing animations. Invariant submanifolds find applications in medical imaging, particularly in analyzing and segmenting anatomical structures from imaging data. By identifying and tracking invariant submanifolds in medical images, researchers and clinicians can accurately characterize and measure distinct anatomical features, leading to precise diagnosis, treatment planning, and analysis of disease progression. Invariant submanifolds can be used in graphic design to create custom shapes and patterns. By manipulating the parameters that define the invariant submanifolds, designers can generate aesthetically pleasing designs that possess specific symmetries or transformations. This can be particularly useful for logo design, textile patterns or architectural elements.

This introduction aims to provide a brief overview of invariant submanifolds, highlighting their importance and relevance in the field of differential geometry and its applications. In Section 2, we will explore some aspects of invariant submanifolds, including definitions and fundamental properties of invariant submanifolds and hyperbolic Kenmotsu manifold. In Section 3, we have derived the main results of invariant submanifold of hyperbolic Kenmotsu manifold. In Section 4, we concluded our results.

2. Preliminaries

In a manifold M with $(2n+1)$ dimensions, possessing the properties of *almost hyperbolic contact manifold* (AHCM), there exists a fundamental tensor field ϕ of type $(1,1)$, along with other key components such as a unit time-like vector field ξ, a 1-form η, we have

$$\phi^2 = \mathcal{I} + \eta \otimes \xi, \quad \eta(\xi) = -1, \quad \eta \circ \phi = 0, \quad \phi(\xi) = 0, \quad \operatorname{rank}(\phi) = 2n, \tag{2.1}$$

it includes identity endomorphism of the tangent bundle of $\widetilde{\mathcal{M}}^{2n+1}$ by $\mathcal{I}$ and the tensor product as $\otimes$. An AHCM $\widetilde{\mathcal{M}}^{2n+1}$ is said to be an *almost hyperbolic contact metric manifold* (AHCMM), is characterized by the property that its semi-Riemannian metric g satisfies the following conditions:

$$g(\phi\mathcal{E}_1,\phi\mathcal{E}_2)=-g(\mathcal{E}_1,\mathcal{E}_2)-\eta(\mathcal{E}_1)\eta(\mathcal{E}_2), \tag{2.2}$$

$$g(\phi\mathcal{E}_1,\mathcal{E}_2)=-g(\mathcal{E}_1,\phi\mathcal{E}_2),\ g(\mathcal{E}_1,\xi)=\eta(\mathcal{E}_1), \tag{2.3}$$

for any arbitrary vector fields $\mathcal{E}_1$ and $\mathcal{E}_2$. A structure denoted as (ϕ,ξ,η,g) defined on the manifold $\widetilde{\mathcal{M}}^{2n+1}$ is referred to as an almost hyperbolic contact metric structure. An AHCMM $\widetilde{\mathcal{M}}^{2n+1}$ is called a hyperbolic *Kenmotsu manifold* (HKM) (Bhatt and Dube [4]) if below mentioned equations are satisfied,

$$(\nabla_{\mathcal{E}_1}\phi)\mathcal{E}_2=g(\phi\mathcal{E}_1,\mathcal{E}_2)\xi-\eta(\mathcal{E}_2)\phi\mathcal{E}_1, \tag{2.4}$$

$$\nabla_{\mathcal{E}_1}\xi=-\mathcal{E}_1-\eta(\mathcal{E}_1)\xi, \tag{2.5}$$

which gives

$$(\nabla_{\mathcal{E}_1}\phi)\mathcal{E}_2=g(\phi\mathcal{E}_1,\phi\mathcal{E}_2)=-g(\mathcal{E}_1,\mathcal{E}_2)-\eta(\mathcal{E}_1)\eta(\mathcal{E}_2). \tag{2.6}$$

Lemma 2.1 ([10])**.** *On a $(2n+1)$-dimensional HKM $\widetilde{\mathcal{M}}^{2n+1}$, we have*

(i) $\mathcal{R}(\mathcal{E}_1,\mathcal{E}_2)\xi=\eta(\mathcal{E}_2)\mathcal{E}_1-\eta(\mathcal{E}_1)\mathcal{E}_2$,

(ii) $\mathcal{R}(\mathcal{E}_1,\xi)\xi=-\mathcal{E}_1-\eta(\mathcal{E}_1)\xi$,

(iii) $\mathcal{R}(\xi,\mathcal{E}_1)\mathcal{E}_2=g(\mathcal{E}_1,\mathcal{E}_2)\xi-\eta(\mathcal{E}_2)\mathcal{E}_1$,

(iv) $\mathcal{S}(\mathcal{E}_1,\xi)=2n\eta(\mathcal{E}_1)$, $\mathcal{S}(\xi,\xi)=-2n$ *and* $\mathcal{Q}\xi=2n\xi$,

where curvature tensor, Ricci tensor and Ricci operator is denoted by $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{Q}$ respectively on $\widetilde{\mathcal{M}}^{2n+1}$.

