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Research Article

Certain Classes of Meromorphically Univalent Functions Associated With Subordinaion and Inclusion Relations

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Abstract. In this present paper, we introduce some new classes of Meromorphic function in punctured unit disk involving integral operator. Our concern is to study some useful characteristic properties for the above defined general integral operator with applications of meromorphic function.

Keywords. Meromorphic functions, Hadamard product, Subordination, Univalent functions, Integral operator

Mathematics Subject Classification (2020). 30C45, 30C55, 33-XX

1. Introduction

Let Σ denotes the class of all meromorphic functions $f(z)$ and $f(z)$ is normalized by (Aouf [1], and Duren [3]):

$$f(z) = z^{-1} + \sum_{k=0}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the punctured unit disk

$$U^* = \{z; z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U/\{0\}.$$

If $f(z)$ and $g(z)$ are analytic functions in U and also let $f(z) \in \Sigma$ given by (1.1) and $g(z) \in \Sigma$ is defined by (Goodman [4,5], and Goyal and Goyal [6]):

$$g(z) = z^{-1} + \sum_{k=0}^{\infty} b_k z^k. \tag{1.2}$$

We define the convolution of functions

$$(f * g)(z) = z^{-1} + \sum_{k=0}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

Again if $f(z)$ and $g(z)$ are analytic functions in unit disk then the function $f(z)$ is subordinate to function $g(z)$, written mathematically as follows

$$f(z) \prec g(z),\ z \in U \quad \text{or} \quad f \prec g,\ z \in U.$$

(Goodman [4,5], Srivastava and Patel [13], and Srivastava and Aouf [14]).
If there exist a Schwarz function $w(z)$ in the unit disk, which is analytic in U with the condition

$$w(0) = 0 \text{ and } |w(z) < 1|,\ z \in U \quad \text{such that} \quad f(z) = g(w(z)),\ z \in U.$$

(Komatu [7], and Miller and Ross [8])
Really it is known as

$$f(z) \prec g(z),\ \ z \in U$$

$$\Longrightarrow \quad f(0) = g(0) \ \text{ and } \ f(U) \subset g(U)$$

If the function $g(z)$ is univalent in unit disk U then we have the following equivalence to Miller and Mocanu [9] and Owa and Srivastava [11].
For a function $f(z) \in \Sigma$ given by (1.1), we define an integral operator

$$\Theta_\xi^\lambda f(z) = \begin{bmatrix} \lambda + \xi - 1 \\ \xi - 1 \end{bmatrix} \lambda z^{-(1+\xi)} \int_0^z \left(1 + \frac{t}{z}\right)^{\lambda-1} t^\xi f(t)\,dt. \tag{1.4}$$

It is clear from (1.4) that

$$\Theta_\xi^\lambda f(z) = z^{-1} + \frac{\Gamma(\lambda+\xi)}{\Gamma(\xi)} \sum_{k=1}^{\infty} \frac{\Gamma(k+\xi)}{\Gamma(k+\lambda+\xi)} a_{k-1} z^{k-1}. \tag{1.5}$$

It is easy to see that this integral operator has the following relation

$$z(\Theta_\xi^\lambda f(z))' = (\lambda+\xi-1)\Theta_\xi^{\lambda-1} f(z) - (\lambda+\xi)\Theta_\xi^\lambda f(z). \tag{1.6}$$

Now we introduce the subclass of the functions class Σ.

Definition 1.1 ([2,10,12,15]). For $\lambda \geq 0$, $\lambda \leq \xi \leq 1$, $\mu \geq 0$, A is a real number, $A \neq B$, and a function $f(z) \in \Sigma$ is said to be in the class $\Sigma(\lambda, \xi, \gamma, A, B)$.

