Communications in Mathematics and Applications
Vol. 16, No. 1, pp. 77–89, 2025
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v16i1.2991

http://www.rgnpublications.com

Research Article

Advancing Fuzzy Algebra: Polynomial Subrings, Zero Divisors, and Related Properties

Dinesh Kute*[1,2], Arundhati Warke[3] and Anil Khairnar[4]

[1] *Symbiosis Institute of Technology (SIT), Symbiosis International (Deemed University) (SIU), Pune, Maharashtra, India*
[2] *Applied Sciences and Humanities, Pimpri Chinchwad College of Engineering, Pune, Maharashtra, India*
[3] *Department of Mathematics, Symbiosis Institute of Technology (SIT), Symbiosis International (Deemed University) (SIU), Pune, Maharashtra, India*
[4] *Department of Mathematics, Abasaheb Garware College, Pune, Maharashtra, India*
***Corresponding author:** dineshkute90@gmail.com

Received: November 9, 2024 **Revised:** January 18, 2025 **Accepted:** January 27, 2025

Abstract. This paper aims to expand classical ring theory into the fuzzy context, thereby strengthening the theoretical framework of fuzzy algebra. It introduces various types of fuzzy subrings, including fuzzy polynomial subrings and fuzzy matrix subrings, and explores their properties using the concept of fuzzy zero divisors. A significant contribution of this work is the development of the McCoy theorem for fuzzy polynomial subrings, leading to the definition of fuzzy McCoy subrings. Further, by building upon the notion of fuzzy zero divisors, the paper defines new classes of fuzzy subrings, such as fuzzy Armendariz subrings and fuzzy reduced subrings, and examines their fundamental properties and interconnections.

Keywords. Fuzzy subring, Fuzzy zero divisor, Fuzzy polynomial subring, Fuzzy reduced subring, Fuzzy Armendariz subring

Mathematics Subject Classification (2020). 08A72, 20N25, 03E72, 13M99

1. Introduction

The foundational work in fuzzy set theory began with Zadeh's seminal article [18] in 1965. This pivotal paper opened up new avenues for research. Recognizing the significant role of abstract algebra in mathematics, many researchers in fuzzy mathematics have since developed fuzzy analogs of established algebraic structures, such as groups, rings, and fields, collectively known as fuzzy abstract algebra. These fuzzy versions offer a more nuanced approach to representing uncertainty and imprecision.

The fuzzification of algebraic structures was initiated by Rosenfeld [17], laying the groundwork for the development of fuzzy subgroups and fuzzy subgroupoids. Building on this foundation, Liu [10] introduced the concepts of fuzzy subrings and fuzzy ideals. Malik and Mordeson [11] further advanced the field by defining fuzzy subfields within a field and investigating their extension properties. Since then, a multitude of studies have focused on various fuzzy substructures in algebraic contexts. For example, Çuvalcıoğlu and Tarsuslu (Yılmaz) [5] established an isomorphism theorem for Intuitionistic Fuzzy Abstract Algebras, while Gulzar *et al*. [6] introduced the notions of the t-intuitionistic fuzzy normalizer and centralizer of t-intuitionistic fuzzy subgroups. Gulzar *et al*. [7] also explored the concept of complex fuzzy subfields and their associated algebraic structures. Alolaiyan *et al*. [3] developed the theory of bipolar fuzzy subrings, including the study of bipolar fuzzy quotient rings, bipolar fuzzy ring homomorphisms, and bipolar fuzzy ring isomorphisms, and examined their core algebraic properties. Muthukumaran and Anandh [14] introduced bipolar-valued multi-fuzzy normal subnear-rings in near-rings and presented several theorems related to this concept. Additionally, Razaq and Alhamzi [2] introduced Pythagorean fuzzy ideals in classical rings, analyzing their extensive algebraic properties, while Alghazzwi *et al*. [1] proposed the novel idea of q-rung orthopair fuzzy subrings and explored their algebraic characteristics. In this evolving landscape, Kute *et al*. [9] have contributed to the study of fuzzy Baer subrings and fuzzy Rickart subrings, investigating their fundamental properties and extending the theoretical foundation of fuzzy algebraic structures. Additionally, a comprehensive review by Kute *et al*. [8] explores fuzzy subring algebra, bridging the gap between theoretical developments and practical applications, thereby enriching the understanding of fuzzified algebraic frameworks.

An Armendariz ring R is defined by the property that for any polynomials $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \in R[x]$ and $g(x) = b_0 + b_1x + b_2x^2 + \ldots + b_nx^n$ in $R[x]$, if $f(x)g(x) = 0$, then $a_ib_j = 0$ for all i and j. This concept was first introduced by Armendariz [4] and later expanded upon by Rege and Chhawchharia [16]. A ring R is called reduced if it has no nonzero nilpotent elements. The theories of zero divisors and nilpotent elements are crucial for understanding and classifying both reduced and Armendariz rings. Specifically, the absence of nilpotent elements is essential in characterizing reduced rings, while the presence of zero divisors plays a significant role in the behavior of polynomial rings over Armendariz rings.

The primary objective of this paper is to extend these ring concepts into the fuzzy domain. We introduce fuzzy polynomial subrings and fuzzy matrix subrings and establish a fuzzy McCoy theorem for polynomial subrings through the concept of fuzzy zero divisors. Furthermore, we define fuzzy nilpotent elements and introduce a new subring called the fuzzy reduced subring. We also explore the concept of fuzzy Armendariz subrings and examine the relationships between fuzzy Armendariz subrings and fuzzy reduced subrings, thereby extending classical ring theory to encompass uncertainty and imprecision.

