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Research Article

On $*$-r-Clean Rings

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Abstract. A ring with an involution $*$ is called $*$-r-clean if every element is a sum of a projection and a regular element (in the sense of von Neumann). In this article, several connections between $*$-r-clean rings and $*$-clean rings are given. Different characterizations of $*$-r-clean rings are presented. The passage of this property to some extensions of rings is studied.

Keywords. Involution, $*$-r-clean ring, $*$-clean ring, Ring extensions

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1. Introduction

In the current paper, let R be an associative but not necessarily commutative ring having identity element, usually written as 1. Standardly, for such a ring R, the letters $U(R)$, $Nil(R)$, $Id(R)$, and $Reg(R)$ denote the set of invertible elements, the set of nilpotent elements, the set of idempotent elements, and the set of (von Neumann) regular elements in R, respectively. Likewise, $J(R)$ denotes the Jacobson radical of R. The ring of $n \times n$ matrices over R and the ring of $n \times n$ upper triangular matrices over R are denoted by $M_n(R)$ and $T_n(R)$, respectively. Recall that a ring is said to be abelian if each of its idempotents is central. Following Nicholson [12], an element of a ring R is called clean if it is the sum of an idempotent and a unit, and R is called clean if every element of R is clean. Ashrafi and Nasibi [3] introduced the concept of r-clean rings. Recall that an element of R is r-clean if it can be written as the sum of an idempotent and a regular element. A ring R is r-clean provided that every element is r-clean. A ring R is a $*$-ring (or ring with involution) if there exists an operation $*: R \to R$ such that for all $x, y \in R$, $(x+y)^* = x^* + y^*$, $(xy)^* = y^*x^*$, and $(x^*)^* = x$. All C^*-algebras and Rickart $*$-rings are $*$-rings.

Furthermore, every commutative ring can be regarded as a $*$-ring with the identity involution $*$. An element e in a $*$-ring R is called a projection if $e^2 = e = e^*$. Following Vaš [14], a $*$-ring R is called $*$-clean if every element of R is the sum of a projection and a unit, and R is strongly $*$-clean if every element of R is the sum of a projection and a unit that commute. Sharma and Basnet in [13] called a $*$-ring R $*$-r-clean if every element of R is the sum of a projection and a regular element.

The main purpose of this paper is to continue to explore r-cleanness for $*$-rings. In Section 2, several conditions for $*$-r-clean rings to be $*$-clean rings are given. It is proved that a $*$-ring R is $*$-clean if R is $*$-r-clean and every idempotent is projection. We show that strongly $*$-r-clean rings and strongly $*$-clean rings are equivalent notions under $*$-abelian Rickart condition. Also, it is proved that R is $*$-clean if and only if R is $*$-r-clean and for each $e^2 = e \in R$, $e - e^* \in J(R)$ and $ee^* = e^*e$. As consequences, various properties of strongly $*$-r-clean rings are derived by investigating extension problems.

2. $*$-r-Clean Rings

Next, we list some properties of the $*$-r-clean rings extracted from [13].

Proposition 2.1. *Let R be a $*$-ring. Then*

(1) *If R is abelian and $a \in R$ is $*$-clean in R and $e \in Id(R)$ is a projection. Then*

  (i) *ae is $*$-clean.*

  (ii) *If $-a$ is also $*$-clean, then $a + e$ is $*$-clean.*

(2) *If $e^2 = e$ is a projection and $a \in eRe$ is strongly $*$-clean in eRe, then a is strongly $*$-clean in R.*

(3) *Let R be an abelian ring, where any idempotent of the form $e = ry$ or yr is a projection, for any $r \in Reg(R)$. Then R is $*$-r-clean if and only if R is $*$-clean.*

The following result is motivated by [3, Theorem 2.2], in fact, it can easily be obtained from Proposition 2.1(3). The direct proof given below is based on the fact that if every idempotent is a projection in a $*$-ring R, then R is abelian (see [10, Lemma 2.1]).