If it satisfies the following subordination condition

$$(1-\gamma)(z\Theta_\xi^\lambda f(z))^\mu + \gamma z \Theta_\xi^{\lambda-1} f(z)(z\Theta_\xi^\lambda f(z))^{\mu-1} \prec \frac{1+Az}{1+Bz}, \quad z \in U. \tag{1.7}$$

For convenience, we can write

$$\Sigma(\lambda, \xi, \gamma, \rho) = \Sigma(\lambda, \xi, \gamma, 1-2\rho, -1).$$

Class of function $f(z) \in \Sigma$ satisfying the condition

$$\mathrm{Re}\{(1-\gamma)(z\Theta_\xi^\lambda f(z))^\mu + \gamma z \Theta_\xi^{\lambda-1} f(z)(z\Theta_\xi^\lambda f(z))^{\mu-1}\} > \rho, \quad z \in U. \tag{1.8}$$

In this paper, we established many properties for the class $\Sigma(\lambda,\xi,\gamma,A,B)$ and derive some results of functions belonging to the class $\Sigma(\lambda,\xi,\gamma,A,B)$ which we have defined here by means of the integral operator Θ_ξ^λ.

2. Preliminary Lemma

To establish our main results, we need the following lemma.

Lemma 2.1 ([9, 11]). *Let the function $h(z)$ be analytic and convex univalent in U with $h(0)=1$, and also let $\phi(z)=1+c_1z+c_2z^2+c_3z^3+\dots$ is analytic in U. If*

$$\phi(z)+\frac{z\phi'(z)}{\delta}\prec h(z),\quad z\in U,\mathrm{Re}(\delta)\ge 0,\delta\neq 0,\tag{2.1}$$

$$\phi(z)\prec\Psi(z)=\frac{\delta}{z^\delta}\int_0^z t^{\delta-1}h(t)dt\prec h(z),\quad z\in U\tag{2.2}$$

and $\Psi(z)$ is the best dominant of (1.1).

3. Main Results

Theorem 3.1. *Let $\lambda\ge 0$, $\xi>-1$, $\xi\in[-1,1]$, $\mu\ge 0$, $A\in R$ and $f(z)\in\Sigma(\lambda,\xi,\gamma,A,B)$. Then*

$$(z\Theta_\xi^\lambda f(z))^\mu\prec\frac{\mu(\lambda+\xi-1)}{\delta}\int_0^1 u^{\frac{\mu(\lambda+\xi-1)}{\delta}}\frac{1+Azu}{1+Bzu}du\prec\frac{1+Az}{1+Bz},\quad z\in U,$$

where

$$(z\Theta_\xi^\lambda f(z))^\mu\prec\frac{\mu\gamma}{\lambda}\int_0^1 t^{\delta-1}dt\prec\frac{1+At}{1+Bt}dt\prec\frac{1+Az}{1+Bz}.$$

Proof. Let the function $f(z)$ defined by (1.1) and setting

$$(z\Theta_\xi^\lambda f(z))^\mu=\phi(z),\quad z\in U.\tag{3.1}$$

Differentiating both side with respect to z, we have

$$\mu+\frac{\mu z^2(\Theta_\xi^\lambda f(z))'}{z\Theta_\xi^\lambda f(z)}=\frac{z\phi'(z)}{\phi(z)}.\tag{3.2}$$

Now from (3.1) and (3.2), we get

$$\mu+\frac{\mu z^2(\Theta_\xi^\lambda f(z))'}{z\Theta_\xi^\lambda f(z)}=\frac{z\phi'(z)}{(z\Theta_\xi^\lambda f(z))^\mu}.\tag{3.3}$$

Now, with the help of (1.6) and (3.3), we have

$$\mu(z\Theta_\xi^\lambda f(z))^\mu+\mu z(z\Theta_\xi^\lambda\xi f(z))^{\mu-1}\{(\lambda+\xi-1)\Theta_\xi^{\lambda-1}f(z)-(\lambda+\xi)\Theta_\xi^\lambda f(z)\}z\phi'(z),$$

$$\gamma[(\Theta_\xi^{\lambda-1}f(z)(z\Theta_\xi^\lambda f(z))^{\mu-1})-(z\Theta_\xi^\lambda f(z))^\mu]=\frac{z\gamma\phi'(z)\gamma}{\mu(\lambda+\xi)},$$

$$\gamma[(\Theta_\xi^{\lambda-1}f(z)(z\Theta_\xi^\lambda f(z))^{\mu-1})-(z\Theta_\xi^\lambda f(z))^\mu]=\frac{z\phi'(z)\gamma}{\mu\left(\frac{\lambda+\xi}{\gamma}\right)}.\tag{3.4}$$