The structure of this paper is organized as follows: Section 2 presents essential preliminaries necessary for establishing the main results. In Section 3, we introduce the concept of fuzzy polynomial subrings over fuzzy subrings, along with the notion of fuzzy matrix subrings, and explore various related results. Section 4 offers an improved and more applicable formulation of fuzzy zero divisors, extending the traditional concept from classical sets to fuzzy sets, and examines their application to fuzzy polynomial subrings over fuzzy subrings. Section 5 states and proves McCoy's theorem for fuzzy polynomial subrings over fuzzy subrings. In Section 6, we define the notion of fuzzy nilpotent elements and subsequently introduce the concept of fuzzy reduced subrings. Additionally, we introduce the concept of fuzzy Armendariz subrings and discuss the fundamental properties of these fuzzy subrings. Finally, Section 7 concludes with closing remarks and final statements.

Throughout this article, R denotes a commutative ring with additive identity 0 and multiplicative identity 1.

2. Preliminaries

We begin by revisiting the basic concept of fuzzy sets, which play a key role in our further analysis.

Definition 2.1 ([18])**.** Let $R \neq \phi$ denote any non-empty set. A function $\eta : R \to [0,1]$ is referred to as a fuzzy subset of R. This function is also called as the membership function. The collection of all fuzzy subsets of R is represented by $FS(R)$.

Definition 2.2 ([17])**.** Let $(G,*)$ be a group and $\eta \in FS(g)$. Then $\eta : G \to [0,1]$ is a fuzzy subgroup of G if for any $g,h \in G$,

(i) $\eta(g*h) \geqslant \min\{\eta(g),\eta(h)\}$,

(ii) $\eta(g^{-1}) = \eta(g)$.

Definition 2.3 ([10])**.** Let $\eta \in FS(R)$. A function $\eta : R \to [0,1]$ is called a fuzzy subring of the ring R if for any $u,v \in R$,

(i) $\eta(u-v) \geqslant \min\{\eta(u),\eta(v)\}$,

(ii) $\eta(uv) \geqslant \min\{\eta(u),\eta(v)\}$.

Definition 2.4 ([10])**.** Let $\eta \in FS(R)$. A function $\eta : R \to [0,1]$ is called a fuzzy left ideal (or fuzzy right ideal) of the ring R if for any $u,v \in R$,

(i) $\eta(u-v) \geqslant \min\{\eta(u),\eta(v)\}$,

(ii) $\eta(uv) \geqslant \eta(v)$ (or $\eta(uv) \geqslant \eta(u)$).

A fuzzy subring η is called a fuzzy ideal if it is both a fuzzy left and a fuzzy right ideal.

Theorem 2.1 ([13])**.** *If $\eta : R \to [0,1]$ is a fuzzy subring of R, then for any $u,v \in R$,*

(i) $\eta(0) \geqslant \eta(u)$,

(ii) $\eta(-u) = \eta(u)$,

(iii) $\eta(u+v) \geqslant \min\{\eta(u),\eta(v)\}$.

3. Fuzzy Polynomial Subrings

Let us introduce the concept of fuzzy polynomial subrings over fuzzy subrings. Additionally, we will define fuzzy matrix subrings and explore various results related to fuzzy polynomial subrings.

Theorem 3.1. *Let η be a fuzzy subring of R. Define $\phi : R[x] \to [0,1]$ such that $\phi(p(x)) = \phi\left(\sum_{i=0}^{n} p_i x^i\right) = \min\{\eta(p_i) \mid i = 0 \text{ to } n\}$. Then, ϕ is a fuzzy subring of $R[x]$. Moreover, ϕ is a fuzzy ideal of ring $R[x]$ if η is a fuzzy ideal of R.*

Proof. Let $p(x) = \sum_{i=0}^{n} p_i x^i$ and $q(x) = \sum_{j=0}^{m} q_j x^j$ be any elements of $R[x]$. Then, $p(x) - q(x) = \sum_{k=0}^{l} (p_k - q_k)x^k$, where $l = \max\{m,n\}$. Consider

$$\begin{aligned}\phi(p(x)-q(x)) &= \phi\left(\sum_{k=0}^{l}(p_k - q_k)x^k\right)\\ &= \min_{0\leqslant k\leqslant l}\{\eta(p_k - q_k)\}\\ &\geqslant \min\left\{\min_{0\leqslant k\leq l}\{\eta(p_k),\eta(q_k)\}\right\}\\ &= \min\left\{\min_{0\leqslant i\leq n}\{\eta(p_i)\},\ \min_{0\leqslant j\leqslant n}\{\eta(q_j)\}\right\}\\ &= \min\{\phi(p(x)),\phi(q(x))\}.\end{aligned}$$