An *invariant submanifold* (IS) $\mathcal{M}$ of a HKM $\widetilde{\mathcal{M}}^{2n+1}$ is characterized by the property that for every point $\mathcal{E}_1\in\mathcal{M}$, the tangent space $\mathcal{T}_{\mathcal{E}_1}\mathcal{M}$ is preserved under the action of the structure vector field ϕ. This implies that the characteristic vector field ξ becomes tangent to $\mathcal{M}$.

For an IS in a HKM, the $(0,i)$-tensor field $\mathcal{T}$ satisfies the condition

$$\pi(\mathcal{E},\xi)=0, \tag{2.7}$$

where $\mathcal{E}$ is any vector tangent to $\widetilde{\mathcal{M}}^{2n+1}$.

On a Riemannian manifold $(\mathcal{M},g)$ with the Levi-Civita connection ∇, we take $\nabla^q\mathcal{T}$ as the covariant differentiation of the qth order of a $(0,i)$-tensor field $\mathcal{T}$, where $i\geq 1$.

A tensor field $\mathcal{T}$ is recurrent if the below mentioned equations satisfied on the entire manifold $\mathcal{M}$ (Roter [11]),

$$(\nabla\mathcal{T})(\mathcal{E}_1,\mathcal{E}_2,\dots,\mathcal{E}_k;\mathcal{E})\mathcal{T}(\mathcal{F}_1,\mathcal{F}_2,\dots,\mathcal{F}_k)=(\nabla\mathcal{T})(\mathcal{F}_1,\dots,\mathcal{F}_k;\mathcal{E})\mathcal{T}(\mathcal{F}_1,\dots,\mathcal{F}_k) \tag{2.8}$$

and

$$(\nabla^2\mathcal{T})(\mathcal{E}_1,\mathcal{E}_2,\dots,\mathcal{E}_k;\mathcal{E},\mathcal{F})\mathcal{T}(\mathcal{F}_1,\mathcal{F}_2,\dots,\mathcal{F}_k)=(\nabla^2\mathcal{T})(\mathcal{F}_1,\mathcal{F}_2,\dots,\mathcal{F}_k;\mathcal{E},\mathcal{F})\mathcal{T}(\mathcal{E}_1,\mathcal{E}_2,\dots,\mathcal{E}_k), \tag{2.9}$$

respectively, where $\mathcal{E},\mathcal{F},\mathcal{E}_1,\mathcal{F}_1,\dots,\mathcal{E}_k,\mathcal{F}_k\in\mathcal{T}\mathcal{M}$. By examining equation (2.8) at a specific point $x\in\mathcal{M}$ in the case where the tensor $\mathcal{T}$ is non-zero, there exists a singular 1-form ϕ and a $(0,2)$-tensor ψ. These are uniquely defined within a neighborhood $\mathcal{U}$ of $\mathcal{E}_1$, satisfying the following equations:

$$\nabla\mathcal{T}=\mathcal{T}\otimes\phi,\quad \phi=d(\log\|\mathcal{T}\|), \tag{2.10}$$

$$\nabla^2\mathcal{T}=\mathcal{T}\otimes\psi. \tag{2.11}$$

These equations hold on $\mathcal{U}$, where the norm of $\mathcal{T}$ and $\|\mathcal{T}\|^2 = g(\mathcal{T},\mathcal{T})$ represented by $\|\mathcal{T}\|$. A tensor $\mathcal{T}$ is considered to be *generalized* 2*-recurrent* (G-2-R) if

$$((\nabla^2\mathcal{T})(\mathcal{E}_1,\dots,\mathcal{E}_k;\mathcal{E},\mathcal{F}) - (\nabla\mathcal{T}\otimes\phi)(\mathcal{E}_1,\dots,\mathcal{E}_k;\mathcal{E},\mathcal{F}))\mathcal{T}(\mathcal{F}_1,\dots,\mathcal{F}_k) \tag{2.12}$$

$$= ((\nabla^2\mathcal{T})(\mathcal{F}_1,\dots,\mathcal{F}_k;\mathcal{E},\mathcal{F}) - (\nabla\mathcal{T}\otimes\phi)(\mathcal{F}_1,\dots,\mathcal{F}_k;\mathcal{E},\mathcal{F}))\mathcal{T}(\mathcal{E}_1,\dots,\mathcal{E}_k). \tag{2.13}$$

The formulas of Gauss and Weingarten are (Chen [6]),

$$\tilde{\nabla}_{\mathcal{E}}\mathcal{F} = \nabla_{\mathcal{E}}\mathcal{F} + \pi(\mathcal{E},\mathcal{F}), \tag{2.14}$$

$$\tilde{\nabla}_{\mathcal{E}}\mathcal{N} = -\mathcal{A}_{\mathcal{N}}\mathcal{E} + \nabla^{\perp}_{\mathcal{E}}\mathcal{N}. \tag{2.15}$$

Consider the tangent vector fields $\mathcal{E},\mathcal{F}$ and denote the normal vector field as $\mathcal{N}$ on $\mathcal{M}$. We denote the second fundamental form as π, the shape operator as $\mathcal{A}$, and normal connection as $\nabla^{\perp}$. The manifold is said to be *totally geodesic* (TG) if π is identically zero.