Adding (3.1) and (3.4),

$$(1-\gamma)(z\Theta_\xi^\lambda f(z))^\mu+\gamma z\Theta_\xi^{\lambda-1}f(z)(z\Theta_\xi^\lambda f(z))^{\mu-1}=\phi(z)+\frac{z\phi'(z)}{\mu^{\frac{\lambda+\xi}{\gamma}}}.\tag{3.5}$$

Since, it is given that $f(z)\in\Sigma(\lambda,\xi,\gamma,A,B)$. We get

$$\phi(z)+\frac{z\phi'(z)}{\mu^{\frac{\lambda+\xi}{\gamma}}}\prec\frac{1+Az}{1+Bz}.$$

Now by the help of Lemma 2.1, we have

$$z(\Theta_\xi^\lambda f(z))^\mu=\phi(z)\prec\frac{z\phi'(z)}{\mu^{\frac{\lambda+\xi}{\gamma}}}z^{\frac{-\mu(\lambda+\xi-1)}{\delta}}\int_0^z t^{\frac{\mu(\lambda+\xi-1)}{\delta}-1}h(t)dt\prec h(z). \tag{3.6}$$

Here $h(t)=\frac{1+At}{1+Bt}$ and if we taking $t=zu$ in eq (3.6), we have

$$(z\Theta_\xi^\lambda f(z))^\mu\prec\frac{\mu(\lambda+\xi-1)}{\delta}\int_0^1 u^{\frac{\mu(\lambda+\xi-1)}{\delta}}\frac{1+Azu}{1+Bzu}du\prec\frac{1+Az}{1+Bz},\quad z\in U. \qquad \square$$

Corollary 3.1. *Let* $\lambda\ge 0,\ \rho\ne 1,\ \mu>0$. *If*

$$(1-\gamma)(z(\Theta_\xi^\lambda f(z))^\mu)+\lambda z\Theta_\xi^\lambda f(z)(z(\Theta_\xi^\lambda f(z))^{\mu-1})\prec\frac{1+(1-2\rho)z}{1-z},\quad z\in U.$$

Then

$$(z(\Theta_\xi^\lambda f(z))^\mu)\prec\rho+\frac{(1-\rho)\lambda\mu}{\lambda}\int_0^1\frac{1+zu}{1-zu}u^{\frac{\delta\mu}{\lambda}-1}du,\quad z\in U.$$

Corollary 3.2. *Let* $\lambda>0,\ \mu\ge 0,\ \Sigma(\lambda,\xi,\gamma,A,B)\subset\Sigma(0,\xi,\gamma,A,B)$. *If*

$$f(z)\in(1-\lambda)(z(\Theta_\xi^\lambda f(z))^\mu)+\lambda z\Theta_\xi^\lambda f(z)(z(\Theta_\xi^\lambda f(z))^{\mu-1})\prec\frac{1+Az}{1+Bz}.$$

Then

$$(z(\Theta_\xi^\lambda f(z))^\mu)\prec\frac{1+Az}{1+Bz},\quad \lambda=0.$$

Theorem 3.2. *Let* $(z)\in\Sigma(0,\xi,\gamma,\rho),\ \mu>0$, *then*

$$f(z)\in\Sigma(\lambda,\xi,\gamma,\rho),\ |z|<\mathrm{Re}(\lambda,\gamma,\mu),$$

where $\mathrm{Re}(\lambda,\gamma,\mu)=\frac{\gamma\mu}{\lambda+(\lambda^2+\gamma^2\mu^2)^{\frac{1}{2}}}$.