Thus,

$$\phi(p(x)-q(x)) \geqslant \min\{\phi(p(x)),\phi(q(x))\}.$$

Next, we know that

$$p(x)q(x) = \sum_{i=0}^{m+n}\left(\sum_{j=0}^{i} p_j q_{i-j}\right)x^i.$$

Hence,

$$\begin{aligned}\phi(p(x)q(x)) &= \phi\left(\sum_{i=0}^{m+n}\left(\sum_{j=0}^{i} p_j q_{i-j}\right)x^i\right)\\ &= \min_{0\leqslant i\leqslant m+n}\left\{\eta\left(\sum_{j=0}^{i} p_j q_{i-j}\right)\right\}\\ &= \min\{\eta(p_0q_0),\eta(p_0q_1+p_1q_0),\eta(p_0q_2+p_1q_1+p_2q_0),\dots\}\\ &\geqslant \min\{\min\{\eta(p_0),\eta(q_0)\},\min\{\min\{\eta(p_0),\eta(q_1)\},\min\{\eta(p_1),\eta(q_0)\}\},\\ &\qquad \min\{\min\{\eta(p_0),\eta(q_2)\},\min\{\eta(p_1),\eta(q_1)\},\min\{\eta(p_2),\eta(q_0)\},\dots\}\\ &= \min\{\min\{\eta(p_0),\eta(p_1),\eta(p_2)\dots\eta(p_n)\},\min\{\eta(q_0),\eta(q_1),\eta(q_2),\dots,\eta(q_m)\}\}\\ &= \min\{\phi(p(x)),\phi(q(x))\}.\end{aligned}$$

Thus, $\phi(p(x)q(x)) \geqslant \min\{phi(p(x)), \phi(q(x))\}$, proving that ϕ is a fuzzy subring of $R[x]$.
If η is a fuzzy ideal of R, we need to show that $\phi(p(x)q(x)) \geqslant \max\{\phi(p(x)), \phi(q(x))\}$. Consider

$$\begin{aligned}
\phi(p(x)q(x)) = \phi\left(\sum_{i=0}^{m+n}\left(\sum_{j=0}^{i} p_j q_{i-j}\right)x^i\right) \\
&= \min\{\eta(p_0q_0), \eta(p_1q_0 + p_0q_1), \eta(p_2q_0 + p_1q_1 + p_0q_2)\} \\
&\geqslant \min\{\max\{\eta(p_0), \eta(q_0)\}, \min\{\max\{\eta(p_0), \eta(q_1)\}, \max\{\eta(p_1), \eta(q_0)\}\}, \\
&\qquad \min\{\max\{\eta(p_2), \eta(q_0)\}, \max\{\eta(p_1), \eta(q_1)\}, \max\{\eta(p_0), \eta(q_2)\}\}, \dots\} \\
&= \max\{\min\{\eta(p_0), \eta(p_1), \dots \eta(p_n)\}, \min\{\eta(q_0), \eta(q_1), \dots \eta(q_m)\}\} \\
&= \max\{\phi(p(x)), \phi(q(x))\}.
\end{aligned}$$

Hence, $\phi(p(x)q(x)) \geqslant \max\{\phi(p(x)), \phi(q(x))\}$. Therefore, ϕ is a fuzzy polynomial ideal of $R[x]$. □

Definition 3.1. The function ϕ defined in Theorem 3.1 is referred to as the fuzzy polynomial subring of $R[x]$ over η.

Definition 3.2. A polynomial $p(x) = \sum_{i=0}^{n} p_i x^i \in R[x]$ is said to be a zero polynomial over ϕ if $\phi(p(x)) = \eta(0)$.

Definition 3.3. A polynomial $p(x) = \sum_{i=0}^{n} p_i x^i \in R[x]$ over ϕ has fuzzy degree k if $k = \max\{i \mid \eta(p_i) \neq \eta(0)\}$.

Following the approach used for fuzzy polynomial subrings in Theorem 3.1, we can state the theorem for fuzzy matrix subrings as follows:

Theorem 3.2. *Let $M_{m\times n}(R)$ represent the set of all $m \times n$ matrices over R, and let η be a fuzzy subring of R. Define $\psi : M_{m\times n}(R) \to [0,1]$ for any $[a_{ij}]_{m\times n} \in M_{m\times n}(R)$ by*

$$\psi([a_{ij}]_{m\times n}) = \min\{\eta(a_{ij})\}, \quad \textit{for all } i \textit{ and } j.$$

Then ψ is a fuzzy matrix subring of $M_{m\times n}(R)$.

4. Fuzzy Zero Divisor

In 2004, Ray [15] pioneered the investigation of fuzzy zero divisors within rings. In this section, we introduce an enhanced and more suitable formulation of fuzzy zero divisors, extending the traditional concept from classical sets to fuzzy sets. Additionally, we explore the application of fuzzy zero divisors to fuzzy polynomial subrings over fuzzy subrings.

Definition 4.1. Let η be a fuzzy subring of R. An element $s \in R$ is called a left (or right) fuzzy zero divisor if there exists $t \in R$ such that $\eta(t) \neq \eta(0)$ but $\eta(st) = \eta(0)$ (or $\eta(ts) = \eta(0)$).
An element s is called a fuzzy zero divisor if it is both a left and a right fuzzy zero divisor with respect to η.

Example 4.1. *Let* $\eta : \mathbb{Z}_3 \times \mathbb{Z}_3 \to [0,1]$ *be defined as,*

$$\eta(x,y) = \begin{cases} 0.8, & \text{if } (x,y) \in \{(0,0),(1,1),(2,2)\}, \\ 0.2, & \text{otherwise}. \end{cases}$$

It can be easily observed that η *is a fuzzy subring, and we have* $\eta((1,2)(2,1)) = \eta(0,0)$. *Thus,* $(1,2)$ *and* $(2,1)$ *are fuzzy zero divisors of* $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Example 4.2. *Let* $\eta : \mathbb{Z} \to [0,1]$ *be defined as,*

$$\eta(x) = \begin{cases} 0.9 & \text{if } x \in \{0, \pm 9, \pm 18, \pm 27, \dots\}, \\ 0.2 & \text{otherwise}. \end{cases}$$

Then η *is a fuzzy subring of* $\mathbb{Z}$, *and we have* $\eta(3 \cdot 6) = \eta(18) = \eta(0)$. *Hence,* 3 *and* 6 *are fuzzy zero divisors of* $\mathbb{Z}$.