Theorem 2.2. *Let R be a $*$-ring. If R is every idempotent is a projection, then R is $*$-r-clean if and only if R is $*$-clean.*

Proof. One direction is trivial. Conversely, let R be $*$-r-clean and $x \in R$. Then $x = e' + r$, where $e' = e'^* \in Id(R)$ and $r \in Reg(R)$. So there exists $y \in R$ such that $ryr = r$. Clearly, $e = ry$ and yr are idempotents. Since R is abelian, we have $(re + (1-e))(ye + (1-e)) = (ye + (1-e))(re + (1-e)) = 1$, thus $u = re + (1-e)$ is a unit. Furthermore, $r = re = eu$. Now, set $f = 1 - e$. Then, $eu + f$ and hence, $-(eu + f)$ is a unit. Since f is an idempotent and hence a projection by hypothesis, so $-r = f + (-(eu + f))$ is $*$-clean. It follows by Proposition 2.1(b) that x is $*$-clean, as required. □

Corollary 2.3. *Let R be a $*$-ring. If every idempotent is a projection, then R is $*$-r-clean if and only if R is strongly $*$-clean.*

The following proposition gives an equivalent condition between r-clean and $*$-r-clean. Recall that an element t of a $*$-ring R is self-adjoint square root of 1 if $t^2 = 1$ and $t^* = t$.

Proposition 2.4. *Let R be a $*$-ring. If $2 \in U(R)$, and for any $u^2 = 1$, $u^* = u \in R$, then R is r-clean if and only if R is $*$-r-clean.*

Proof. Assume that R is r-clean. For any $a \in R$ and $a = e + r$, where $e \in Id(R)$ and $r \in Reg(R)$. Then $(2e-1)^2 = 1$. So we have $2e = 2e^*$, and hence $2(e - e^*) = 0$. Since $2 \in U(R)$, it implies that $e = e^*$ and so a is $*$-r-clean. Thus R is $*$-r-clean. The other direction is obvious. □

Corollary 2.5. *Let R be a $*$-ring. If $2 \in U(R)$, and for any $u^2 = 1$, $u^* = u \in R$, then R is r-clean if and only if R is $*$-clean.*

Proof. From the proof of Proposition 2.4, we get every idempotent is projection, so by Theorem 2.2, R is $*$-clean, as required. □

Theorem 2.6. *Let R be a $*$-ring and $2 \in U(R)$. Then following are equivalent:*

(1) *R is $*$-r-clean.*

(2) *Every element of R is a sum of a regular and a self-adjoint square root of* 1.

Proof. (1)⇒(2): Let $a \in R$. Then $\frac{1+a}{2} = p + r$ for a projection $p \in Id(R)$ and a $r \in Reg(R)$. It follows that $a = (2p-1) + 2r$ where $(2p-1)^* = 2p-1, (2p-1)^2 = 1$ and $2r \in Reg(R)$.

(2)⇒(1): Given any $a \in R$, then there exist $y, w \in R$ satisfying $2a - 1 = y + w$ with $y^* = y$, $y^2 = 1$ and $w \in Reg(R)$. Thus, $a = \frac{y+1}{2} + \frac{w}{2}$ is a $*$-r-clean expression since $\left(\frac{y+1}{2}\right)^* = \frac{y+1}{2}$, $\left(\frac{y+1}{2}\right)^2 = \frac{y+1}{2}$ and $\frac{w}{2} \in Reg(R)$. □

Theorem 2.7. *Let R be a $*$-ring. Then R is $*$-r-clean if and only if every element $x \in R$ can be written as $x = r - e$, where $r \in Reg(R)$ and $e \in Id(R)$ is a projection.*

Proof. Let R be $*$-r-clean and $x \in R$. Then as R is $*$-r-clean, so $-x = r + e$, where $r \in Reg(R)$ and $e = e^* \in Id(R)$. Hence $x = (-r) - e$, where $-r \in Reg(R)$ and $e = e^* \in Id(R)$.