Proof. We setting

$$((z\Theta_\xi^\lambda f(z))^\mu)=(1-\rho)h(z)+\rho. \tag{3.7}$$

Taking logarithm and differentiating, we have

$$(z\Theta_\xi^\lambda f(z))^\mu+z(z\Theta_\xi^\lambda f(z))^{\mu-1}(z\Theta_\xi^\lambda f(z))'=\frac{(1-\rho)zh'(z)}{\mu}. \tag{3.8}$$

From (3.6) and (3.8), we get

$$(z\Theta_\xi^\lambda f(z))^\mu+z(z\Theta_\xi^\lambda f(z))^{\mu-1}(z\Theta_\xi^\lambda f(z))'=\frac{(1-\rho)zh'(z)}{\mu(\lambda+\xi-1)},$$

$$(z\Theta_\xi^{\lambda-1} f(z))^{\mu-1}(z\Theta_\xi^{\lambda-1} f(z))'-(z\Theta_\xi^\lambda f(z))^\mu=\frac{(1-\rho)zh'(z)}{\gamma\mu} \tag{3.9}$$

and

$$(z\Theta_\xi^\lambda f(z))^\mu-\rho=(1-\rho)h(z). \tag{3.10}$$

By (3.9) and (3.10), we have

$$\mathrm{Re}\left(h(z)+\frac{\gamma zh'(z)}{\mu(\lambda+\xi-1)}\right)\ge\mathrm{Re}\left(h(z)-\frac{\gamma}{\mu(\lambda+\xi-1)}\right)$$

$$\geq \operatorname{Re} h(z)\left(1-\frac{2r}{(1-r^2)\mu(\lambda+\xi-1)}\right)$$
$$> 0.$$

The right hand side of the above inequality is positive and $r < \operatorname{Re}(\lambda,\gamma,\mu) = \frac{\gamma\mu}{\lambda+(\lambda^2+\gamma^2\mu^2)^{\frac{1}{2}}}$.
Hence it follows that

$$f(z) \in \Sigma(\lambda,\xi,\gamma,\rho)|z| < \operatorname{Re}(\lambda,\gamma,\mu).$$

□

Theorem 3.3. *Let* $0 \leq \gamma_2 \leq \gamma_1$, *then*

$$\Sigma(\lambda,\xi,\gamma_1,\rho,A,B) \subset \Sigma(\lambda,\xi,\gamma_2,\rho,A,B).$$

Proof. Let $f(z) \in \Sigma(\lambda,\xi,\gamma,A,B)$ and $f(z)$ is given by (1.1) then we get

$$f(z) \in \Sigma(\lambda,\xi,\gamma_2,A,B)$$

and also from Theorem 3.1,

$$(1-\gamma_2)(z\Theta_\xi^\lambda f(z))^\mu + z\gamma_2\Theta_\xi^{\lambda-1} f(z)(z\Theta_\xi^\lambda f(z))^{\mu-1} = \phi(z) + \frac{z\phi'(z)}{\mu^{\frac{\lambda+\xi}{\gamma}}}.$$

Since, it is given that $f(z) \in \Sigma(\lambda,\xi,\gamma,A,B)$. We get

$$\phi(z) + \frac{z\phi'(z)}{\mu^{\frac{\lambda+\xi}{\gamma}}} \prec \frac{1+Az}{1+Bz}.$$

Suppose $f(z) \in \Sigma(\lambda,\xi,\gamma,A,B)$ and

$$(1-\gamma_1)(z\Theta_\xi^\lambda f(z))^\mu + z\gamma_1\Theta_\xi^{\lambda-1} f(z)(z\Theta_\xi^\lambda f(z))^{\mu-1} \prec \frac{1+Az}{1+Bz}.$$

Also, we have from Theorem 3.1,

$$(z\Theta_\xi^\lambda f(z))^\mu \prec \frac{1+Az}{1+Bz} = \frac{\gamma_2}{\gamma_1}(1-\gamma_1)(z\Theta_\xi^\lambda f(z))^\mu + z\gamma_1\Theta_\xi^{\lambda-1} f(z)(z\Theta_\xi^\lambda f(z))^{\mu-1} + \left(1-\frac{\gamma_2}{\gamma_1}\right).$$

Therefore,

$$(z\Theta_\xi^\lambda f(z))^\mu \prec \frac{1+Az}{1+Bz} + \left(1-\frac{\gamma_2}{\gamma_1}\right)\frac{1+Az}{1+Bz}.$$