The relationship between the π and $\mathcal{A}_{\mathcal{N}}$ is represented as:

$$\tilde{g}(\pi(\mathcal{E},\mathcal{F}),\mathcal{N}) = g(\mathcal{A}_{\mathcal{N}}\mathcal{E},\mathcal{F}), \tag{2.16}$$

where $\tilde{g}$ denotes metric on $\mathcal{M}$. This equation relates the components of π and $\mathcal{A}_{\mathcal{N}}$ with respect to the tangent vector fields $\mathcal{E}$ and $\mathcal{F}$.

The first and second covariant derivatives of the π can be obtained as follows:

$$(\tilde{\nabla}_{\mathcal{E}}\pi)(\mathcal{F},\mathcal{Z}) = \nabla^{\perp}_{\mathcal{E}}(\pi(\mathcal{E},\mathcal{Z})) - \pi(\nabla_{\mathcal{E}}\mathcal{F},\mathcal{Z}) - \pi(\mathcal{F},\nabla_{\mathcal{E}}\mathcal{Z}), \tag{2.17}$$

$$(\tilde{\nabla}^2\pi)(\mathcal{Z},\mathcal{W},\mathcal{E},\mathcal{F}) = (\tilde{\nabla}_E\tilde{\nabla}_{\mathcal{F}}\pi)(\mathcal{Z},\mathcal{W}) \tag{2.18}$$

$$= \nabla^{\perp}_{\mathcal{E}}((\tilde{\nabla}_{\mathcal{F}}\pi)(\mathcal{Z},\mathcal{W})) - (\tilde{\nabla}_{\mathcal{F}}\pi)(\nabla_{\mathcal{E}}\mathcal{Z},\mathcal{W}) - (\tilde{\nabla}_{\mathcal{E}}\pi)(\mathcal{Z},\nabla_{\mathcal{F}}\mathcal{W}) - (\tilde{\nabla}_{\nabla_{\mathcal{E}}\mathcal{F}}\pi)(\mathcal{Z},\mathcal{W}) \tag{2.19}$$

respectively, where the van der Waerden-Bortolotti connection of $\mathcal{M}$ denoted by $\tilde{\nabla}$ (Chen [6]). $\mathcal{M}$ is said to have parallel π if $\tilde{\nabla}\pi = 0$ [6]. Further, we define endomorphism $\mathcal{R}(\mathcal{E},\mathcal{F})$ and $\mathcal{E}\wedge_B\mathcal{F}$ of $\chi(\mathcal{M})$ by

$$\mathcal{R}(\mathcal{E},\mathcal{F})\mathcal{Z} = \nabla_{\mathcal{E}}\nabla_{\mathcal{F}}\mathcal{Z} - \nabla_{\mathcal{F}}\nabla_{\mathcal{E}}\mathcal{Z} - \nabla_{[\mathcal{E},\mathcal{F}]}\mathcal{Z} \tag{2.20}$$

and

$$(\mathcal{E}\wedge_{\mathcal{B}}\mathcal{F})\mathcal{Z} = \mathcal{B}(\mathcal{F},\mathcal{Z})\mathcal{E} - \mathcal{B}(\mathcal{E},\mathcal{Z})\mathcal{F}, \tag{2.21}$$

respectively, where $\mathcal{E},\mathcal{F},\mathcal{Z}\in\chi(\mathcal{M})$ and symmetric $(0,2)$-tensor field on $(\mathcal{M},g)$ is denoted by $\mathcal{B}$. The tensor $\mathcal{Q}(\mathcal{B},\mathcal{T})$ is defined by

$$\mathcal{Q}(\mathcal{B},\mathcal{T})(\mathcal{E}_1,\dots,\mathcal{E}_k;\mathcal{E},\mathcal{F}) = -(\mathcal{T}(\mathcal{E}\wedge_B F)\mathcal{E}_1,\dots,\mathcal{E}_k) - \dots - \mathcal{T}(\mathcal{E}_1,\dots,\mathcal{E}_{k-1}(\mathcal{E}\wedge_B\mathcal{F})\mathcal{E}_k). \tag{2.22}$$

By substituting $\mathcal{T} = \pi$, $\tilde{\nabla}\pi$, $\mathcal{B} = g$, and $\mathcal{B} = \mathcal{S}$ into the given formula, these tensors $\mathcal{Q}(g,\pi)$, $\mathcal{Q}(\mathcal{S},\pi)$, $\mathcal{Q}(g,\tilde{\nabla}\pi)$ and $\mathcal{Q}(\mathcal{S},\tilde{\nabla}\pi)$ can be calculated.

