Communications in Mathematics and Applications
Vol. 6, No. 2, pp. 49–54, 2015
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

On Hyponormal Toeplitz Operators with Trigonometric Polynomial Symbols

Research Article

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Abstract. This paper gives a necessary and sufficient conditions for the hyponormality of a Toeplitz operator T_φ on the trigonometric polynomial symbol of the type $\varphi(z)=\sum_{n=-N}^{N}a_nz^n$ under some certain assumptions of the Fourier coefficients of φ.

Keywords. Toeplitz operators; Hyponormal operators; Trigonometric polynomial; Symmetry

MSC. 47B35; 47B20

Received: November 18, 2015 **Accepted:** December 9, 2015

1. Introduction

A bounded linear operator T on a Hilbert space is said to be hyponormal if its self commutator $[T^*,T]:=T^*T-TT^*$ is positive (semi definite). Given $\varphi\in L^\infty(\mathbb{T})$, the Toeplitz operator with symbol φ is the operator T_φ on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T}=\partial\mathbb{D}$ defined by $T_\varphi f:=P(\varphi\cdot f)$, where $f\in H^2(\mathbb{T})$ and P denotes the orthogonal projection that maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$.

If $\varphi=\sum_{n=-N}^{N}a_nz^n$, then the author along with M. Hazarika in [5] and [6], and a handful number of authors, including [3], [4], [7], [8], [10] a few among them, had exhaustively investigated the hyponormality of Toeplitz operators T_φ under some certain assumptions about the Fourier coefficients of φ. In this paper, we continue to investigate the hyponormality of T_φ by relaxing some of the assumptions restricted on the Fourier coefficients of φ. If φ is a trigonometric polynomial of the form $\varphi(z)=\sum_{n=-m}^{N}a_nz^n$, where a_m and a_N are non-zero, then Farenick and Lee in [4] showed that the conditions $m\le N$ and $|a_{-m}|\le|a_N|$ are necessary for the hyponormality of T_φ. Also, they proved that for $\varphi(z)=\sum_{n=-m}^{N}a_nz^n$, if $|a_m|=|a_N|\neq 0$ then

T_φ is hyponormal if and only if the coefficients of φ satisfy the following 'symmetry' condition:

$$\overline{a}_N \begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ a_{-m} \end{pmatrix} = a_{-m} \begin{pmatrix} \overline{a}_{N-m+1} \\ \overline{a}_{N-m+2} \\ \vdots \\ \overline{a}_N \end{pmatrix}. \tag{1.1}$$

But, the case for arbitrary polynomial φ with $|a_m| \neq |a_N|$, though solved in principle by Cowen's theorem [2] or Zhu's theorem [10], in practice not so easy. In [8], the hyponormality of T_φ was studied when $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$ satisfies the full symmetric condition (1.1) and the case for partial symmetric condition with the following assumptions:

$$\overline{a}_N \begin{pmatrix} a_{-m} \\ a_{-(m+1)} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-m} \begin{pmatrix} \overline{a}_m \\ \overline{a}_{m+1} \\ \vdots \\ \overline{a}_N \end{pmatrix}, \quad \text{for } m-1 \leq \frac{N}{2}. \tag{1.2}$$

Also, a complete criterion for the hyponormality of T_φ is found in [7] with the condition (1.2). In [5] and [6], the author along with M. Hazarika gave a set of necessary and sufficient conditions for the hyponormality of T_φ under different sets of restrictions on the Fourier coefficients of φ up to third degree. In this paper, our main aim is to give a general criterion for the hyponormality of T_φ when the Fourier coefficients of $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$ satisfy the following condition:

$$\overline{a}_N \begin{pmatrix} a_{-4} \\ a_{-5} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N} \begin{pmatrix} \overline{a}_4 \\ \overline{a}_5 \\ \vdots \\ \overline{a}_N \end{pmatrix}.$$

This criterion obviously can establish some of the earlier results found in [5], [6] and [8] with some relaxed conditions. Here we shall employ the following variant of Cowen's theorem that was proposed by Nakazi and Takahashi [9], and Schur's algorithm due to Zhu [10].