Hence

$$f(z) \in \Sigma(\lambda,\xi,\gamma_2,A,B).$$

□

Theorem 3.4. *Suppose* $\lambda \geq 0$, $\xi > -1 \leq B \leq A \leq 1$, $\mu > 0$, $f(z) \in \Sigma(\lambda,\xi,\gamma,A,B)$. *Then*

$$\frac{\gamma\mu}{\lambda}\int_0^1 \left(\frac{1-Au}{1-Bu}\right) u^{\frac{\mu\gamma}{\lambda}-1} du < \operatorname{Re}(z\Theta_\xi^\lambda f(z))^\mu < \frac{\gamma\mu}{\lambda}\int_0^1 \left(\frac{1+Au}{1+Bu}\right) u^{\frac{\mu\gamma}{\lambda}-1} du$$

and inequality is sharp defined by the extremal function

$$\Theta_\xi^\lambda f(z) = \frac{1}{z}\left(\frac{\gamma\mu}{\lambda}\int_0^1 \left(\frac{1+Azu}{1+Bzu}\right) u^{\frac{\mu\gamma}{\lambda}-1} du\right)^{\frac{1}{\mu}}.$$

Proof. By the definition of subordination and Theorem 3.1, we have

$$(z\Theta_\xi^\lambda f(z))^\mu < \frac{\gamma\mu}{\lambda}\int_0^1 \left(\frac{1+Azu}{1+Bzu}\right) u^{\frac{\mu\gamma}{\lambda}-1} du \prec \frac{1+Az}{1+Bz}$$

with $f(z) \in \Sigma(\lambda,\xi,\gamma,A,B)$, $A > B$,

$$\begin{aligned}
\mathrm{Re}(z\Theta_\xi^\lambda f(z))^\mu &< \sup\ \mathrm{Re}\left\{\frac{\gamma\mu}{\lambda}\int_0^1\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu\gamma}{\lambda+\xi-1}-1}du\right\}\\
&\le \frac{\gamma\mu}{\lambda}\left\{\int_0^1 \sup\ \mathrm{Re}\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu\gamma}{\lambda+\xi-1}-1}du\right\}\\
&< \frac{\gamma\mu}{\lambda}\left\{\int_0^1\left(\frac{1+Au}{1+Bu}\right)u^{\frac{\mu\gamma}{\lambda+\xi-1}-1}du\right\}.
\end{aligned}$$

Now by the maximum modulus principal,

$$\begin{aligned}
\mathrm{Re}(z\Theta_\xi^\lambda f(z))^\mu &> \inf\ \mathrm{Re}\left\{\frac{\gamma\mu}{\lambda}\int_0^1\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu\gamma}{\lambda+\xi-1}-1}du\right\}\\
&\ge \frac{\gamma\mu}{\lambda}\left\{\int_0^1 \inf\ \mathrm{Re}\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu\gamma}{\lambda+\xi-1}-1}du\right\}\\
&> \frac{\gamma\mu}{\lambda}\left\{\int_0^1\left(\frac{1-Au}{1-Bu}\right)u^{\frac{\mu\gamma}{\lambda+\xi-1}-1}du\right\}.
\end{aligned}$$

□

Theorem 3.5. *Suppose $\lambda \ge 0$, $-1 \le B \le A \le 1$, $\mu > 0$, $f(z) \in \Sigma(\lambda,\xi,\gamma,A,B)$. Then*

$$\frac{\gamma\mu}{\lambda}\int_0^1\left(\frac{1+Au}{1+Bu}\right)u^{\frac{\mu\gamma}{\lambda}-1}du < \mathrm{Re}(z\Theta_\xi^\lambda f(z))^\mu < \frac{\gamma\mu}{\lambda}\int_0^1\left(\frac{1-Au}{1-Bu}\right)u^{\frac{\mu\gamma}{\lambda}-1}du$$

and inequality is sharp defined by the extremal function.