Definition 2.1. An immersion is considered to be *semiparallel* (SP) (Deprez [7]), 2*-semiparallel* (2-SP) (Özgür and Murathan [1]), *pseudoparallel* (PP) (Asperti *et al.* [3]), 2*-pseudoparallel* (2-PP) (Özgür and Murathan [1]) and *Ricci-generalized pseudoparallel* (RGPP) (Murathan *et al.* [9]) respectively, for every vector fields $\mathcal{E},\mathcal{F}$ tangent to $\mathcal{M}$ the following specified conditions are satisfied,

$$\tilde{\mathcal{R}}\cdot\pi = 0, \tag{2.23}$$

$$\tilde{\mathcal{R}}\cdot\tilde{\nabla}\pi = 0, \tag{2.24}$$

$$\tilde{\mathcal{R}}\cdot\pi = \mathcal{K}_1\mathcal{Q}(g,\pi), \tag{2.25}$$

$$\tilde{\mathcal{R}}\cdot\widetilde{\nabla}\pi=\mathcal{K}_1\mathcal{Q}(g,\widetilde{\nabla}\pi), \tag{2.26}$$

$$\tilde{\mathcal{R}}\cdot\pi=\mathcal{K}_2\mathcal{Q}(\mathcal{S},\pi), \tag{2.27}$$

where $\tilde{R}$ represents the curvature tensor corresponding to the connection $\widetilde{\nabla}$.

Definition 2.2. An immersion is 2-RGPP if it satisfies the equation

$$\tilde{\mathcal{R}}\cdot\widetilde{\nabla}\pi=\mathcal{K}_2\mathcal{Q}(\mathcal{S},\widetilde{\nabla}\pi), \tag{2.28}$$

where $\mathcal{K}_1$ and $\mathcal{K}_2$ are functions that depend on π and $\widetilde{\nabla}\pi$. By (2.23), we have

$$(\tilde{\mathcal{R}}(\mathcal{E},\mathcal{F})\cdot\pi)(\mathcal{X}_1,\mathcal{X}_2)=\mathcal{R}^{\perp}(\mathcal{E},\mathcal{F})\pi(\mathcal{X}_1,\mathcal{X}_2)-\pi(\mathcal{R}(\mathcal{E},\mathcal{F})\mathcal{X}_1,\mathcal{X}_2)-\pi(\mathcal{X}_1,\mathcal{R}(\mathcal{E},\mathcal{F})\mathcal{X}_2), \tag{2.29}$$

for every vector fields $\mathcal{E},\mathcal{F},\mathcal{X}_1$ and $\mathcal{X}_2$ tangent to $\mathcal{M}$. Similarly,

$$\begin{aligned}(\tilde{R}(\mathcal{E},\mathcal{F})\cdot\widetilde{\nabla}\pi)(\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3)&=\mathcal{R}^{\perp}(\mathcal{E},\mathcal{F})(\widetilde{\nabla}\pi)(\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3)-(\widetilde{\nabla}\pi)(R(\mathcal{E},\mathcal{F})\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3)\\&\quad-(\widetilde{\nabla}\pi)(\mathcal{X}_1,\mathcal{R}(\mathcal{E},\mathcal{F})\mathcal{X}_2,\mathcal{X}_3)-(\widetilde{\nabla}\pi)(\mathcal{X}_1,\mathcal{X}_2,\mathcal{R}(\mathcal{E},\mathcal{F})\mathcal{X}_3),\end{aligned} \tag{2.30}$$

for all vector fields $\mathcal{E},\mathcal{F},\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3$ tangent to $\mathcal{M}$, where $(\widetilde{\nabla}\pi)(\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3)=(\widetilde{\nabla}_{\mathcal{X}_1}\pi)(\mathcal{X}_2,\mathcal{X}_3)$ (Arslan *et al.* [2]).

3. Totally Geodesic Properties on Invariant Submanifold (IS) of Hyperbolic Kenmotsu Manifold (HKM)

In this study, we focus our attention on examining the *invariant submanifold* (IS) of HKM that satisfy certain conditions. By investigating these specific submanifolds, we aim to gain a deeper understanding of the properties and characteristics of HKM,

(i) $\tilde{\mathcal{R}}\cdot\widetilde{\nabla}\pi=0$,

(ii) $\tilde{\mathcal{R}}\cdot\pi=\mathcal{K}_1\mathcal{Q}(g,\pi)$,

(iii) $\tilde{\mathcal{R}}\cdot\widetilde{\nabla}\pi=\mathcal{K}_1\mathcal{Q}(g,\widetilde{\nabla}\pi)$,

(iv) $\tilde{\mathcal{R}}\cdot\pi=\mathcal{K}_2\mathcal{Q}(\mathcal{S},\pi)$

(v) $\tilde{\mathcal{R}}\cdot\widetilde{\nabla}\pi=\mathcal{K}_2\mathcal{Q}(\mathcal{S},\widetilde{\nabla}\pi)$.