Theorem 4.1. *Let* η *be a fuzzy subring of* R. *If* $\eta(x) \neq \eta(0)$ *for any* $0 \neq x \in R$, *then the set of zero divisors of* R *is similar to the set of fuzzy zero divisors of* R *with respect to* η.

Remark 4.1. In Examples 4.1 and 4.2, it is noteworthy that we identified fuzzy zero divisors that are distinct from the zero divisors of the ring.

We now extend the notion of fuzzy zero divisors to fuzzy polynomial subrings over fuzzy subrings with the following example

Example 4.3. *Let* $\eta : \mathbb{Q} \to [0,1]$ *be defined as,*

$$\eta(x) = \begin{cases} 0.9 & \text{if } x \in \mathbb{Z}, \\ 0.3 & \text{otherwise}, \end{cases}$$

and let $\phi : \mathbb{Q}[x] \to [0,1]$ *be a fuzzy polynomial subring over* η. *Consider the polynomial* $p(x) = \frac{3}{2}x + 3$. *Since there exists a non-zero polynomial* $q(x) = 2x + \frac{2}{3} \in \mathbb{Q}[x]$ *such that* $\phi(q(x)) = 0.3 \neq \eta(0)$. *Moreover,*

$$\phi(p(x)q(x)) = \phi\left(\left(\frac{3}{2}x + 3\right)\left(2x + \frac{2}{3}\right)\right) = \phi(3x^2 + 7x + 2) = 0.9 = \eta(0).$$

Therefore, $p(x)$ *is a fuzzy zero divisor in the ring* $\mathbb{Q}[x]$.

5. McCoy Theorem for Fuzzy Polynomial Subrings

In 1942, McCoy [12] applied the condition of zero divisors to polynomial rings, specifically stating that $f(x)$ is a zero divisor in $R[x]$ if and only if there exists a non-zero $k \in R$ such that $kf(x) = 0$.

Now, we are ready to state and prove McCoy's theorem for fuzzy polynomial subrings over fuzzy subrings.

Theorem 5.1. *If* η *is a fuzzy ideal of* R *and* ϕ *is a fuzzy polynomial subring of a polynomial ring* $R[x]$ *over* η. *Then a polynomial* $p(x) = \sum_{i=0}^{n} p_i x^i \in R[x]$ *is fuzzy zero divisor in* $R[x]$ *if and only if there exist* $0 \neq k \in R$ *with* $\eta(k) \neq \eta(0)$ *such that* $\phi(kp(x)) = \eta(0)$.

Proof. Suppose there exist $0 \neq k \in R$ such that $\phi(kp(x)) = \eta(0)$ and $\eta(k) \neq \eta(0)$. Then $p(x)$ has fuzzy zero divisors in $R[x]$.
Conversely, suppose $p(x) = \sum_{i=0}^{n} p_i x^i$ is fuzzy zero divisor in $R[x]$. In this case, we can find a minimal fuzzy degree polynomial $q(x) = \sum_{j=0}^{m} q_j x^j \in R[x]$ such that $\phi(p(x)q(x)) = \eta(0)$ with $\eta(q_m) \neq \eta(0)$. Hence, fuzzy degree of the polynomial $p(x)$ is m. Now it is enough to prove that $m = 0$. On the contrary, suppose that $m \geqslant 1$. If $\phi(p_i q(x)) = \eta(0)$, for all i, then $\eta(p_i q_m) = \eta(0)$, for all i. Thus, $\phi(p(x)q_m) = \eta(0)$, a contradiction to the minimality of fuzzy degree of polynomial $q(x)$. Hence, the set $\{i \mid \phi(p_i q(x)) \neq \eta(0)\}$ is non-empty. Let $k = \max\{i \mid \phi(p_i q(x)) \neq \eta(0)\}$. Now consider,

$$\phi(p(x)q(x)) = \phi\left(\left(\sum_{i=0}^{k} p_i x^i\right)q(x) + \left(\sum_{i=k+1}^{n} p_i x^i\right)q(x)\right) = \eta(0). \tag{5.1}$$

From Definition 3.2, we get $\phi\left(\left(\sum_{i=0}^{k} p_i x^i\right)q(x)\right) = \eta(0)$. This gives $\eta(p_k q_m) = \eta(0)$ and hence fuzzy degree of the polynomial $p_k q(x)$ is less than m. Also, $\phi(p_k q(x)) \neq \eta(0)$. Let $\phi(p(x)p_k q(x)) = \phi(p_k(p(x)q(x))) \geqslant \max\{\phi(p_k), \phi(p(x)q(x))\} = \eta(0)$. Hence $\phi(p(x)p_k q(x)) = \eta(0)$, a contradiction to minimality of fuzzy degree of $q(x)$. Hence, $m = 0$. □

Definition 5.1. Let η be a fuzzy ideal of R, and let ϕ be a fuzzy polynomial subring of the polynomial ring $R[x]$ over η. A fuzzy ideal η is called a fuzzy McCoy subring of R, if every pair of non-zero polynomials $p(x), q(x) \in R[x]$ with $p(x)q(x) = 0$, there exists a non-zero element $k \in R$ such that $\eta(k) \neq \eta(0)$, and both $\phi(p(x)k) = \eta(0)$ and $\phi(kq(x)) = \eta(0)$.