Proof. By the definition of subordination and Theorem 3.1, we have

$$\begin{aligned}
(z\Theta_\xi^\lambda f(z))^\mu &< \frac{\gamma\mu}{\lambda}\int_0^1\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu\gamma}{\lambda}-1}du\\
&\prec \frac{1+Az}{1+Bz} \quad \text{with } f(z) \in \Sigma(\lambda,\xi,\gamma,A,B),\ A < B,
\end{aligned}$$

$$\begin{aligned}
\mathrm{Re}(z\Theta_\xi^\lambda f(z))^\mu &< \sup\ \mathrm{Re}\left\{\frac{\gamma\mu}{\lambda}\int_0^1\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu\gamma}{\lambda+\xi-1}-1}du\right\}\\
&\le \frac{\gamma\mu}{\lambda}\left\{\int_0^1 \sup\ \mathrm{Re}\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu\gamma}{\lambda+\xi-1}-1}du\right\}\\
&< \frac{\gamma\mu}{\lambda}\left\{\int_0^1\left(\frac{1-Au}{1-Bu}\right)u^{\frac{\mu\gamma}{\lambda+\xi-1}-1}du\right\}.
\end{aligned} \tag{3.11}$$

Now by the maximum modulus principal

$$\begin{aligned}
\mathrm{Re}(z\Theta_\xi^\lambda f(z))^\mu &> \inf\ \mathrm{Re}\left\{\frac{\gamma\mu}{\lambda}\int_0^1\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu\gamma}{\lambda+\xi-1}-1}du\right\}\\
&\ge \frac{\gamma\mu}{\lambda}\left\{\int_0^1 \inf\ \mathrm{Re}\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu\gamma}{\lambda+\xi-1}-1}du\right\}\\
&> \frac{\gamma\mu}{\lambda}\left\{\int_0^1\left(\frac{1+Au}{1+Bu}\right)u^{\frac{\mu\gamma}{\lambda+\xi-1}-1}du\right\}.
\end{aligned} \tag{3.12}$$

From (3.11) and (3.12) the inequality holds good. □

Corollary 3.3. *Suppose $\lambda \ge 0$, $\mu > 0$, $0 \le \rho < 1$, $f(z) \in \Sigma(\lambda,\xi,\gamma,A,B)$. Then*

$$\frac{\gamma\mu}{\lambda}\int_0^1\frac{1-(1-2\rho)u}{1+u}u^{\frac{\mu\gamma}{\lambda}-1}du < \mathrm{Re}(z\Theta_\xi^\lambda f(z))^\mu < \frac{\gamma\mu}{\lambda}\int_0^1\frac{1+(1-2\rho)u}{1-u}u^{\frac{\mu\gamma}{\lambda}-1}du$$

and inequality is equivalent to

$$\rho+\frac{(1-\rho)\gamma\mu}{\lambda}\int_0^1\frac{1-u}{1+u}u^{\frac{\mu\gamma}{\lambda}-1}du<\mathrm{Re}(z\Theta_\xi^\lambda f(z))^\mu<\rho+\frac{(1-\rho)\gamma\mu}{\lambda}\int_0^1\frac{1+u}{1-u}u^{\frac{\mu\gamma}{\lambda}-1}du.$$

Theorem 3.6. *Suppose $\lambda\ge 0$, $-1\le B\le A\le 1$, $\mu>0$, $f(z)\in\Sigma(\lambda,\xi,\gamma,A,B)$. Then*

$$\left\{\frac{\gamma\mu}{\lambda}\int_0^1\frac{1-Au}{1-Bu}u^{\frac{\mu\gamma}{\lambda}-1}du\right\}^{\frac{1}{2}}<\mathrm{Re}(z\Theta_\xi^\lambda f(z))^{\frac{1}{2}}<\left\{\frac{\gamma\mu}{\lambda}\int_0^1\frac{1+Au}{1+Bu}u^{\frac{\mu\gamma}{\lambda}-1}du\right\}^{\frac{1}{2}},\quad z\in U$$

and the inequality is sharp with the extremal function defined as

$$\Theta_\xi^\lambda f(z)=\frac{1}{z}\left(\frac{\gamma\mu}{\lambda}\int_0^1\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu\gamma}{\lambda}-1}du\right)^{\frac{1}{\mu}}.$$

Proof. By the definition of subordination and Theorem 3.1, we have

$$(z\Theta_\xi^\lambda f(z))^\mu<\frac{\gamma\mu}{\lambda}\int_0^1\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu\gamma}{\lambda}-1}du\prec\frac{1+Az}{1+Bz}\quad\text{with } f(z)\in\Sigma(\lambda,\xi,\gamma,A,B),\ A>B.$$