Cowen's theorem. *Suppose that $\varphi \in L^\infty(\mathbb{T})$ is arbitrary and write $\mathscr{E}(\varphi) = \{k \in H^\infty(\mathbb{T}) : \|k\|_\infty \leq 1$ and $\varphi - k\overline{\varphi} \in H^\infty(\mathbb{T})\}$. Then T_φ is hyponormal if and only if $\mathscr{E}(\varphi)$ is nonempty.*

2. Schur's Function Φ_n and Kehe Zhu's Theorem

In this section, we give a brief description of Schur's function Φ_n and Zhu's idea in determining the hyponormality of Toeplitz operator by applying it.

Suppose that $f(z) = \sum_{j=0}^{\infty} c_j z^j$ is in the closed unit ball of $H^\infty(\mathbb{T})$ i.e. $||f||_\infty \leq 1$. If $f_0 = f$, define by induction a sequence $\{f_n\}$ of functions in the closed unit ball of $H^\infty(\mathbb{T})$ as follows:

$$f_{n+1}(z) = \frac{f_n(z) - f_n(0)}{z(1 - \overline{f_n(0)} f_n(z))}, \quad |z| < 1, \ n = 0, 1, 2, \ldots.$$

We know that $f_n(0)$ depends only on the values of $c_0, c_1, c_2, \ldots, c_n$, so we can write $f_n(0) = \Phi_n(c_0, \ldots, c_n)$, $n = 0, 1, 2, \ldots$, which gives that Φ_n is a function of $(n+1)$ complex variables. Now, we call the Φ_n's *Schur's functions*. Now, we can proceed to explain Zhu's theorem as follows:

Theorem 2.1 ([10]). *If $\varphi(z)=\sum_{n=-N}^{N} a_n z^n$, where $a_N \neq 0$ and if*

$$\begin{pmatrix} \overline{c}_0 \\ \overline{c}_1 \\ \vdots \\ \overline{c}_{N-1} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \cdots & a_{N-1} & a_N \\ a_2 & a_3 & \cdots & a_N & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_N & 0 & \cdots & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \overline{a}_{-1} \\ \overline{a}_{-2} \\ \vdots \\ \overline{a}_{-N} \end{pmatrix}, \tag{2.1}$$

then T_φ is hyponormal if and only if $|\Phi_n(c_0,\ldots,c_n)| \le 1$ for each $n=0,1,\ldots,N-1$.

Till date no closed form of general Schur's function Φ_n is derived. But Schur's algorithm enables us to determine Schur's function Φ_n up to any desired lavel for $n \ge 1$. In [10], Zhu has listed the first three Schur's functions:

$$\Phi_0(c_0) = c_0\ , \tag{2.2}$$

$$\Phi_1(c_0,c_1) = \frac{c_1}{1-|c_0|^2}\ , \tag{2.3}$$

$$\Phi_2(c_0,c_1,c_2) = \frac{c_2(1-|c_0|^2)+\overline{c}_0 {c_1}^2}{(1-|c_0|^2)^2-|c_1|^2}\ . \tag{2.4}$$

In [5], using Schur's algorithm $\Phi_3(c_0,c_1,c_2,c_3)$ was evaluated as:

$$\Phi_3(c_0,c_1,c_2,c_3) = \frac{\begin{pmatrix} ((1-|c_0|^2)^2-|c_1|^2)((1-|c_0|^2)c_3+\overline{c}_0 c_1 c_2) \\ +(c_2(1-|c_0|^2)+\overline{c}_0 {c_1}^2)(\overline{c}_0(1-|c_0|^2)c_1+\overline{c}_1 c_2) \end{pmatrix}}{((1-|c_0|^2)^2-|c_1|^2)^2-|c_2(1-|c_0|^2)+\overline{c}_0 {c_1}^2|^2}\ . \tag{2.5}$$

If we define a function $k(z)=\sum_{j=0}^{\infty} c_j z^j$ in the closed unit ball of $H^\infty(\mathbb{T})$ such that $\varphi - k\overline{\varphi} \in H^\infty(\mathbb{T})$, then $c_0,c_1,c_2,\ldots,c_{N-1}$ are nothing but the values given in the equation (2.1). Thus, Zhu's theorem states that if $k(z)=\sum_{j=0}^{\infty} c_j z^j$ satisfies $\varphi - k\overline{\varphi} \in H^\infty(\mathbb{T})$, then the values of c_j's for $j \ge N$ do not make any impact in determining the hyponormality of T_φ. In [8], Zhu's theorem was reformulated in a simplified form as follows:

Theorem 2.2 ([8]). *If $\varphi(z)=\sum_{n=-m}^{N} a_n z^n$, where $m \le N$ and $a_N \neq 0$, then T_φ is hyponormal if and only if*

$$|\Phi_n(c_0,\ \ldots,c_n)| \le 1, \quad \textit{for each } n=0,1,\ldots,N-1,$$

where c_n's are given by the following recurrence relation:

$$\begin{cases} c_0 = c_1 = \ldots = c_{N-m-1} = 0\ , \\ c_{N-m} = \dfrac{a_{-m}}{\overline{a}_N}\ , \\ c_n = (\overline{a}_N)^{-1}\left(a_{-N+n} - \sum_{j=N-m}^{n-1} c_j \overline{a}_{N-n+j}\right) \quad \textit{for } n = N-m+1,\ldots,N-1\ . \end{cases} \tag{$*$}$$

3. Main theorem

To establish our theorem we need the following lemma:

Lemma 3.1 ([6], [8]). *Suppose that $k(z)=\sum_{j=0}^{\infty} c_j z^j$ is in the closed unit ball of $H^\infty(\mathbb{T})$ and that $\{\Phi_n\}$ is the sequence of Schur's functions associated with $\{c_n\}$. If $c_1=c_2=\ldots=c_{n-1}=0$ and $c_n\neq 0$, then we have that*

$$\Phi_0=c_0,\ \Phi_1=\ldots=\Phi_{n-1}=0;\ \Phi_n=\frac{c_n}{1-|c_0|^2}\,, \tag{3.1}$$

$$\Phi_{n+1}=\frac{c_{n+1}}{(1-|c_0|^2)(1-|\Phi_n|^2)}\,, \tag{3.2}$$

$$\Phi_{n+2}=\frac{(1-|\Phi_n|^2)c_{n+2}c_n+|\Phi_n|^2c_{n+1}^2}{c_n(1-|c_0|^2)(1-|\Phi_n|^2)^2(1-|\Phi_{n+1}|^2)}\,. \tag{3.3}$$

Now we begin our main theorem:

Theorem 3.1. *Let $\varphi(z)=\sum_{n=-N}^{N} a_n z^n$ (with $|a_{-N}|\le|a_N|$ and $N\ge 4$) be a trigonometric polynomial which satisfies the following partial symmetry condition:*

$$\overline{a}_N\begin{pmatrix} a_{-4}\\ a_{-5}\\ \vdots\\ a_{-N}\end{pmatrix}=a_{-N}\begin{pmatrix} \overline{a}_4\\ \overline{a}_5\\ \vdots\\ \overline{a}_N\end{pmatrix}$$

with $\overline{a}_N a_{-i}\neq a_{-N}\overline{a}_i$ where $i=1,2,3$.
Let

$$\alpha=\frac{\overline{a}_N a_{-3}-a_{-N}\overline{a}_3}{|a_N|^2-|a_{-N}|^2}\ ;\quad \beta=\frac{\overline{a}_N a_{-2}-a_{-N}\overline{a}_2}{|a_N|^2-|a_{-N}|^2}\ ;\quad \gamma=\frac{\overline{a}_N a_{-1}-a_{-N}\overline{a}_1}{|a_N|^2-|a_{-N}|^2}\,.$$

Then,

(a) *For $N=4$, T_φ is hyponormal if and only if*

(1) $|a_{-4}|\le|a_4|$;

(2) $|\alpha|\le 1$;

(3) $|\beta\overline{a}_4-\alpha\overline{a}_3+\alpha^2\overline{a}_{-4}|\le|a_4|(1-|\beta|^2)$;

(4) $|(1-|\alpha|^2)(\gamma\overline{a}_4^2+\alpha(\overline{a}_3^2-\overline{a}_2\overline{a}_4+\beta\overline{a}_4-\alpha\overline{a}_3)-\beta\overline{a}_3\overline{a}_4)+(\beta\overline{a}_4-\alpha\overline{a}_3+\alpha^2\overline{a}_{-4})(\alpha\overline{a}_{-4}+\overline{\alpha}(\beta\overline{a}_4-\alpha\overline{a}_3))|\le\left(|a_4|(1-|\alpha|^2)\right)^2-|\beta\overline{a}_4-\alpha\overline{a}_3+\alpha^2\overline{a}_{-4}|^2$.