Theorem 3.1. *Let $\mathcal{M}$ be an IS of a HKM $\widetilde{\mathcal{M}}^{2n+1}$. The necessary and sufficient condition for $\mathcal{M}$ to be 2-SP is TG.*

Proof. Let $\mathcal{M}$ be 2-SP then $\tilde{\mathcal{R}}\cdot\widetilde{\nabla}\pi=0$. We modify $\mathcal{E}=\mathcal{F}=\xi$ in (2.30), we arrive at the equation

$$\mathcal{R}^{\perp}(\xi,\mathcal{F})(\widetilde{\nabla}\pi)(\mathcal{X}_1,\xi,\mathcal{X}_3)-(\widetilde{\nabla}\pi)(\mathcal{R}(\xi,\mathcal{F})\mathcal{X}_1,\xi,\mathcal{X}_3)-(\widetilde{\nabla}\pi)(\mathcal{X}_1,\mathcal{R}(\xi,\mathcal{F})\xi,\mathcal{X}_3)-(\widetilde{\nabla}\pi)(\mathcal{X}_1,\xi,\mathcal{R}(\xi,\mathcal{F})\mathcal{X}_3)=0. \tag{3.1}$$

In view of (2.17), (2.5), Lemma 2.1 (ii),(iii) and (2.7), we have the following equalities:

$$\begin{aligned}(\widetilde{\nabla}\pi)(\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3)&=(\widetilde{\nabla}_{\mathcal{X}_1}\pi)(\xi,\mathcal{X}_3)\\&=\nabla^{\perp}_{\mathcal{X}_1}\pi(\xi,\mathcal{X}_3)-\pi(\nabla_{\mathcal{X}_1}\xi,\mathcal{X}_3)-\pi(\xi,\nabla_{\mathcal{X}_1}\mathcal{X}_3)\\&=\pi(\mathcal{X}_1,\mathcal{X}_3),\end{aligned} \tag{3.2}$$

$$\begin{aligned}(\widetilde{\nabla}\pi)(\mathcal{R}(\xi,\mathcal{F})\mathcal{X}_1,\xi,\mathcal{X}_3)&=(\widetilde{\nabla}_{\mathcal{R}(\xi,\mathcal{F})\mathcal{X}_1}\pi)(\xi,\mathcal{X}_3)\\&=\nabla^{\perp}_{\mathcal{R}(\xi,\mathcal{F})\mathcal{X}_1}\pi(\xi,\mathcal{X}_3)-\pi(\nabla_{\mathcal{R}(\xi,\mathcal{F})\mathcal{X}_1}\xi,\mathcal{X}_3)-\pi(\xi,\nabla_{\mathcal{R}(\xi,\mathcal{F})\mathcal{X}_1}\mathcal{X}_3)\\&=-\eta(\mathcal{X}_1)\pi(\mathcal{F},\mathcal{X}_3),\end{aligned} \tag{3.3}$$

$$
\begin{aligned}
(\widetilde{\nabla}\pi)(\mathcal{X}_1,\mathcal{R}(\xi,\mathcal{F})\xi,\mathcal{X}_3) &= (\widetilde{\nabla}_{\mathcal{X}_1}\pi)(\mathcal{R}(\xi,\mathcal{F})\xi,\mathcal{X}_3)\\
&= \nabla^{\perp}\pi(\mathcal{R}(\xi,\mathcal{F})\xi,\mathcal{X}_3)-\pi(\nabla_{\mathcal{X}_1}\mathcal{R}(\xi,\mathcal{F})\xi,\mathcal{X}_3)-\pi(\mathcal{R}(\xi,\mathcal{F})\xi,\nabla_{\mathcal{X}_1}\mathcal{X}_3)\\
&= \nabla^{\perp}_{\mathcal{X}_1}\pi(-\eta(\mathcal{F})\xi-\mathcal{F},\mathcal{X}_3)-\pi(\nabla_{\mathcal{X}_1}(-\eta(\mathcal{F})\xi-\mathcal{F}),\mathcal{X}_3))-\pi(\mathcal{F},\nabla_{\mathcal{X}_1}\mathcal{X}_3), \qquad (3.4)\\
(\widetilde{\nabla}\pi)(\mathcal{X}_1,\xi,\mathcal{R}(\xi,\mathcal{F})\mathcal{X}_3) &= (\widetilde{\nabla}_{\mathcal{X}_1}\pi)(\xi,\mathcal{R}(\xi,\mathcal{F})\mathcal{X}_3)\\
&= \nabla^{\perp}_{\mathcal{X}_1}\pi(\xi,\mathcal{R}(\xi,\mathcal{F})\mathcal{X}_3)-\pi(\nabla_{\mathcal{X}_1}\xi,\mathcal{R}(\xi,\mathcal{F})\mathcal{X}_3)-\nabla(\xi,\nabla_{\mathcal{X}_1}\mathcal{R}(\xi,\mathcal{F})\mathcal{X}_3)\\
&= -\eta(\mathcal{X}_3)\pi(\mathcal{X}_1,\mathcal{F}). \qquad (3.5)
\end{aligned}
$$