Since $-1\le B\le A\le 1$,

$$0\le\frac{1-A}{1-B}<\mathrm{Re}(z\Theta_\xi^\lambda f(z))^\mu<\frac{1+A}{1+B}.$$

□

Theorem 3.7. *Let $\lambda\ge 0$, $-1\le B\le A\le 1$, $\mu>0$, $f(z)\in\Sigma(\lambda,\xi,\gamma,A,B)$. Then*

$$\left\{\frac{\gamma\mu}{\lambda}\int_0^1\frac{1+Au}{1+Bu}u^{\frac{\mu\gamma}{\lambda}-1}du\right\}^{\frac{1}{2}}<\mathrm{Re}(z\Theta_\xi^\lambda f(z))^{\frac{\mu}{2}}<\left\{\frac{\gamma\mu}{\lambda}\int_0^1\frac{1-Au}{1-Bu}u^{\frac{\mu\gamma}{\lambda}-1}du\right\}^{\frac{1}{2}},\quad z\in U,$$

and the inequality is sharp with the extremal function defined as

$$\Theta_\xi^\lambda f(z)=\frac{1}{z}\left(\frac{\gamma\mu}{\lambda}\int_0^1\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu\gamma}{\lambda}-1}du\right)^{\frac{1}{\mu}}.$$

Proof. By the definition of subordination and Theorem 3.1, we have

$$(z\Theta_\xi^\lambda f(z))^\mu\prec\frac{1+Az}{1+Bz}\quad\text{with } f(z)\in\Sigma(\lambda,\xi,\gamma,A,B),\quad A<B,$$

and for $\frac{\mu}{2}=\frac{1}{2}$, we have

$$\begin{aligned}\mathrm{Re}(z\Theta_\xi^\lambda f(z))^{\frac{\mu}{2}}&<\sup\ \mathrm{Re}\left\{\frac{\gamma\mu}{\lambda}\int_0^1\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu\gamma}{\lambda}-1}du\right\}^{\frac{1}{2}}\\&<\frac{\gamma\mu}{\lambda}\left\{\int_0^1\sup\ \mathrm{Re}\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu\gamma}{\lambda}-1}du\right\}^{\frac{1}{2}}\\&<\frac{\gamma\mu}{\lambda}\left\{\int_0^1\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu\gamma}{\lambda}-1}du\right\}^{\frac{1}{2}},\\\mathrm{Re}(z\Theta_\xi^\lambda f(z))^{\frac{\mu}{2}}&>\inf\ \mathrm{Re}\left\{\frac{\gamma\mu}{\lambda}\int_0^1\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu\gamma}{\lambda}-1}du\right\}^{\frac{1}{2}}\\&\ge\frac{\gamma\mu}{\lambda}\left\{\int_0^1\inf\ \mathrm{Re}\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu\gamma}{\lambda}-1}du\right\}^{\frac{1}{2}}\\&>\frac{\gamma\mu}{\lambda}\left\{\int_0^1\left(\frac{1-Azu}{1-Bzu}\right)u^{\frac{\mu\gamma}{\lambda}-1}du\right\}^{\frac{1}{2}}.\end{aligned}$$

□

Theorem 3.8. *Let $\lambda\ge 0$, $-1\le B\le A\le 1$, $\mu>0$, $f(z)\in\Sigma(\lambda,\xi,\gamma,A,B)$. Then*

(i) *If* $\lambda = 0$ *when* $|z| = r < 1$, *we have*

$$\frac{1}{r}\left(\frac{1-Au}{1-Bu}\right)^{\frac{1}{\mu}} \le |z\Theta_{\xi}^{\lambda} f(z)| \le \frac{1}{r}\left(\frac{1+Au}{1+Bu}\right)^{\frac{1}{\mu}}$$

and the inequality is sharp with the extremal function defined by

$$|z\Theta_{\xi}^{\lambda} f(z)| = \frac{1}{r}\left(\frac{1+Au}{1+Bu}\right)^{\frac{1}{\mu}}.$$

