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Research Article

Generating Functions of a New Class of Semi-Orthogonal Polynomials $X_n(x;a,\alpha)$ Using Lie Group Theory

R. R. Jagtap*[1], P. G. Andhare[1] and S. B. Gaikwad[2]

[1] *Department of Mathematics, R. B. Narayanrao Borawake College (affiliated to Savitribai Phule Pune University), Shrirampur, Ahilyanagar, Maharashtra, India*
[2] *Department of Mathematics, New Arts, Commerce and Science College (affiliated to Savitribai Phule Pune University), Ahilyanagar, Maharashtra, India*
***Corresponding author:** rrjmaths@gmail.com

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Abstract. In this paper, by applying the group theoretic method introduced by Weisner, we determined new generating relations of a new class of semi-orthogonal polynomials $X_n(x;a,\alpha)$. By giving proper analytical reasoning to the index m of the semi-orthogonal polynomial, we derived three linear partial differential operators with the help of the ascending and descending differential recurrence relation of the polynomial. These linear partial differential operators generate a Lie group.

Keywords. X_n polynomials, Generating functions, Weisner method

Mathematics Subject Classification (2020). 33C45, 33C50

1. Introduction

Louis Weisner [14,15] has developed a group theoretic method to obtain the generating functions for a large class of functions under certain conditions. He also showed the group theoretic crucialness in the study of Hypergeometric functions, Hermite functions and Bessel functions and their generating functions. For a given set, the necessary and sufficient condition is that it must have descending and ascending recurrence relations.

Andhare and Choudhary [1] obtained new generating relations for a class of polynomials $Y_m(a,x)$ by using Lie group theory. By using the Weisner Method, Bhagavan and Tadikonda [3] obtained three new generating relations for the Chebyshev polynomials. Srinivasulu and Bhagavan [12] obtained new generating relations for the two variable Hypergeometric polynomial $R_n(\beta;\alpha;x,y)$ by Weisner method. Many researcher used this method to obtain the generating functions in the theory of special functions for several semi-orthogonal polynomials (see, Chongdar [4], Elkhazendar *et al.* [5], Grosswald [6], Manocha [7], McBride [8], Miller Jr. [9], Pathan *et al.* [10], Srinivasulu and Bhagavan [12], Srivastava and Manocha [13], and Weisner [14, 15]).

In the study of pure and other branches of applied mathematics, mathematical physics and approximation theory orthogonal polynomials (Rainville [11]) and semi-orthogonal polynomials play vital role. It has applications in various branches of engineering and science.

Bajpai [2] investigated a new class of semi-orthogonal polynomials $X_n(x;a,y)$. The semi-orthogonal polynomials $X_n(x;a,y)$ are defined as,

$$X_n(x;a,y) = {}_2F_0\left(-n,a;-;-\frac{x}{y}\right), \tag{1.1}$$

where $n = 0,1,2,\ldots$.

These X_n polynomials have relation with Bessel polynomials (Bajpai [2], and Chongdar [4]), Hermite polynomials and Laguerre polynomials.

Replacing y by α and n by m in (1.1), we get

$$X_m(x;a,\alpha) = {}_2F_0\left(-m,a;-;-\frac{x}{\alpha}\right) = \sum_{k=0}^{\infty} \frac{(-m)_k(a)_k\left(-\frac{x}{\alpha}\right)^k}{k!}. \tag{1.2}$$

For the semi-orthogonal polynomial $X_n(x;a,\alpha)$, we get following pure recurrence relation, where $m = 0,1,2,\ldots$,

$$X_{m+1}(x;a,\alpha) = \frac{1}{\alpha}\left[(\alpha + mx + ax)X_m(x;a,\alpha) - mxX_{m-1}(x;a,\alpha)\right], \tag{1.3}$$

and also it satisfies the following differential recurrence relations,

$$\frac{d}{dx}X_m(x;a,\alpha) = \frac{m}{x}X_m(x;a,\alpha) - \frac{m}{x}X_{m-1}(x;a,\alpha), \tag{1.4}$$

$$\frac{d}{dx}X_m(x;a,\alpha) = \frac{\alpha}{x^2}X_{m+1}(x;a,\alpha) - \frac{\alpha}{x^2}(\alpha + ax)X_m(x;a,\alpha). \tag{1.5}$$

