Communications in Mathematics and Applications
Vol. 16, No. 1, pp. 45–54, 2025
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v16i1.2957

Research Article

Mixed Hegselmann-Krause Dynamics III

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Received: December 14, 2024 **Revised:** February 5, 2025 **Accepted:** February 20, 2025

Abstract. The mixed Hegselmann-Krause (HK) model encompasses both the Deffuant and HK models. Building upon our previous work (Mixed Hegselmann-Krause dynamics II, *Discrete and Continuous Dynamical Systems - B* **28**(5) (2023), 2981 – 2993), we delve into the mixed HK model within a heterogeneous interaction framework. This involves either pair interaction, where all interacting pairs approach each other equally at their rate, or group interaction at each time step. Our research focuses on identifying circumstances conducive to consensus formation within this heterogeneous interaction paradigm. Furthermore, we delve into pair interaction scenarios where interacting pairs can approach each other at distinct rates. This differs from the Deffuant model, where an interacting pair can only approach each other at the same rate under a homogeneous threshold. Our investigation also aims to elucidate the conditions under which consensus can be attained under pair interaction with distinct approaching rates.

Keywords. Mixed Hegselmann-Krause dynamics, Hegselmann-Krause model, Deffuant model, Social network, Heterogeneous interaction mode

Mathematics Subject Classification (2020). 91C20, 91D25, 91D30, 93D50, 94C15

1. Introduction

The Hegselmann-Krause (HK) model and the Deffuant model are prominent in discussions of group interaction and pair interaction. Both models set a positive confidence threshold ϵ to categorize whether two individuals are opinion neighbors or not. Two individuals are opinion neighbors if their opinion distance does not exceed the confidence threshold ϵ. Unlike discrete opinion dynamics (Cox and Durrett [4]), the HK model and the Deffuant model belong to continuous opinion dynamics (Bernardo *et al*. [1], Bhattacharyya *et al*. [2], Chen *et al*. [3], Deffuant *et al*. [5], Fortunato [6], Hegselmann and Krause [7], Lorenz [10, 11], Parasnis *et al*. [12], Proskurnikov and Tempo [13], Shang [14, 15], and Vasca *et al*. [16]). In the HK model, an individual updates its opinion by averaging the opinions of its opinion neighbors. In the Deffuant model, a pair of socially connected individuals are selected to update their opinion by approaching each other equally in opinion at a rate $\mu \in [0,1/2]$ if and only if they are opinion neighbors. The mixed HK model was proposed by Li [8], arguing that it encompasses the HK model. In the sequel, Li [9] further argued that the mixed HK model includes not only the HK model but also the Deffuant model. Li [9] discusses the mixed HK model under both pair interaction and group interaction.

As a follow-up to the work presented in [9], our study explores scenarios where the update rule can be dictated by either pair interaction or group interaction at each time step, resulting in a heterogeneous interaction mode. In a nutshell, the mixed HK model operates such that a set of individuals collectively decides the update during group interaction, while pairs of individuals determine the update during pair interaction. In detail, let $[n] = \{1,\dots,n\}$ be the collection of all individuals, let $G(t) = ([n], E(t))$ be the social graph at time t, let $\mathcal{G}(t) = ([n], \mathcal{E}(t))$ be the opinion graph at time t, let $\widetilde{G}(t) = ([n], \widetilde{E}(t))$ be the social graph for update at time t, let random variable $\alpha_i(t) \in [0,1]$ be the degree of stubbornness of individual i at time t, and let U_t, $t \geq 0$ with the update at time $t+1$ under same interaction mode, be independent and identically distributed random variables with support S. For update under pair interaction, $U_t = \{(i,j) \in [n]^2 : i \neq j \text{ and } (\alpha_i(t) < 1 \text{ or } \alpha_j(t) < 1)\}$ and $S \subset \{$all matching in the complete graph of order $n\}$, whereas for update under group interaction, $U_t = \{i \in [n] : \alpha_i(t) < 1\}$, $S \subset \mathcal{P}([n])$, the power set of $[n]$. $\widetilde{E}(t) = E(t)$ when the update at time $t+1$ is under group interaction, whereas $\widetilde{E}(t) = U_t \cap E(t)$ when the update at time $t+1$ is under pair interaction. The update at time $t+1$ depends on the profile $\widetilde{G}(t) \cap \mathcal{G}(t)$, the intersection of the social graph for update and the opinion graph at time t. $\mathcal{N}_i(t) = \{j \in [n] : i = j \text{ or } (i,j) \in \widetilde{E}(t) \cap \mathcal{E}(t)\}$ is the collection of social and opinion neighbors of individual i at time t for update. The update mechanism is given by:

$$x_i(t+1) = \alpha_i(t)x_i(t) + \frac{1-\alpha_i(t)}{|\mathcal{N}_i(t)|} \sum_{j \in \mathcal{N}_i(t)} x_j(t), \tag{1}$$

where $x_i(t) \in \mathbb{R}^d$ is the opinion of individual i at time t with $x_i(0) \in \mathbb{R}^d$ a random variable. In other words, an individual can determine its degree of stubbornness and mix its opinion with the average opinion of its social and opinion neighbors for update. When eq. (1) is under pair interaction for the update at time $t+1$, the relationship between the degree of stubbornness $\alpha_i(t)$ and the approaching rate $\mu_i(t)$ of individual i is given by $\alpha_i(t) = 1 - 2\mu_i(t)$. There are some definitions as follows:

$$\binom{[n]}{2} = \text{the collection of all two-element subsets of } [n],$$

$$Z_1(t) = \sum_{i,j \in [n]} (\|x_i(t) - x_j(t)\|^2 \wedge \epsilon^2) \vee \epsilon^2 \mathbb{1}\{(i,j) \notin E(t)\},$$

$$Z_2(t) = \sum_{i,j \in [n]} \|x_i(t) - x_j(t)\|^2,$$

$$Z_t = \begin{cases} Z_1(t), & \text{if the update at time } t \text{ is under group interaction,} \\ Z_2(t), & \text{if the update at time } t \text{ is under pair interaction,} \end{cases}$$

$$W_t = \sum_{i\in[n]} \|x_i(t) - c\|, \quad \text{for } c \in \mathbb{R}^d.$$

Additionally, we delve into pair interaction with distinct approaching rates, where interacting pairs approach each other at different rates. Note that the degree of stubbornness is controllable. To address an interacting pair approaching each other at distinct rates, we make the following assumption:

$$\alpha_i(t) \le \alpha_j(t) \text{ whenever } \|x_i(t) - c\| \ge \|x_j(t) - c\| \text{ and } (i,j) \in \widetilde{E}(t), \quad \text{for all } c \in \mathbb{R}^d. \tag{2}$$

Namely, the further an individual is from c, the smaller its degree of stubbornness, and thus the larger its approaching rate.

2. Main Results

Theorem 2.1 shows the circumstances under which a consensus occurs under the interplay of pair interaction, where all interacting pairs approach each other equally at their rate, and group interaction. The assumption of the complete opinion graph occurring at some time step does not weaken the assumption. Note that opinion graphs preserve completeness, and the social graph is controllable. Diverse profiles can be created under the assumption of Theorem 2.1.

Theorem 2.1. *Assume that $\bigcup_{a\in S} a$ contains $[n]$ for update under group interaction and $\binom{[n]}{2}$ for update under pair interaction, all interacting pairs approach each other equally at their rate for update under pair interaction, the opinion graph is complete at some time step and that one of the statements holds:*

(i) $0 \le \limsup_{t\to\infty} \sup\{\alpha_i(t) : i \in [n] \text{ and } \alpha_i(t) < 1\} < 1$ *and the social graph is connected after some finite time.*

(ii) $0 \le \sup_{t\ge 0} \sup\{\alpha_i(t) : i \in [n] \text{ and } \alpha_i(t) < 1\} < 1$ *and the social graph is connected infinitely many times.*

Then, there is a consensus if the following conditions are met at each time t after some finite time:

(i) *The social graph is complete at time t when updates at times t and $t+1$ occur under pair interaction with equal approaching rates and group interaction.*

(ii) $E(t) \subset E(t+1)$ *when updates at times t and $t+1$ occur under group interaction.*

Theorem 2.2 reveals conditions under which a consensus can be achieved under pair interaction, where interacting pairs can approach each other at distinct approaching rates.