(ii) *If* $\lambda \neq 0$ *when* $|z| = r < 1$, *we have*

$$\frac{1}{r}\left\{\frac{\gamma\mu}{\lambda}\int_0^1 \frac{1-Aru}{1-Bru} u^{\frac{\mu\gamma}{\lambda}-1} du\right\}^{\frac{1}{\mu}} \le |z\Theta_{\xi}^{\lambda} f(z)| \le \frac{1}{r}\left\{\frac{\gamma\mu}{\lambda}\int_0^1 \frac{1+Aru}{1+Bru} u^{\frac{\mu\gamma}{\lambda}-1} du\right\}^{\frac{1}{\mu}}, \quad z \in U$$

and the inequality is sharp with the extremal function defined by

$$|z\Theta_{\xi}^{\lambda} f(z)| = \frac{1}{r}\left(\frac{1+Au}{1+Bu}\right)^{\frac{1}{\mu}}.$$

Proof. (i): Since $\lambda = 0$ and $f(z) \in \Sigma(0,\xi,\gamma,A,B)$, $-1 \le B \le A \le 1$ then

$$(z\Theta_{\xi}^{\lambda} f(z))^{\mu} \prec \frac{1+Az}{1+Bz}.$$

Now by the definition of subordination, we have

$$(z\Theta_{\xi}^{\lambda} f(z))^{\mu} \prec \frac{1+A(w)z}{1+Bw(z)},$$

where $w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \dots$ is analytic in U and $|w(z)| \le |z|$ when $|z| = r < 1$.
Now,

$$(z\Theta_{\xi}^{\lambda} f(z))^{\mu} = \left|\frac{1+A(w)z}{1+Bw(z)}\right| \le \frac{1+A|w(z)|}{1+B|w(z)|} \le \frac{1+Ar}{1+Br},$$

$$|(z\Theta_{\xi}^{\lambda} f(z))^{\mu}| \ge \mathrm{Re}(z\Theta_{\xi}^{\lambda} f(z))^{\mu} \ge \frac{1-Ar}{1-Br}.$$

(ii): If $\lambda \neq 0$ then by Theorem 3.1, we get

$$(z\Theta_{\xi}^{\lambda} f(z))^{\mu} \prec \frac{\mu\gamma}{\lambda}\int_0^1 u^{\frac{\gamma\mu}{\lambda}-1}\frac{1+Azu}{1+Bzu} du \prec \frac{1+Az}{1+Bz}, \quad z \in U.$$

Now by the definition of subordination

$$(z\Theta_{\xi}^{\lambda} f(z))^{\mu} = \frac{\mu\gamma}{\lambda}\int_0^1 u^{\frac{\gamma\mu}{\lambda}-1}\frac{1+Aw(z)}{1+Bw(z)} du,$$

where $w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \dots$ is analytic in U and $|w(z)| \le |z|$ when $|z| = r < 1$,

$$\begin{aligned} |(z\Theta_{\xi}^{\lambda} f(z))^{\mu}| &\le \frac{\mu\gamma}{\lambda}\int_0^1 u^{\frac{\gamma\mu}{\lambda}-1}\left|\frac{1+Auw(z)}{1+Buw(z)}\right| du \\ &\le \frac{\mu\gamma}{\lambda}\int_0^1 u^{\frac{\gamma\mu}{\lambda}-1}\frac{1+Au|w(z)|}{1+B|w(z)|} du \\ &\le \frac{\mu\gamma}{\lambda}\int_0^1 u^{\frac{\gamma\mu}{\lambda}-1\frac{1+Aur}{1+Bur}} du \end{aligned}$$

and

$$|(z\Theta_{\xi}^{\lambda} f(z))^{\mu}| \ge \mathrm{Re}(z\Theta_{\xi}^{\lambda} f(z))^{\mu} \ge \frac{\mu\gamma}{\lambda}\int_0^1 u^{\frac{\gamma\mu}{\lambda}-1}\frac{1-Aur}{1-Bur} du.$$

The inequality is sharp with the extremal function defined by

$$|z\Theta_{\xi}^{\lambda} f(z)| = \frac{1}{r}\left(\frac{1+Au}{1+Bu}\right)^{\frac{1}{\mu}} \text{ and } f(z) \in \Sigma(\lambda,\xi,\gamma,A,B). \qquad \square$$

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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