From (1.4) and (1.5), the following differential equation can be determined which is the type of linear and ordinary

$$\left[x^2\frac{d^2}{dx^2} + [\alpha + (a - m + 1)x]\frac{d}{dx} - ma\right]X_m(x;a,\alpha) = 0. \tag{1.6}$$

If we use the operator notation,

$$X\left(x,\frac{d}{dx},-ma\right) = x^2\frac{d^2}{dx^2} + [\alpha + (a - m + 1)x]\frac{d}{dx} - ma, \tag{1.7}$$

by using (1.7), the equation (1.6) can be rewritten as

$$X\left(x,\frac{d}{dx},-ma\right)X_m(x;a,\alpha) = 0. \tag{1.8}$$

2. Linear Differential Operators

For the semi-orthogonal polynomial $X_m(x;a,\alpha)$, we define the first order partial differential operators A, B and C which are linear, such that

$$A[X_m(x;a,\alpha)y^{-ma}] = a_m X_m(x;a,\alpha)y^{-ma}, \tag{2.1}$$

$$B[X_m(x;a,\alpha)y^{-ma}] = b_m X_{m-1}(x;a,\alpha)y^{-ma+a}, \tag{2.2}$$

$$C[X_m(x;a,\alpha)y^{-ma}] = c_m X_{m+1}(x;a,\alpha)y^{-ma-a} \tag{2.3}$$

and

$$E[X_m(x;a,\alpha)y^{-ma}] = -X_m(x;a,\alpha)y^{-ma}, \tag{2.4}$$

where, a_m, b_m and c_m are functions of m and which do not dependent on x and y, but not needed independent of the parameters a and α. We want to determine the first order linear differential operators A, B and C by a method given by Srivastava and Manocha [13].

Proof of Equation (2.1). Let $A = R_1(x,y)\frac{\partial}{\partial x} + R_2(x,y)\frac{\partial}{\partial y} + R_3(x,y)$, where each R_i $(i = 1,2,3)$ are functions of x and y, but not needed independent of the parameters a and α,

$$\begin{aligned}
A[X_m(x;a,\alpha)y^{-ma}] &= a_m X_m(x;a,\alpha)y^{-ma} \\
&= \left\{R_1(x,y)\frac{\partial}{\partial x} + R_2(x,y)\frac{\partial}{\partial y} + R_3(x,y)\right\}\{X_m(x;a,\alpha)y^{-ma}\} \\
&= R_1(x,y)y^{-ma}\frac{d}{dx}X_m(x;a,\alpha) + R_2(x,y)y^{-ma}\left(\frac{-ma}{y}\right)X_m(x;a,\alpha) \\
&\quad + R_3(x,y)y^{-ma}X_m(x;a,\alpha) \\
&= R_1(x,y)y^{-ma}\left\{\frac{m}{x}X_m(x;a,\alpha) - \frac{m}{x}X_{m-1}(x;a,\alpha)\right\} \\
&\quad - \frac{ma}{y}R_2(x,y)y^{-ma}X_m(x;a,\alpha) + R_3(x,y)y^{-ma}X_m(x;a,\alpha) \\
&= \frac{-m}{x}R_1(x,y)y^{-ma}X_{m-1}(x;a,\alpha) \\
&\quad + \left\{\frac{m}{x}R_1(x,y) - \frac{ma}{y}R_2(x,y) + R_3(x,y)\right\}y^{-ma}X_m(x;a,\alpha).
\end{aligned}$$