Substituting (3.2) to (3.5) into (3.1), we obtain

$$
\begin{aligned}
&-\mathcal{R}^{\perp}(\xi,\mathcal{F})\pi(\mathcal{X}_1,\mathcal{X}_3)+\eta(\mathcal{X}_1)\pi(\mathcal{F},\mathcal{X}_3)-\nabla^{\perp}_{\xi}\pi(\{-\mathcal{F}-\eta(\mathcal{F})\xi\},\mathcal{X}_3)\\
&\quad+\pi(\nabla_{\mathcal{X}_1}\{-\mathcal{F}-\eta(\mathcal{F})\xi\},\mathcal{X}_3)+\pi(\mathcal{F},\nabla_{\mathcal{X}_1}\mathcal{X}_3)+\eta(\mathcal{X}_3)\pi(\mathcal{X}_1,\mathcal{F})=0. \qquad (3.6)
\end{aligned}
$$

Replacing $\mathcal{X}_3$ by ξ and using (2.5), (2.7) in (3.6), we get $\pi(\mathcal{X}_1,\mathcal{F})=0$. The converse part of the above theorem is also trivial. □

Theorem 3.2. *Let $\mathcal{M}$ be an IS of a HKM $\widetilde{\mathcal{M}}^{2n+1}$. The necessary and sufficient condition for $\mathcal{M}$ is PP is TG.*

Proof. Let $\mathcal{M}$ be PP then $\tilde{\mathcal{R}}\cdot\pi=\mathcal{K}_1\mathcal{Q}(g,\pi)$. Setting $\mathcal{E}=\mathcal{X}_2=\xi$ in (2.22), (2.30) and adding, we reach

$$
\begin{aligned}
&\mathcal{R}^{\perp}(\xi,\mathcal{F})\pi(\mathcal{X}_1,\xi)-\pi(\mathcal{R}(\xi,\mathcal{F})\mathcal{X}_1,\xi)-\pi(\mathcal{X}_1,\mathcal{R}(\xi,\mathcal{F})\xi)\\
&\quad=-\mathcal{K}_1\{g(\xi,\xi)\pi(\mathcal{X}_1,\mathcal{F})-g(\xi,\mathcal{X}_1)\pi(\xi,\mathcal{F})+g(\xi,\mathcal{F})\pi(\xi,\mathcal{X}_1)-g(\mathcal{F},\mathcal{X}_1)\pi(\xi,\xi)\}. \qquad (3.7)
\end{aligned}
$$

Making use of equations (2.1), Lemma 2.1(ii) and (2.7) in (3.7), we get $\pi(\mathcal{X}_1,\mathcal{F})=0$ and if $\mathcal{K}_1\neq 1$. The converse part of the above theorem is also trivial. □

Theorem 3.3. *Let $\mathcal{M}$ be an IS of a HKM $\widetilde{\mathcal{M}}^{2n+1}$. The necessary and sufficient condition for $\mathcal{M}$ to be 2-PP is TG.*

Proof. Let $\mathcal{M}$ be 2-PP then $\tilde{\mathcal{R}}\cdot\widetilde{\nabla}\pi=\mathcal{K}_1\mathcal{Q}(g,\widetilde{\nabla}\pi)$. Taking $\mathcal{E}=\mathcal{X}_2=\xi$ into (2.22), (2.30) and adding, from equations (2.1) and (2.7), we arrive at following equation,

$$
(\mathcal{R}^{\perp}(\mathcal{E},\mathcal{F})\cdot\widetilde{\nabla}\pi)(\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3)=\mathcal{K}_1\mathcal{Q}(g,\widetilde{\nabla}\pi)(\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3;\mathcal{E},\mathcal{F}),
$$