Example 5.1. *Let $\eta : \mathbb{R} \to [0,1]$ be a fuzzy ideal defined as,*

$$\eta(x) = \begin{cases} 0.8, & \text{if } x \in \mathbb{Z}, \\ 0.1, & \text{otherwise.} \end{cases}$$

Clearly, $\phi : \mathbb{R}[x] \to [0,1]$ is a fuzzy polynomial ideal of $\mathbb{R}[x]$. Let $f(x) = \frac{3}{2}x^2 + 3x + 6$ and $g(x) = \frac{2}{3}x + 2 \in \mathbb{R}[x]$ be such that $\phi(f(x)g(x)) = \eta(0)$. Then, for $0 \neq k = \frac{2}{3} \in \mathbb{R}$, we have $\phi(kf(x)) = \phi\left(\frac{2}{3}\left(\frac{3}{2}x^2 + 3x + 6\right)\right) = \eta(0)$. Hence, $f(x)$ is a fuzzy zero divisor in the ring $\mathbb{R}[x]$.

6. Fuzzy Armendariz Subrings and Fuzzy Reduced Subrings

In this section, we define the notion of a fuzzy nilpotent element and consequently introduce the concept of a fuzzy reduced subring. Additionally, we introduce the notion of a fuzzy Armendariz subring and discuss the fundamental properties of both these fuzzy subrings.

Definition 6.1. Let η be a fuzzy subring of ring R. An element $x \in R$ is called a fuzzy nilpotent element if there exists a natural number n such that $\eta(x^n) = \eta(0)$.

Example 6.1. *Let $\eta : \mathbb{Z}_6 \to [0,1]$ be a fuzzy subring defined as,*

$$\eta(x) = \begin{cases} 0.9, & \text{if } x \in \{0,2,4\}, \\ 0.5, & \text{otherwise.} \end{cases}$$

Then, $0, 2,$ *and* 4 *are fuzzy nilpotent elements in* $\mathbb{Z}_6$ *with respect to* η*.*

Remark 6.1. Every nilpotent element of a ring is also a fuzzy nilpotent element, but the converse is not necessarily true (see Example 6.1).

Theorem 6.1. *If* η *is a fuzzy ideal of* R *and* $x, y \in R$ *are fuzzy nilpotent elements with respect to* η*, then* $x + y$ *is also a fuzzy nilpotent element with respect to* η*.*

Proof. Let $\eta(x^m) = \eta(0)$ and $\eta(y^n) = \eta(0)$, for some $m, n \in \mathbb{N}$. Consider $t = mn + 1$. We will show that $\eta((x + y)^t) = \eta(0)$. Using the binomial expansion, we have

$$(x + y)^t = \sum_{k=0}^{t} \binom{t}{k} x^{t-k} y^k.$$

For any term $\binom{t}{k} x^{t-k} y^k$, if $t - k \le m$ and $k \le n$, then, $t = (t - k) + k \le m + n \le mn$. However, since $t = mn + 1$, this inequality implies:

$$t \le m + n \ \text{ and } \ mn < t \quad \Longrightarrow \quad t - k > m \ \text{ or } \ k > n.$$

Therefore, $\eta\left(\binom{t}{k} x^{t-k} y^k\right) \ge \max\left\{\eta\left(\binom{t}{k}\right), \eta(x^{t-k}), \eta(y^k)\right\} = \eta(0)$. Thus,

$$\eta((x + y)^t) = \eta\left(\sum_{k=0}^{t} \binom{t}{k} x^{t-k} y^k\right) \ge \eta(0).$$

Since $\eta((x + y)^t) = \eta(0)$, it follows that $x + y$ is a fuzzy nilpotent element of R with respect to η. □

Theorem 6.2. *If* η *is a fuzzy ideal of* R *and* $x \in R$ *is a fuzzy nilpotent element with respect to* η*, then for any* $y \in R$*, both* xy *and* yx *are fuzzy nilpotent elements with respect to* η*.*

Proof. Let $\eta(x^n) = \eta(0)$, for some $n \in \mathbb{N}$. Now for any $y \in R$, consider $\eta((xy)^n) = \eta(x^n y^n) \geqslant \max\{\eta(x^n), \eta(y^n)\} = \eta(0)$. Similarly, we can show that $\eta((yx)^n) = \eta(0)$. Hence, xy and yx are fuzzy nilpotent elements with respect to η, for all $x \in R$. □

Definition 6.2. A fuzzy subring η of ring R is called a fuzzy reduced subring of R if it has no non zero fuzzy nilpotent elements. Equivalently, if for any $x \in R$, $\eta(x^n) = \eta(0)$ then $\eta(x) = \eta(0)$.

Theorem 6.3. *Let* η *be a fuzzy ideal of* R*. If* η *is a fuzzy reduced subring of* R *and* ϕ *is a fuzzy polynomial subring of polynomial ring* $R[x]$ *over* η*. Let* $p(x) = \sum_{i=0}^{m} a_i x^i$ *and* $q(x) = \sum_{j=0}^{n} b_j x^j$ *be two polynomials in* $R[x]$ *such that* $\phi(p(x)q(x)) = \eta(0)$ *then* $\eta(a_i b_j) = \eta(0)$*, for all* i *and* j*.*