(b) *For $N\ge 5$, T_φ is hyponormal if and only if*

(1) $|a_{-N}|\le|a_N|$;

(2) $|\alpha|\le 1$;

(3) $\left|\beta-\alpha\left(\frac{\overline{a}_{N-1}}{\overline{a}_N}\right)\right|\le 1-|\alpha|^2$;

(4) $|1-|\alpha|^2\left(\gamma(\overline{a}_N)^2+\alpha\left(\overline{a}_{N-1}^2-\overline{a}_{N-2}\overline{a}_N\right)-\beta\overline{a}_N\overline{a}_{N-1}\right)+\overline{\alpha}(\beta\overline{a}_N-\alpha\overline{a}_{N-1})^2| \le \left(|a_N|(1-|\alpha|^2)\right)^2-|\beta\overline{a}_N-\alpha\overline{a}_{N-1}|^2$

Proof (*Throughput our proof we shall be using* $A=|a_N|^2-|a_{-N}|^2$).

(a) For $N=4$: If c_0, c_1, c_2 and c_3 are the solutions of the recurrence relation $(*)$, then a straightforward calculation gives that:

$$c_0=\frac{a_{-4}}{\overline{a}_4};\quad c_1=(\overline{a}_4)^{-2}A\alpha;\quad c_2=(\overline{a}_4)^{-3}A(\beta\overline{a}_4-\alpha\overline{a}_3);$$

$$c_3=(\overline{a}_4)^{-2}A\left(\gamma\overline{a}_4^2+\alpha(\overline{a}_3^2-\overline{a}_2\overline{a}_4)-\beta\overline{a}_3\overline{a}_4\right)$$

Now putting the values of c_0, c_1, c_2 and c_3 in the equations (2.2), (2.3), (2.4) and (2.5) and applying the Theorem 2.1 we get the results.

(b) For $N\ge 5$: By Theorem 1, T_φ is hyponormal if and only if there is a function k in the closed unit ball of $H^\infty(\mathbb{T})$ such that $\varphi-k\overline{\varphi}\in H^\infty(\mathbb{T})$ which implies that k should necessarily satisfy the following property:

$$k\left(\sum_{n=1}^{N}\overline{a}_n z^{-n}\right)-\sum_{n=1}^{N}a_{-n}z^{-n}\in H^\infty(\mathbb{T}). \tag{3.4}$$

The equation (3.4) gives a way to compute the Fourier coefficients $\widehat{k}(0),\widehat{k}(1),\dots,\widehat{k}(N-1)$ of k uniquely. Let us denote $\widehat{k}(n)=c_n$ for $n=0,1,\dots,N-1$ and then the values of c_n's can be calculated uniquely in terms of the coefficients of φ as follows:

$$c_0=\frac{a_{-N}}{\overline{a}_N},$$

$$c_1=c_2=\ldots=c_{N-5}=c_{N-4}=0,$$

$$c_{N-3}=A(\overline{a}_N)^{-2}\alpha,$$

$$c_{N-2}=A(\overline{a}_N)^{-3}(\beta\overline{a}_N-\alpha\overline{a}_{N-1}),$$

$$c_{N-1}=A(\overline{a}_N)^{-4}\left(\gamma\overline{a}_N^2+\alpha(\overline{a}_{N-1}^2-\overline{a}_{N-2}\overline{a}_N)-\beta\overline{a}_N\overline{a}_{N-1}\right).$$

Thus, $k_p(z)=c_0+c_{N-3}z^{N-3}+c_{N-2}z^{N-2}+c_{N-1}z^{N-1}$ is the unique analytic polynomial of degree less than N satisfying $\varphi-k_p\overline{\varphi}\in H^\infty(\mathbb{T})$. Now, by putting the values of c_0, c_{N-3}, c_{N-2} and c_{N-1} in (3.1), (3.2) and (3.3) and applying the Theorem 2.1 we get the required results. □

Conclusion

Theorem 3.1 is a generalised form of all the Theorem 6 and Theorem 8 in [8]; Theorem 3.1, Theorem 3.2 and Theorem 3.4 in [6] and Theorem 3.1 in [5] as these theorems can be established very easily through this theorem.

Acknowledgement

Author is thankful to the learned referee for his/her deep observations and suggestions.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author read and approved the final manuscript.

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