which gives,

$$
\begin{aligned}
&\mathcal{R}^{\perp}(\mathcal{E},\mathcal{F})(\widetilde{\nabla}_{\mathcal{X}_1}\pi)(\mathcal{X}_2,\mathcal{X}_3)-(\widetilde{\nabla}_{\mathcal{R}(\mathcal{E},\mathcal{F})\mathcal{X}_1}\pi)(\mathcal{X}_2,\mathcal{X}_3)-(\widetilde{\nabla}_{\mathcal{X}_1}\pi)(\mathcal{R}(\mathcal{E},\mathcal{F})\mathcal{X}_2,\mathcal{X}_3)(\widetilde{\nabla}_{\mathcal{X}_1}\pi)(\mathcal{X}_2,\mathcal{R}(\mathcal{E},\mathcal{F})\mathcal{X}_3)\\
&\quad=-\mathcal{K}_1\{(\widetilde{\nabla}_{(\mathcal{E}\wedge_g\mathcal{F})\mathcal{X}_1)}\pi)(\mathcal{X}_2,\mathcal{X}_3)+(\widetilde{\nabla}_{\mathcal{X}_1}\pi)((\mathcal{E}\wedge_g\mathcal{F})\mathcal{X}_2,\mathcal{X}_3)+(\widetilde{\nabla}_{\mathcal{X}_1}\pi)(\mathcal{F},(\mathcal{E}\wedge_g\mathcal{F})\mathcal{X}_3)\}. \qquad (3.8)
\end{aligned}
$$

Simplifying the above equation, and then substituting $\mathcal{E}=\mathcal{X}_3=\xi$ and after necessary arrangements we get,

$$
\begin{aligned}
&\mathcal{R}^{\perp}(\xi,\mathcal{F})(\widetilde{\nabla}_{\mathcal{X}_1}\pi)(\mathcal{X}_2,\xi)-(\widetilde{\nabla}_{\mathcal{R}(\xi,\mathcal{F})\mathcal{X}_1}\pi)(\mathcal{X}_2,\xi)-(\widetilde{\nabla}_{\mathcal{X}_1}\pi)(\mathcal{R}(\xi,\mathcal{F})\mathcal{X}_2,\xi)(\widetilde{\nabla}_{\mathcal{X}_1}\pi)(\mathcal{X}_2,\mathcal{R}(\xi,\mathcal{F})\xi)\\
&\quad=-\mathcal{K}_1\{g(\mathcal{F},\mathcal{X}_1)(\widetilde{\nabla}_{\xi}\pi)(\mathcal{X}_2,\xi)-\eta(\mathcal{X}_1)(\widetilde{\nabla}_{\mathcal{F}}\pi)(\mathcal{X}_2,\xi)+(\widetilde{\nabla}_{\mathcal{X}_1}\pi)(g(\mathcal{F},\mathcal{X}_2)\xi-\eta(\mathcal{X}_2)\mathcal{F},\xi)\\
&\qquad+(\widetilde{\nabla}_{\mathcal{X}_1}\pi)(\mathcal{X}_2,\eta(\mathcal{F})\xi+\mathcal{F})\}. \qquad (3.9)
\end{aligned}
$$

After simplifying by taking $\mathcal{X}_2=\xi$ and using (2.7), (2.5) in above equation, we get $(\mathcal{K}_1-2)\cdot\pi(\mathcal{X}_1,\mathcal{F})=0$, provided $\mathcal{K}_1\neq 2$. The converse part of the above theorem is also trivial. □

Theorem 3.4. *Let $\mathcal{M}$ be an IS of a HKM $\widetilde{\mathcal{M}}^{2n+1}$. The necessary and sufficient condition for $\mathcal{M}$ to be RGPP is TG.*

Proof. Let $\mathcal{M}$ be RGPP, i.e., $\tilde{R}\cdot\pi = \mathcal{K}_2\mathcal{Q}(S,\pi)$.
If we choose $\mathcal{E}=\xi$ and $\mathcal{X}_2=\xi$ in (2.22) and (2.7) and adding, gives

$$\begin{aligned}&\mathcal{R}^{\perp}(\xi,\mathcal{F})\pi(\mathcal{X}_1,\xi)-\pi(\mathcal{R}(\xi,\mathcal{F})\mathcal{X}_1,\xi)-\pi(\mathcal{X}_1,\mathcal{R}(\xi,\mathcal{F})\xi)\\ &\quad=-\mathcal{K}_2\{\pi(\mathcal{S}(\mathcal{F},\mathcal{X}_1)\xi-\mathcal{S}(\xi,\mathcal{X}_1)\mathcal{F},\xi)+\pi(\mathcal{S}(\xi,\mathcal{F})\xi-\mathcal{S}(\xi,\xi)\mathcal{F},\mathcal{X}_1)\}.\end{aligned}\tag{3.10}$$

Making use of Lemma 2.1(ii),(iv), (2.7) in (3.10), we get $(2n\mathcal{K}_2-1)\pi(\mathcal{X}_1,\mathcal{F})=0$, where $\mathcal{K}_2\neq\frac{1}{2n}$. The converse part of the above theorem is also trivial. □

Theorem 3.5. *Let $\mathcal{M}$ be an IS of a HKM $\widetilde{\mathcal{M}}^{2n+1}$. The necessary and sufficient condition for $\mathcal{M}$ is 2 -RGPP is TG.*