Theorem 2.2. *Assume that* (2) *holds, the opinion graph is complete at some time step, the social graph is connected infinitely many times,*

$$\bigcup_{a\in S} a \supset \binom{[n]}{2} \text{ under pair interaction and } \inf_{t\ge 0} \min_{i,j\in[n];i\ne j} |\alpha_i(t) - \alpha_j(t)| > 0.$$

Then, there is a consensus under pair interaction.

The pair interaction in the mixed HK model indicates that an interacting pair approach each other toward the center of their opinions. In fact, Theorem 2.2 works for an interacting pair approaching each other toward any point on the line segment of their opinions.

3. Interplay Between Pair Interaction With Equal Approaching Rates and Group Interaction

In this section, we investigate the interplay between pair interaction with equal approaching rates and group interaction. Pair interaction with equal approaching rates indicates all interacting pairs approaching each other equally at their rate. We construct the function Z_t to discuss $Z_t - Z_{t+1}$ under the transitions from pair interaction or group interaction to pair interaction or group interaction. It turns out that for Z_t to be nonincreasing, there is no restriction on the opinion graph for update under group interaction. There is no restriction on the social graph for update under pair interaction. In particular, there is no restriction on the social graph and opinion graph from pair interaction to pair interaction.

Lemma 3.1. *If updates at times t and $t+1$ are under pair interaction with equal approaching rates and group interaction and the social graph is complete at time t, then*

$$Z_t - Z_{t+1} \geq 4 \sum_{i\in[n]} \|x_i(t) - x_i(t+1)\|^2.$$

Proof. Let $x_i = x_i(t)$, $x_i^\star = x_i(t+1)$ and $\mathcal{N}_i = \mathcal{N}_i(t)$ for all $i \in [n]$. Observe that $j \notin \mathcal{N}_i$ and the social graph complete at time t imply individuals i and j are not opinion-connected. If updates at times t and $t+1$ are under pair interaction and group interaction and the social graph is complete at time t, then

$$\begin{aligned}
Z_t - Z_{t+1} &= Z_2(t) - Z_1(t+1) \\
&\geq \sum_{i,j\in[n]} (\|x_i - x_j\|^2 \wedge \epsilon^2 - \|x_i^\star - x_j^\star\|^2 \wedge \epsilon^2) \\
&= \sum_{i\in[n]} \sum_{j\in\mathcal{N}_i} (\|x_i - x_j\|^2 - \|x_i^\star - x_j^\star\|^2 \wedge \epsilon^2) + \sum_{i\in[n]} \sum_{j\in\mathcal{N}_i^c} (\epsilon^2 - \|x_i^\star - x_j^\star\|^2 \wedge \epsilon^2) \\
&\geq \sum_{i\in[n]} \sum_{j\in\mathcal{N}_i} (\|x_i - x_j\|^2 - \|x_i^\star - x_j^\star\|^2) \\
&= \sum_{i\in[n]} \sum_{j\in\mathcal{N}_i} (\|x_i - x_i^\star\|^2 + \|x_j - x_j^\star\|^2 + 2\langle x_i^\star - x_j^\star, x_j^\star - x_j\rangle + 2\langle x_i - x_i^\star, x_i^\star - x_j\rangle) \\
&= 2 \sum_{i\in[n]} |\mathcal{N}_i| \|x_i - x_i^\star\|^2 + 2 \sum_{i\in[n]} \sum_{j\in\mathcal{N}_i} \langle x_i^\star - x_i, x_j^\star - x_j\rangle \\
&\quad + 2 \sum_{j\in[n]} \sum_{i\in\mathcal{N}_j} \langle x_i - x_j^\star, x_j^\star - x_j\rangle + 2 \sum_{i\in[n]} \sum_{j\in\mathcal{N}_i} \langle x_i - x_i^\star, x_i^\star - x_j\rangle \\
&\geq 2 \sum_{i\in[n]} |\mathcal{N}_i| \|x_i - x_i^\star\|^2 + 2 \sum_{i\in[n]} \|x_i^\star - x_i\|^2 \\
&\quad + \sum_{i\in[n]} \sum_{j\in\mathcal{N}_i - \{i\}} [(\|x_i^\star - x_i\| - \|x_j^\star - x_j\|)^2 - \|x_i^\star - x_i\|^2 - \|x_j^\star - x_j\|^2] \\
&\quad + 4 \sum_{i\in[n]} \frac{\alpha_i}{1-\alpha_i} |\mathcal{N}_i| \mathbb{1}\{\alpha_i < 1\} \|x_i - x_i^\star\|^2
\end{aligned}$$