Conversely, suppose that every element $x \in R$ can be written as $x = r - e$, where $r \in Reg(R)$ and $e = e^* \in Id(R)$. So for every element $x \in R$, we can write $-x = r - e$, where $r \in Reg(R)$ and $e = e^* \in Id(R)$. Hence $x = (-r) + e$, where $-r \in Reg(R)$ and $e = e^* \in Id(R)$. □

Proposition 2.8. *Let R be a $*$-r-clean. Then R is $*$-clean if one of the following conditions holds:*

(1) *For each $e^2 = e \in R, e - e^* \in J(R)$ and $ee^* = e^*e$.*

(2) *For each $e^2 = e \in R, e - e^* \in Nil(R)$ and $ee^* = e^*e$.*

Proof. Assume that (1) holds. Write $j := e - e^*$. Then

$$e(1-e^*) = j(1-e^*) \in J(R)$$

and

$$e^*(1-e) = -j(1-e) \in J(R).$$

As $ee^* = e^*e$, it readily follows that both $e(1-e^*)$ and $e^*(1-e)$ are also idempotents, which in turn implies that $e = ee^* = e^*e = e^*$ since $J(R)$ contains no nonzero idempotent. So every idempotent of R is a projection and the result follows from Theorem 2.2. The proof of (2) is similar to (1). □

Recall that in [14], a $*$-ring R is $*$-abelian if every projection of R is central.

Corollary 2.9. *Let R be a $*$-abelian and $*$-r-clean. Then R is $*$-clean if one of the following conditions holds:*

(1) *For each $e^2 = e \in R$, $e - e^* \in J(R)$.*

(2) *For each $e^2 = e \in R, e - e^* \in Nil(R)$.*

Proof. This proposition is a direct consequence of the above proposition. We nevertheless give another independent proof.

(1) Since $e - e^* \in J(R)$, one obtains by a straightforward check, that

$$x = 1 + (e - e^*)^*(e - e^*) \in U(R).$$

Then, $xe = ee^*e = ex$ and $x^* = x$, so that $xe^* = e^*x$ and $e = x^{-1}ee^*e$. Therefore, $x^{-1}ee^* = x^{-1}x^{-1}ee^*ee^* = x^{-1}ee^*x^{-1}ee^*$. Putting $p := x^{-1}ee^*$, we calculate that $p = p^2 = p^*$. Since $(1-e)p = (1-e)x^{-1}ee^* = 0$, one has that $p = ep$. However, $pe = (x^{-1}ee^*)e = x^{-1}(ee^*e) = x^{-1}xe = e$, so we get $eR = pR$. Since R is $*$-abelian, we easily obtain that $e = pe = ep = p$ and then every idempotent is a projection. Thus, the asked assertion follows directly from Theorem 2.2.
The proof of (2) similar to (1) since $1 + Nil(R) \subseteq U(R)$. □

Following Nicholson [12], a ring R is exchange if for every $a \in R$, there exists $e^2 = e \in aR$ such that $1 - e \in (1 - a)R$. Clean rings are exchange, the converse holds whenever the rings are abelian. Recall that a ring R is said to have idempotent stable range one (written $isr(R) = 1$) provided that for any $a, b \in R$, $aR + bR = R$ implies that $a + be \in U(R)$ for some $e \in Id(R)$ (see [6, 15]). If e is an arbitrary element of R (not necessarily an idempotent), then R is said to have stable range one. Clearly, if $isr(R) = 1$, then R is clean and has stable range one. Cui and Wang [7] extended the notion of $isr(R) = 1$ to $*$-versions. A $*$-ring R is said to have projection stable range one (written $psr(R) = 1$) if for any $a, b \in R$, $aR + bR = R$ implies there exists a projection $p \in R$ such that $a + bp \in U(R)$.