Proof. We are given that $\phi(p(x)q(x)) = \eta(0)$ and hence $\eta(a_0 b_0) = \eta(0)$. We will prove the theorem by induction on $i + j$. Let $0 < l \le m + n$. Assume that $\eta(a_r b_s) = \eta(0), 0 \le r + s < l$. We need to show that $\eta(a_r b_s) = \eta(0)$ for $r + s = l$. Consider,

$$\begin{aligned}
\eta((a_r b_s)^2) &= \eta\left(\left(\sum_{i<r} a_i b_j\right) a_r b_s + (a_r b_s)^2 + \left(\sum_{i>r} a_i b_j\right) a_r b_s - \left(\sum_{i<r} a_i b_j\right) a_r b_s - \left(\sum_{i>r} a_i b_j\right) a_r b_s\right) \\
&\geqslant \min\left\{\eta\left(\left(\sum_{i<r} a_i b_j\right) a_r b_s + (a_r b_s)^2 + \left(\sum_{i>r} a_i b_j\right) a_r b_s\right), \eta\left(\left(\sum_{i<r} a_i b_j\right) a_r b_s + \left(\sum_{i>r} a_i b_j\right) a_r b_s\right)\right\}
\end{aligned}$$

$$= \min\left\{\eta\left((a_r b_s)\left(\sum_{i<r} a_i b_j + a_r b_s + \sum_{i>r} a_i b_j\right)\right), \eta\left(\left(\sum_{i<r} a_i b_j\right) a_r b_s + \left(\sum_{i>r} a_i b_j\right) a_r b_s\right)\right\}.$$

Since,

$$\begin{aligned}\eta\left(a_r b_s\left(\sum_{i<r} a_i b_j + a_r b_s + \sum_{i>r} a_i b_j\right)\right) &\geqslant \max\left\{\eta(a_r b_s), \eta\left(\sum_{i<r} a_i b_j + a_r b_s + \sum_{i>r} a_i b_j\right)\right\}\\ &= \max\{\eta(a_r b_s), \eta(p(x)q(x))\}\\ &= \max\{\eta(a_r b_s), \eta(0)\}\\ &= \eta(0).\end{aligned}$$

Thus,

$$\eta\left((a_r b_s)\left(\sum_{i<r} a_i b_j + a_r b_s + \sum_{i>r} a_i b_j\right)\right) = \eta(0).$$

Now,

$$\eta\left(\left(\sum_{i<r} a_i b_j\right) a_r b_s + \left(\sum_{i>r} a_i b_j\right) a_r b_s\right) \geqslant \min\left\{\eta\left(\left(\sum_{i<r} a_i b_j\right) a_r b_s\right), \eta\left(\left(\sum_{i>r} a_i b_j\right) a_r b_s\right)\right\}.$$

By induction, if $i < r$ and $i + s < r + s = l$, we have $\eta(a_i b_s) = \eta(0)$. For $i > r$ and $i + j = l = r + s$, we have $j < s$ then $r + j < r + s = l$. Hence, $\eta(a_r b_j) = \eta(0)$. Thus, $\eta\left(\left(\sum_{i<r} a_i b_j\right) a_r b_s\right) = \eta(0)$ and $\eta\left(\left(\sum_{i>r} a_i b_j\right) a_r b_s\right) = \eta(0)$ and hence $\eta\left(\left(\sum_{i<r} a_i b_j\right) a_r b_s + \left(\sum_{i>r} a_i b_j\right) a_r b_s\right) = \eta(0)$. Therefore, $\eta((a_r b_s)^2) = \eta(0)$, and since R is fuzzy reduced subring, $\eta(a_r b_s) = \eta(0)$. □

Theorem 6.4. *Let η be a fuzzy ideal and a fuzzy reduced subring of R, and let ϕ be a fuzzy polynomial subring of $R[x]$ over η. If $f(x) \in R[x]$ is such that $\phi(f(x) - f^2(x)) = \eta(0)$, then $f(x)$ is a fuzzy constant polynomial over ϕ.*

Proof. Let $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ and $\phi(f(x) - f^2(x)) = \phi(f(x)(1 - f(x))) = \eta(0)$, this implies $\phi\left(\left(\sum_{i=0}^{n} a_i x^i\right)\left[(1 - a_0) + \sum_{i=1}^{n} a_i x^i\right]\right) = \eta(0)$. By Theorem 6.3, it follows that $\eta(a_i a_j) = \eta(0)$, for all $i, j \geq 1$. Consequently, $\eta(a_i^2) = \eta(0)$ for all $i \geq 1$, and thus $\eta(a_i) = \eta(0)$ for all $i \geq 1$. Hence, $f(x)$ is a fuzzy constant polynomial over ϕ . □

Definition 6.3. Let η be a fuzzy subring of R and ϕ be a fuzzy polynomial subring of $R[x]$ over η. Then η is called a fuzzy Armendariz subring of R if, whenever $p(x) = \sum_{i=0}^{m} a_i x^i$, $q(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ such that $\phi(p(x)q(x)) = \eta(0)$ then $\eta(a_i b_j) = \eta(0)$ for all i and j.

Theorem 6.5. *Suppose η is a fuzzy Armendariz subring of R and ϕ is a fuzzy polynomial subring of $R[x]$ over η. If $p_1(x), p_2(x), \ldots, p_n(x) \in R[x]$ are such that $\phi(p_1(x)p_2(x)\cdots p_n(x)) = \eta(0)$, then $\eta(a_1 a_2 \cdots a_n) = \eta(0)$ for all coefficients a_i of $p_i(x)$, for all i.*