Now equating the coefficients of $X_m(x;a,\alpha)$ and $X_{m-1}(x;a,\alpha)$ on both sides, we get

$$R_1(x,y) = 0, \quad -\frac{ma}{y}R_2(x,y) + R_3(x,y) = a_m.$$

Choosing,

$$R_2(x,y) = -\frac{y}{a}, \quad R_3(x,y) = 0.$$

We get

$$A = -\frac{y}{a}\frac{\partial}{\partial y}.$$

□

Proof of Equation (2.2). Let $B = R_1(x,y)\frac{\partial}{\partial x} + R_2(x,y)\frac{\partial}{\partial y} + R_3(x,y)$, where, each R_i $(i = 1,2,3)$ are functions of x and y, but not needed independent of the parameters a and α,

$$\begin{aligned}
B[X_m(x;a,\alpha)y^{-ma}] &= b_m X_{m-1}(x;a,\alpha)y^{-ma+a} \\
&= \left\{R_1(x,y)\frac{\partial}{\partial x} + R_2(x,y)\frac{\partial}{\partial y} + R_3(x,y)\right\}\{X_m(x;a,\alpha)y^{-ma}\}
\end{aligned}$$

$$
\begin{aligned}
&= R_1(x,y)y^{-ma}\frac{d}{dx}X_m(x;a,\alpha) + R_2(x,y)y^{-ma}\left(\frac{-ma}{y}\right)X_m(x;a,\alpha) \\
&\quad + R_3(x,y)y^{-ma}X_m(x;a,\alpha) \\
&= R_1(x,y)y^{-ma}\left\{\frac{m}{x}X_m(x;a,\alpha) - \frac{m}{x}X_{m-1}(x;a,\alpha)\right\} \\
&\quad - \frac{ma}{y}R_2(x,y)y^{-ma}X_m(x;a,\alpha) + R_3(x,y)y^{-ma}X_m(x;a,\alpha) \\
&= \frac{-m}{xy^a}R_1(x,y)y^{-ma+a}X_{m-1}(x;a,\alpha) \\
&\quad + \left\{\frac{m}{x}R_1(x,y) - \frac{ma}{y}R_2(x,y) + R_3(x,y)\right\}y^{-ma}X_m(x;a,\alpha).
\end{aligned}
$$

Choosing,

$$R_1(x,y) = xy^a, \quad R_2(x,y) = \frac{y^{a+1}}{a}, \quad R_3(x,y) = 0.$$

We get

$$B = xy^a\frac{\partial}{\partial x} + \frac{y^{a+1}}{a}\frac{\partial}{\partial y}. \qquad \square$$

Proof of Equation (2.3). Let $C = R_1(x,y)\frac{\partial}{\partial x} + R_2(x,y)\frac{\partial}{\partial y} + R_3(x,y)$, where each R_i $(i = 1,2,3)$ are functions of x and y, but not needed independent of the parameters a and α,

$$
\begin{aligned}
C[X_m(x;a,\alpha)y^{-ma}] &= c_m X_{m+1}(x;a,\alpha)y^{-ma-a} \\
&= \left\{R_1(x,y)\frac{\partial}{\partial x} + R_2(x,y)\frac{\partial}{\partial y} + R_3(x,y)\right\}\{X_m(x;a,\alpha)y^{-ma}\}, \\
&= R_1(x,y)y^{-ma}\frac{d}{dx}X_m(x;a,\alpha) + R_2(x,y)y^{-ma}\left(\frac{-ma}{y}\right)X_m(x;a,\alpha) \\
&\quad + R_3(x,y)y^{-ma}X_m(x;a,\alpha) \\
&= R_1(x,y)y^{-ma}\left\{\frac{\alpha}{x^2}X_{m+1}(x;a,\alpha) - \frac{\alpha}{x^2}(\alpha+ax)X_m(x;a,\alpha)\right\} \\
&\quad - \frac{ma}{y}R_2(x,y)y^{-ma}X_m(x;a,\alpha) + R_3(x,y)y^{-ma}X_m(x;a,\alpha) \\
&= \frac{\alpha y^a}{x^2}R_1(x,y)y^{-ma-a}X_{m+1}(x;a,\alpha) \\
&\quad + \left\{-\frac{(\alpha+ax)}{x^2}R_1(x,y) - \frac{ma}{y}R_2(x,y) + R_3(x,y)\right\}y^{-ma}X_m(x;a,\alpha).
\end{aligned}
$$