Proposition 2.10. *Let R be a $*$-ring. Then the following are equivalent:*

(1) *$psr(R) = 1$ and R is $*$-abelian.*

(2) *For any $a, b \in R$, $aR + bR = R$ implies there exists a projection $p \in comm(a)$ such that $a + bp \in U(R)$.*

(3) *$isr(R) = 1$ and every idempotent of R is a projection.*

(4) *R is clean (or exchange) and every idempotent of R is a projection.*

(5) *R is $*$-clean and $*$-abelian.*

(6) *R is strongly $*$-clean.*

(7) *For every $a \in R$, there exists a projection $p \in aR$ such that $1 - p \in (1 - a)R$.*

(8) *R is r-clean and every idempotent of R is a projection.*

Proof. The proof follows from [7, Proposition 4.2] and Theorem 2.2. □

A $*$-ring R is called $*$-regular if for any $x \in R$ there exists a projection $e \in R$ such that $xR = eR$. An involution $*$ of R is said to be proper if $xx^* = 0$ implies $x = 0$ for all $x \in R$. A ring element is right (left) regular if it does not have nontrivial right (left) annihilators. It is regular if it is left and right regular. Note that we use the term regular in the definition of r-clean and $*$-r-clean in the sense that $a \in R$ is regular if $a = axa$ for some $x \in R$, i.e., von Neumann regular. It is known [4, Proposition 3] that a $*$-ring R is $*$-regular if and only if R is von Neumann regular and the involution $*$ is proper.

Corollary 2.11. *Let R be a $*$-regular ring. Then R is $*$-r-clean if and only if R is $*$-clean.*

Proof. ($\Leftarrow$): This is obvious.

($\Rightarrow$): By Theorem 2.2, it suffices to show that every idempotent $e \in R$ is a projection. Since R is $*$-regular, there exists a projection f such that $eR = fR$. Since e, f are central idempotents, it follows that $e = f$. So every idempotent in R is a projection. □

Corollary 2.12. *Let R be a abelian Rickart $*$-ring. Then the following are equivalent:*

(1) *R is $*$-r-clean.*

(2) *R is r-clean.*

(3) *R is (strongly) $*$-clean.*

(4) *R is (strongly) clean.*

Proof. If R is an abelian Rickart $*$-ring, then every idempotent is projection by [14, Lemma 3]. □

According to [14, Lemma 3] (2), if R is a Rickart $*$-ring, then R is abelian if and only if R is $*$-abelian.

Corollary 2.13. *Let R be a $*$-abelian Rickart $*$-ring. Then the following are equivalent:*

(1) *R is $*$-r-clean.*

(2) *R is r-clean.*

(3) *R is (strongly) $*$-clean.*

(4) *R is (strongly) clean.*

Lemma 2.14. *Let R be a $*$-ring with no nonzero divisors. Then R is $*$-clean if R is $*$-r-clean.*

Proof. For every $x \in R$, we write $x = e + r$, where $e = e^* \in Id(R)$ and $r \in Reg(R)$. Then there exists $y \in R$ such that $ryr = r$. Now, if $r = 0$, then $x = e = (2e-1)+(1-e)$ is $*$-clean. But if $r \neq 0$, then since R is a ring with no zero divisors and $ryr = r$, so $r \in U(R)$. Hence x again is $*$-clean. □

Corollary 2.15. *Let R be a $*$-ring with no zero divisors. Then R is $*$-r-clean if and only if R is local.*

Proof. It follows from [11, Lemma 14] and Lemma 2.14. □

3. Some Extensions of $*$-r-Clean Rings

This section studies the behavior of the $*$-r-clean property under various ring constructions such as power series rings, polynomial rings and quotient rings.

Proposition 3.1. *Let R be a $*$-ring. If every idempotent of R is a projection, then R is $*$-r-clean if and only if $R[[x]]$ is $*$-r-clean.*

Proof. (⇒): Suppose R is $*$-r-clean. Then R is r-clean, so $R[[x]]$ is r-clean by [2, Theorem 15]. Since every idempotent of R is a projection, we have R is abelian by [10, Lemma 2.1] and every idempotent e of $R[[x]]$ is contained in R by [9, Lemma 1]. Hence every idempotent of $R[[x]]$ is a projection. So $R[[x]]$ is $*$-r-clean.