Proof. Let $\mathcal{M}$ be 2-RGPP, i.e., $\tilde{\mathcal{R}}\cdot\widetilde{\nabla}\pi=\mathcal{K}_2\mathcal{Q}(\mathcal{S},\widetilde{\nabla}\pi)$. Changing $\mathcal{E}$ and $\mathcal{X}_2$ with ξ in (2.22), (2.30) and adding, which in view of (3.7) and (2.7) gives

$$\begin{aligned}&\mathcal{R}^{\perp}(\xi,\mathcal{F})(\widetilde{\nabla}_{\mathcal{X}_1}\pi)(\xi,\mathcal{X}_3)-(\widetilde{\nabla}_{\mathcal{R}(\xi,\mathcal{F})\mathcal{X}_1}\pi)(\xi,\mathcal{X}_3)-(\widetilde{\nabla}_{\mathcal{X}_1}\pi)(\mathcal{R}(\xi,\mathcal{F})\xi,\mathcal{X}_3)-(\widetilde{\nabla}_{\mathcal{X}_1}\pi)(\xi,\mathcal{R}(\xi,\mathcal{F})\mathcal{X}_3)\\ &\quad=-\mathcal{K}_2\{(\widetilde{\nabla}_{(\xi\wedge_s\mathcal{F})\mathcal{X}_1}\pi)(\xi,\mathcal{X}_3)+(\widetilde{\nabla}_{\mathcal{X}_1}\pi)((\xi\wedge_s\mathcal{F})\xi,\mathcal{X}_3)+(\widetilde{\nabla}_{\mathcal{X}_1}\pi)(\xi,(\xi\wedge_s\mathcal{F})\mathcal{X}_3)\}.\end{aligned}\tag{3.11}$$

After calculating each term of the above equation and performing necessary rearrangements, we get

$$\begin{aligned}&-\mathcal{R}^{\perp}(\xi,\mathcal{F})\pi(\mathcal{X}_1,\mathcal{X}_3)+\eta(\mathcal{X}_1)\pi(\mathcal{F},\mathcal{X}_3)+(\tilde{\nabla}_{\mathcal{X}_1}\pi)(\mathcal{F},\mathcal{X}_3)+\eta(\mathcal{F})\pi(\mathcal{X}_1,\mathcal{X}_3)+\eta(\mathcal{X}_3)\pi(\mathcal{X}_1,\mathcal{F})\\ &\quad=-\mathcal{K}_2\{-2n\eta(\mathcal{X}_1)\pi(\mathcal{F},\mathcal{X}_3)+2n(\tilde{\nabla}_{\mathcal{X}_1}\pi)(\mathcal{F},\mathcal{X}_3)+2n\eta(\mathcal{F})\pi(\mathcal{X}_1,\mathcal{X}_3)-2n\eta(\mathcal{X}_3)\pi(\mathcal{X}_1,\mathcal{F})\}.\end{aligned}\tag{3.12}$$

By putting $\mathcal{X}_3=\xi$ and using equations (2.5), (2.7) in (3.12), we find that $(4n\mathcal{K}_2-1)\pi(\mathcal{F},\mathcal{X}_1)=0$, provided $\mathcal{K}_2\neq-\frac{1}{4n}$. The converse part of the above theorem is also trivial. □

Corollary 3.1. *Let $\mathcal{M}$ be an IS of a HKM $\widetilde{\mathcal{M}}^{2n+1}$. The statements below are equivalent:*

(i) *π is parallel,*
(ii) *π is recurrent,*
(iii) *$\mathcal{M}$ has parallel third fundamental form,*
(iv) *π is G-2-R*
(v) *$\mathcal{M}$ is SP,*
(vi) *$\mathcal{M}$ is 2-SP,*
(vii) *$\mathcal{M}$ is PP and if $\mathcal{K}_1\neq 1$,*
(viii) *$\mathcal{M}$ is 2-PP,*
(ix) *$\mathcal{M}$ is RGPP and if $\mathcal{K}_2\neq\frac{1}{2n}$,*
(x) *$\mathcal{M}$ is 2-RGPP,*
(xi) *$\mathcal{M}$ is TG.*

4. Conclusion

Our study delved into the geodesic properties of IS, specifically within the framework of HKM. We examined and established the relation between the various conditions related to the π. These conditions encompassed being 2-SP, PP, 2-PP, RGPP and 2-RGPP.

These results have implications in both theoretical and applied mathematics, as well as in various fields where the understanding and utilization of IS in HKM play a significant role. Such knowledge can be beneficial in areas such as differential geometry, mathematical physics and even in the development of novel algorithms and computational techniques.

Overall, our study contributes to the advancement of our understanding of IS in HKM and provides a foundation for future research and exploration in this area of mathematics.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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