Now equating the coefficients of $X_m(x;a,\alpha)$ and $X_{m+1}(x;a,\alpha)$ on both sides and choosing

$$R_1(x,y) = \frac{x^2}{\alpha y^a}, \quad R_2(x,y) = 0, \quad R_3(x,y) = \frac{\alpha+ax}{\alpha y^a},$$

we get

$$C = \frac{x^2}{\alpha y^a}\frac{\partial}{\partial x} + \frac{\alpha+ax}{\alpha y^a}.$$

Therefore, we get the first order linear differential operators

$$A = -\frac{y}{a}\frac{\partial}{\partial y}; \quad B = xy^a\frac{\partial}{\partial x} + \frac{y^{a+1}}{a}\frac{\partial}{\partial y}; \quad C = \frac{x^2}{\alpha y^a}\frac{\partial}{\partial x} + \frac{\alpha+ax}{\alpha y^a}; \quad E = -1. \tag{2.5}$$

These linear differential operators satisfy the commutation relations

$$\left.\begin{array}{l}[A,B]=AB-BA=-B,\\ [A,C]=AC-CA=C,\\ [B,C]=BC-CB=-1=E\\ \text{and}\\ [A,E]=[B,E]=[C,E]=0.\end{array}\right\} \tag{2.6}$$

These commutator relations exhibits that the linear differential operators A, B, C, E generate a Lie group and the operator C commutes with the operators B and A.

The extended form of group generated by each of operators B and C can be expressed as,

$$e^{b'B}f(x,y)=f\left(\frac{xy^{-a}}{y^{-a}-b'},(y^{-a}-b')^{\frac{-1}{a}}\right), \tag{2.7}$$

$$e^{c'C}f(x,y)=e^{c'y^{-a}}\left(1-\frac{c'xy^{-a}}{\alpha}\right)^{-a}f\left(\frac{\alpha x}{\alpha-c'xy^{-a}},y\right)=e^{c'y^{-a}}\left(1-\frac{c'xy^{-a}}{\alpha}\right)^{-a}f\left(\frac{x}{1-\frac{c'xy^{-a}}{\alpha}},y\right), \tag{2.8}$$

$$e^{c'C}e^{b'B}f(x,y)=e^{c'y^{-a}}\left(1-\frac{c'xy^{-a}}{\alpha}\right)^{-a}f\left(\frac{\alpha xy^{-a}}{(y^{-a}-b')(\alpha-c'xy^{-a})},(y^{-a}-b')^{\frac{-1}{a}}\right),$$

$$e^{c'C}e^{b'B}f(x,y)=e^{c'y^{-a}}\left(1-\frac{c'xy^{-a}}{\alpha}\right)^{-a}f\left(\frac{xy^{-a}}{(y^{-a}-b')\left(1-\frac{c'xy^{-a}}{\alpha}\right)},(y^{-a}-b')^{\frac{-1}{a}}\right). \tag{2.9}$$

□

3. Generating Function Relations

By assigning different values to b' and c', the generating relations are determined for the following three cases:

Case 1. $b'=1$, $c'=0$.

Case 2. $b'=0$, $c'=1$.

Case 3. $b'c'\neq 0$.

Case 1. Putting $b'=1$ in equation (2.7),

$$e^{B}f(x,y)=f\left(\frac{xy^{-a}}{y^{-a}-1},(y^{-a}-1)^{\frac{-1}{a}}\right),$$

$$\exp B\{y^{-ma}X_m(x;a,\alpha)\}=(1-t)^m X_m\left(\frac{x}{1-t};a,\alpha\right),$$

$$\sum_{p=0}^{\infty}\frac{(-m)_p X_{m-p}(x;a,\alpha)t^p}{p!}=(1-t)^m X_m\left(\frac{x}{1-t};a,\alpha\right), \tag{3.1}$$

where $t=y^{-a}$.