(⇐): Suppose $R[[x]]$ is $*$-r-clean. Then $R[[x]]$ is r-clean, so R is r-clean by [2, Theorem 15]. Thus R is $*$-r-clean since every idempotent of R is a projection. □

According to [11, Proposition 13], the polynomial ring $R[x]$ is never clean. Hence, $R[x]$ is not $*$-clean for any involution $*$.

Lemma 3.2. *Let R be a commutative ring. Then the polynomial ring $R[x]$ is never r-clean if $R \neq 0$.*

Proof. Suppose that $x = e + r$ where e is an idempotent and $r \in Reg(R[x])$. If $e = e_0 + e_1x + \cdots + e_nx^n$ and $r = r_0 + r_1x + \cdots + r_nx^n$. If $e \neq e_0$ then e has the form $e = e_0 + x^m g$ where $m \geq 1$ and $g = a + bx + \cdots$ where $a \neq 0$. Comparing coefficients of x^m in $e^2 = e$ gives $2a = a$, a contradiction. Hence $e = e_0$, so $e_0 - x = -r$ is a regular in $R[x]$. Then there exists a $h = a_0 + a_1x + \cdots + a_nx^n \in R[x]$ such that $(e_0 - x)h(e_0 - x) = e_0 - x$. But obviously this is a contradiction. □

Proposition 3.3. *The polynomial ring $R[x]$ is never r-clean if $R \neq 0$.*

Proof. Assume that x is r-clean and write $x = e + r$ where e is an idempotent and r is a regular, then we consider the subring S of R generated by 1 and r. Then $e \in S[x]$ and x is r-clean in $S[x]$. Since S is commutative, Lemma 3.2 implies that this is a contradiction. □

Corollary 3.4. *The polynomial ring $R[x]$ is never $*$-r-clean if $R \neq 0$.*

If R is a $*$-ring, then $U(R)^* = U(R)$ and hence $J(R)^* = J(R)$. Thus, $*$ induces an involution of $R/J(R)$ which is again denoted by $*$, where $\bar{a}^* = \overline{a^*}$. A ring is reduced if it has no nonzero nilpotent elements. A ring R is semiregular if $R/J(R)$ is regular and idempotents can be lifted modulo $J(R)$.

Proposition 3.5. *Let R be a semiregular $*$-ring. If every idempotent of R is a projection, then R is $*$-clean.*

Proof. Let R be a semiregular ring. As $R/J(R)$ is regular, so is r-clean. But every idempotent of R is a projection, so $R/J(R)$ is $*$-r-clean and hence $*$-clean by Theorem 2.2. □

An ideal I of a $*$-ring R is said to be $*$-ideal provided that $I^* \subseteq I$. If I is a $*$-ideal of a $*$-ring, it is easy to check that R/I is also a $*$-ring.

Theorem 3.6. *Let I be regular and $*$-ideal of a $*$-ring R. If projections can be lifted modulo I, then R is $*$-r-clean, if and only if R/I is $*$-r-clean.*

Proof. If R is $*$-r-clean, since the homomorphism image of a projection (resp., regualr) is also a projection (resp., regular), then R/I is $*$-r-clean.

If R/I is $*$-r-clean, then for any $a \in R, a + I$ is $*$-r-clean. Thus there exists a projection $e + I \in Id(R/I)$ such that $(a - e) + I \in Reg(R/I)$. Hence $((a - e) + I)(x + I)((a - e) + I) = (a - e) + I$ for some $x \in R$. So $(a-e)-(a-e)x(a-e) \in I$. Now, since I is regular, so $a-e \in Reg(R)$ by [5, Lemma 1]. Since projections can be lifted modulo I, we may assume that e is a projection of R. Therefore a is $*$-r-clean, as required. □

Lemma 3.7. *Let R be a $*$-abelian ring. If $J(R)$ is nil, then every projection of the factor-ring $R/J(R)$ can be lifted to a projection of R.*