Proof. Let a_i be any coefficient of $p_i(x)$, for all i. Suppose $\phi(p_1(x)p_2(x)\cdots p_n(x)) = \eta(0)$. Therefore, $\phi(p_1(x)(p_2(x)\cdots p_n(x))) = \eta(0)$. By Definition 6.3, we have $\eta(a_1 t) = \eta(0)$ for any

coefficient t of the polynomial $p_2(x)\cdots p_n(x)$. Hence, $\phi(a_1p_2(x)\cdots p_n(x))=\eta(0)$.
Now, $\phi((a_1p_2(x))(p_3(x)\cdots p_n(x)))=\eta(0)$, and hence $\eta((a_1a_2)s)=\eta(0)$ for any coefficient s of the polynomial $p_3(x)\cdots p_n(x)$. Thus, $\phi(a_1a_2(p_3(x)\cdots p_n(x)))=\eta(0)$.
By continuing this process, we eventually obtain $\eta(a_1a_2\cdots a_n)=\eta(0)$. □

Example 6.2. *Let $U_{n\times n}(R)$ denote the set of upper triangular matrices over the ring R, and let η be a fuzzy subring of R with $\eta(a)\neq\eta(0)$ for some $a\in R$. Consider ψ as a fuzzy matrix subring of $U_{n\times n}(R)$ over η and ϕ as a fuzzy polynomial subring of $U_{n\times n}(R)[x]$ over ψ. Define the following polynomials in $U_{n\times n}(R)[x]$:*

$$f(x)=\begin{bmatrix}a&0\\0&0\end{bmatrix}+\begin{bmatrix}a&-a\\0&0\end{bmatrix}x,\qquad g(x)=\begin{bmatrix}0&0\\0&a\end{bmatrix}+\begin{bmatrix}0&a\\0&a\end{bmatrix}x.$$

Suppose that $f(x)g(x)=0$. Hence, $\phi(f(x)g(x))=\eta(0)$. However,

$$\psi\left(\begin{bmatrix}a&0\\0&0\end{bmatrix}\begin{bmatrix}0&a\\0&a\end{bmatrix}\right)=\psi\left(\begin{bmatrix}0&a^2\\0&0\end{bmatrix}\right)=\eta(a^2)\neq\eta(0).$$

Therefore, ψ is not a fuzzy Armendariz subring of $U_{n\times n}(R)$.

Theorem 6.6. *Let η be a fuzzy ideal of R. If η is a fuzzy reduced subring of R, then η is also a fuzzy Armendariz subring of R.*

Proof. Straightforward. □

Theorem 6.7. *Let η be a fuzzy ideal of R. If η is a fuzzy reduced subring of R, then any fuzzy matrix subring ψ of $A=\left\{\begin{bmatrix}r&s&u\\0&r&t\\0&0&r\end{bmatrix}\middle|\, a,b,c,d\in R\right\}$ over η is a fuzzy Armendariz subring of A.*

Proof. Notice that for any $\begin{bmatrix}r&s&u\\0&r&t\\0&0&r\end{bmatrix},\begin{bmatrix}r'&s'&u'\\0&r'&t'\\0&0&r'\end{bmatrix}\in A$. The addition and multiplication of matrices in A are defined as:

$$\begin{bmatrix}r&s&u\\0&r&t\\0&0&r\end{bmatrix}+\begin{bmatrix}r'&s'&u'\\0&r'&t'\\0&0&r'\end{bmatrix}=(r+r',s+s',u+u',t+t'),$$

$$\begin{bmatrix}r&s&u\\0&r&t\\0&0&r\end{bmatrix}\cdot\begin{bmatrix}r'&s'&u'\\0&r'&t'\\0&0&r'\end{bmatrix}=(rr',rs'+sr',ru'+st'+ur',rt'+tr').$$

Let $f(x)$ and $g(x)$ be polynomials in $A[x]$ given by:

$$f(x)=\sum_{i=0}^{n}\begin{bmatrix}r_i&s_i&u_i\\0&r_i&t_i\\0&0&r_i\end{bmatrix}x^i=(f_0(x),f_1(x),f_2(x),f_3(x)),$$

where $f_0(x)=\sum\limits_{i=0}^{n}r_ix^i$, $f_1(x)=\sum\limits_{i=0}^{n}s_ix^i$, $f_2(x)=\sum\limits_{i=0}^{n}u_ix^i$ and $f_3(x)=\sum\limits_{i=0}^{n}t_ix^i$, and

$g(x)=\sum_{i=0}^{n}\begin{bmatrix} r_i' & s_i' & u_i' \\ 0 & r_i' & t_i' \\ 0 & 0 & r_i' \end{bmatrix} x^i=(g_0(x),g_1(x),g_2(x),g_3(x))$ such that $\phi(f(x)g(x))=\eta(0)$.

Therefore, $\phi(f_0(x)g_0(x),f_0(x)g_1(x)+f_1(x)g_0(x),f_0(x)g_2(x)+f_1(x)g_3(x)+f_2(x)g_0(x),f_0(x)g_3(x)+f_3(x)g_0(x))=\eta(0)$. From Theorem 6.3, since $\phi(f_i(x)g_j(x))=\eta(0)$, it follows that $\eta(r_is_j)=\eta(0)$, for all i and j.

Therefore,

$$\psi\left(\begin{bmatrix} r_i & s_i & u_i \\ 0 & r_i & t_i \\ 0 & 0 & r_i \end{bmatrix}\cdot\begin{bmatrix} r_j' & s_j' & u_j' \\ 0 & r_j' & t_j' \\ 0 & 0 & r_j' \end{bmatrix}\right)=\eta(0),$$

for all i and j, and hence ψ satisfies the condition to be a fuzzy Armendariz subring of A over η. □

7. Conclusion

Our exploration of subrings within the framework of fuzzy polynomial subrings, fuzzy reduced subrings, and fuzzy Armendariz subrings highlights their critical importance in the field of algebra. In this paper, we have delved into the theory of fuzzy zero divisors in rings and extended this analysis to fuzzy polynomial subrings. We also presented and proved McCoy's theorem for fuzzy polynomial subrings. Furthermore, we introduced and thoroughly examined the key properties of fuzzy Armendariz subrings and fuzzy reduced subrings. We anticipate that our findings will contribute to the advancement of the theory of fuzzy subrings and foster further research in the study of fuzzy algebraic structures.