Case 2. Putting $b'=0$, $c'=1$ in equation (2.8),

$$e^{C}f(x,y)=e^{y^{-a}}\left(1-\frac{xy^{-a}}{\alpha}\right)f\left(\frac{x}{1-\frac{xy^{-a}}{\alpha}},y\right),$$

$$\exp C\left\{y^{-ma}X_m(x;a,\alpha)\right\}=\exp(y^{-a})\left(1-\frac{xy^{-a}}{\alpha}\right)^{-a}f\left(\frac{x}{1-\frac{xy^{-a}}{\alpha}},y\right),$$

$$\sum_{p=0}^{\infty}\frac{X_{m+p}(x;a,\alpha)z^p}{p!}=\exp(z)\left(1-\frac{xz}{\alpha}\right)^{-a}X_m\left(\frac{x}{1-\frac{xz}{\alpha}};a,\alpha\right),$$

$$\exp(z)\left(1-\frac{xz}{\alpha}\right)^{-a}X_m\left(\frac{x}{1-\frac{xz}{\alpha}};a,\alpha\right)=\sum_{p=0}^{\infty}\frac{X_{m+p}(x;a,\alpha)z^p}{p!},\tag{3.2}$$

where $z=y^{-a}$.

Case 3. From equation (2.9) for $b'c'\neq 0$, putting $b'=w$ and $c'=1$, we get

$$e^Ce^{wB}f(x,y)=e^{y^{-a}}\left(1-\frac{xy^{-a}}{\alpha}\right)^{-a}f\left(\frac{xy^{-a}}{(y^{-a}-w)\left(1-\frac{xy^{-a}}{\alpha}\right)},(y^{-a}-w)^{\frac{-1}{\alpha}}\right),$$

$$\exp(C)\exp(wB)f(x,y)=e^{y^{-a}}\left(1-\frac{xy^{-a}}{\alpha}\right)^{-a}f\left(\frac{xy^{-a}}{(y^{-a}-w)\left(1-\frac{xy^{-a}}{\alpha}\right)},(y^{-a}-w)^{\frac{-1}{\alpha}}\right),$$

$$\exp(C)\exp(wB)\{y^{-ma}X_m(x;a,\alpha)\}=e^{y^{-a}}\left(1-\frac{xy^{-a}}{\alpha}\right)^{-a}((y^{-a}-w)^{\frac{-1}{\alpha}})^{-ma}\cdot X_m\left(\frac{xy^{-a}}{(y^{-a}-w)\left(1-\frac{xy^{-a}}{\alpha}\right)};a,\alpha\right),$$

$$\exp(C)\exp(wB)\{y^{-ma}X_m(x;a,\alpha)\}=e^{y^{-a}}\left(1-\frac{xy^{-a}}{\alpha}\right)^{-a}(y^{-a}-w)^m\cdot X_m\left(\frac{xy^{-a}}{(y^{-a}-w)\left(1-\frac{xy^{-a}}{\alpha}\right)};a,\alpha\right),$$

$$\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-m)_pw^p(y^{-a})^{m+q-p}X_{m+q-p}(x;a,\alpha)}{p!q!}=e^{y^{-a}}\left(1-\frac{xy^{-a}}{\alpha}\right)^{-a}(y^{-a}-w)^m\cdot X_m\left(\frac{xy^{-a}}{(y^{-a}-w)\left(1-\frac{xy^{-a}}{\alpha}\right)};a,\alpha\right).$$

Putting $y^{-a}=z$ in the above equation,

$$\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-m)_pw^pz^{m+q-p}X_{m+q-p}(x;a,\alpha)}{p!q!}=e^z\left(1-\frac{xz}{\alpha}\right)^{-a}(z-w)^mX_m\left(\frac{xz}{(z-w)\left(1-\frac{xz}{\alpha}\right)};a,\alpha\right)$$

or we can write,

$$e^z\left(1-\frac{xz}{\alpha}\right)^{-a}(z-w)^mX_m\left(\frac{xz}{(z-w)\left(1-\frac{xz}{\alpha}\right)};a,\alpha\right)=\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-m)_pw^pz^{m+q-p}X_{m+q-p}(x;a,\alpha)}{p!q!}.\tag{3.3}$$

Equations (3.1), (3.2) and (3.3) are new generating relations for the new class of semi-orthogonal polynomials $X_m(x;a,\alpha)$.

4. Conclusion

Three new generating relations are obtained for the new class of semi-orthogonal polynomials $X_m(x;a,\alpha)$ by using the Weisner's method. This method is a very powerful technique to obtain generating relations from the differential recurrence relations of the ascending and the descending type of the semi-orthogonal polynomial $X_m(x;a,\alpha)$.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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