Proof. Let $\bar{e} = \bar{e}^* = \bar{e}^2 \in R/J(R)$. Since all idempotents lift modulo $J(R)$ as $J(R)$ is supposed to be nil, we may without loss of generality assume that $e^2 = e \in R$. Since $\bar{e}^* = \overline{e^*}$, one concludes that $e - e^* \in J(R)$. Due to the proof of Corollary 2.9, there exists $p^2 = p^* = p \in R$ such that $eR = pR$. So, $e = pe$ and hence $p = ep = p^* = pe^*$, which in turn assures that

$$e - p = pe - pe^* = p(e - e^*) \in J(R).$$ □

Proposition 3.8. *Suppose R is a $*$-ring. Then R is $*$-r-clean if R is $*$-abelian, $J(R)$ is nil and $R/J(R)$ is $*$-r-clean and every idempotent of $R/J(R)$ is a projection.*

Proof. Since $R/J(R)$ is $*$-r-clean and every idempotent of $R/J(R)$ is a projection, it follows from Theorem 2.2 that $R/J(R)$ is itself $*$-clean. Let $a \in R$, we have $\bar{a} = \bar{p} + \bar{u}$ and, by Lemma 3.7, $p - e \in J(R)$ for some projection $e \in R$. Then $a - e \in U(R)$. Thus R is $*$-clean and hence $*$-r-clean. □

Theorem 3.9. *Let R be a $*$-ring. If $p_1, p_2, \dots, p_n$ are orthogonal central projections with $1 = p_1 + p_2 + \dots + p_n$, and $p_i R$ is $*$-r-clean for each i, then R is $*$-r-clean.*

Proof. Since the idempotents p_i are central $(1 \le i \le n)$, the ring R is a direct product of its subrings $p_i R$, i.e., $R \cong p_1 R \times p_2 R \times \cdots \times p_n R$. This yields the result. □

Let R be a $*$-ring and G be a group. Then $*$ induces an involution of the group ring RG, denoted by $*$, where $\left(\sum_g a_g g\right)^* = \sum_g a_g^* g^{-1}$ (see [10, Lemma 2.12]). A group G is called locally finite if every finitely generated subgroup of G is finite. A group G is a 2-group if the order of each element of G is a power of 2 .

Proposition 3.10. *Let R be a commutative semiperfect $*$-ring, let G be a group and $(eRe)G$ be $*$-r-clean for each local idempotent e in R. Then RG is $*$-r-clean.*

Proof. Since R is semiperfect, so by [1, Theorem 27.6], R has a complete orthogonal set $e_1, \dots, e_n$ of idempotents with $e_i R e_i$ a local ring for each i. So e_i is a local idempotent for each i. Now by hypothesis, $(e_i R e_i)G$ is $*$-r-clean. Since $(e_i R e_i)G \cong e_i(RG)e_i$ for each i, it follows that $e_i(RG)e_i$ is $*$-r-clean. Hence RG is $*$-r-clean by Theorem 3.9. □

The next example shows that a ring can be $*$-r-clean but not $*$-clean.

Example 3.11. Consider the ring R of the Bergmann example. For the convenience of the reader we recall the construction of this ring. Let K be a field with $char(K) \neq 2$, and $A = K[[X]]$. Let $Q = K[[X, X^{-1}]]$ be the field of fractions of A. Define $R = \{r \in End(A_K) : \exists\, q \in Q$ and $\exists n > 0$ with $r(a) = qa$ for all $a \in X^n A\}$. By [8], R is a regular ring, which not clean. In particular R is r-clean but not clean. Consider the regular ring $T = R \times R$ and define the involution $*$ on T by $(a,b)^* = (b,a)$. The projections of T are the elements (e,e) where e is an idempotent in R, the units of T are the elements of the form (u,v) where u and v are unit in R. The ring T is regular and hence $T = Reg(T) + (0,0)$, so that T is $*$-r-clean. Since R is not clean, it is clear that T is not $*$-clean.

Let $R = T_2(\mathbb{Z}/2\mathbb{Z})$ with $*$ is exchange of the diagonal elements. According to [10, Example 2.6], this ring is clean but not $*$-clean. One raises the following:

Question 1. *Can we find a ring R which is r-clean but not $*$-r-clean?*

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Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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