Acknowledgments

Sincere thanks to Prof. B. N. Waphare from the Center for Advanced Studies in Mathematics, Department of Mathematics, Savitribai Phule Pune University, Pune, and Prof. Krishnat Masalkar from Department of Mathematics, Abasaheb Garware College, Pune, for their valuable suggestions.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] D. Alghazzwi, A. Ali, A. Almutlg, E. Abo-Tabl and A. A. Azzam, A novel structure of q-rung orthopair fuzzy sets in ring theory, *AIMS Mathematics* **8**(4) (2023), 8365 – 8385, DOI: 10.3934/math.2023422.

[2] A. Razaq and G.Alhamzi, On Pythagorean fuzzy ideals of a classical ring, *AIMS Mathematics* **8**(2) (2023), 4280 – 4303, DOI: 10.3934/math.2023213.

[3] H. Alolaiyan, M. H. Mateen, D. Pamucar, M. K. Mahmmod and F. Arslan, A certain structure of bipolar fuzzy subrings, *Symmetry* **13**(8) (2021), 1397, DOI: 10.3390/sym13081397.

[4] E. P. Armendariz, A note on extensions of Baer and P. P.-rings, *Journal of the Australian Mathematical Society* **18**(4) (1974), 470 – 473, DOI: 10.1017/S1446788700029190.

[5] G. Çuvalcıoğlu and S. Tarsuslu (Yılmaz), Isomorphism theorems on intuitionistic fuzzy abstract algebras, *Communications in Mathematics and Applications* **12(1)** (2021), 109 – 126, DOI: 10.26713/cma.v12i1.1446.

[6] M. Gulzar, D. Alghazzawi, M. Mateen and N. Kausar, A certain class of t-intuitionistic fuzzy subgroups, *IEEE Access* **8** (2020), 163260 – 163268, DOI: http://doi.org/10.1109/ACCESS.2020.3020366.

[7] M. Gulzar, F. Dilawar, D. Alghazzawi and M. Mateen, A note on complex fuzzy subfield, *Indonesian Journal of Electrical Engineering and Computer Science* **21**(2) (2021), 1048 – 1056, DOI: 10.11591/ijeecs.v21.i2.pp1048-1056.

[8] D. Kute, A. Warke, A. Khairnar and L. Sharma, A review of fuzzy subring algebra: Theories and case studies, *Nanotechnology Perceptions* **20**(6) (2024), 3364 – 3383, URL: https://nano-ntp.com/index.php/nano/article/view/3483.

[9] D. Kute, A. Warke, A. Khairnar, and L. Sharma, Fuzzy baer subrings: A fuzzified extension of Baer rings, *Advances in Nonlinear Variational Inequalities* **28**(2) (2025), 399 – 409, URL: https://internationalpubls.com/index.php/anvi/article/view/2352.

[10] W.-J. Liu, Fuzzy invariant subgroups and fuzzy ideals, *Fuzzy Sets and Systems* **8**(2) (1982), 133 – 139, DOI: 10.1016/0165-0114(82)90003-3.

[11] D. S. Malik and J. N. Mordeson, Fuzzy subfields, *Fuzzy Sets and Systems* **37**(3) (1990), 383 – 388, DOI: 10.1016/0165-0114(90)90034-4.

[12] N. H. McCoy, Remarks on divisors of zero, *The American Mathematical Monthly* **49**(5) (1942), 286 – 295, DOI: 10.2307/2303094.

[13] S. Melliani, I. Bakhadach and L. S. Chadli, Fuzzy rings and fuzzy polynomial rings, in: *Homological and Combinatorial Methods in Algebra* (SAA 2016), A. Badawi, M. Vedadi, S. Yassemi and A. Yousefian Darani (editors), Springer Proceedings in Mathematics & Statistics, Vol. 228, Springer, Cham., (2018), DOI: 10.1007/978-3-319-74195-6_8.

[14] S. Muthukumaran and B. Anandh, Bipolar valued multi fuzzy normal subnear-ring of a near-ring, *Communications in Mathematics and Applications* **14**(1) (2023), 397 – 403, DOI: 10.26713/cma.v14i1.2000.

[15] A. K. Ray, A note on fuzzy characteristic and fuzzy divisor of zero of a ring, *Novi Sad Journal of Mathematics* **34**(1) (2004), 39 – 45, URL: https://sites.dmi.uns.ac.rs/nsjom/Papers/34_1/nsjom_34_1_039_045.pdf.

[16] M. B. Rege and S. Chhawchharia, Armendariz rings, *Proceedings of the Japan Academy Series A Mathematical Sciences* **73**(1) (1997), 14 – 17, DOI: 10.3792/pjaa.73.14.

[17] A. Rosenfeld, Fuzzy groups, *Journal of Mathematical Analysis and Applications* **35**(3) (1971), 512 – 517, DOI: 10.1016/0022-247X(71)90199-5.

[18] L. A. Zadeh, Fuzzy sets, *Information and Control* **8**(3) (1965), 338 – 353, DOI: 10.1016/S0019-9958(65)90241-X.