$$
\begin{aligned}
&= 4\sum_{i\in[n]}\left(1+\frac{\alpha_i}{1-\alpha_i}|\mathcal{N}_i|\mathbb{1}\{\alpha_i<1\}\right)\|x_i-x_i^\star\|^2 \\
&\geq 4\sum_{i\in[n]}\|x_i-x_i^\star\|^2. \qquad \square
\end{aligned}
$$

Observe that

$$
\sum_{i\in[n]}\sum_{j\in\mathcal{N}_i}(\|x_i-x_j\|^2-\|x_i^\star-x_j^\star\|^2)\geq 4\sum_{i\in[n]}\|x_i-x_i^\star\|^2 \tag{3}
$$

if the update at time $t+1$ is under group interaction.

Lemma 3.2. *If updates at times t and $t+1$ are under group interaction and pair interaction with equal approaching rates and the opinion graph is complete at time t, then*

$$
Z_t-Z_{t+1}\geq 2n\sum_{i\in[n]}\|x_i(t)-x_i(t+1)\|^2.
$$

Proof. If updates at times t and $t+1$ are under group interaction and pair interaction and the opinion graph is complete at time t, then

$$
\begin{aligned}
Z_t-Z_{t+1} &= Z_1(t)-Z_2(t+1) \\
&\geq \sum_{i,j\in[n]}(\|x_i-x_j\|^2-\|x_i^\star-x_j^\star\|^2). \qquad (4)
\end{aligned}
$$

If $(p,\hat{p})$ is an interacting pair with approaching rate $\mu>0$, then

$$
\|x_p-x_{\hat{p}}\|^2-\|x_p^\star-x_{\hat{p}}^\star\|^2=4\frac{1-\mu}{\mu}\|x_p-x_p^\star\|^2.
$$

If $(p,\hat{p})$ and $(q,\hat{q})$ are distinct interacting pairs with positive approaching rates μ and ν, then

$$
\begin{aligned}
&\|x_p-x_q\|^2-\|x_p^\star-x_q^\star\|^2+\|x_p-x_{\hat{q}}\|^2-\|x_p^\star-x_{\hat{q}}^\star\|^2+\|x_{\hat{p}}-x_q\|^2-\|x_{\hat{p}}^\star-x_q^\star\|^2 \\
&+\|x_{\hat{p}}-x_{\hat{q}}\|^2-\|x_{\hat{p}}^\star-x_{\hat{q}}^\star\|^2 \\
&\quad=4\frac{1-\mu}{\mu}\|x_p-x_p^\star\|^2+4\frac{1-\nu}{\nu}\|x_q-x_q^\star\|^2.
\end{aligned}
$$

If $(p,\hat{p})$ is an interacting pair with positive approaching rate μ and q is in none of the interacting pairs, then

$$
\|x_p-x_q\|^2-\|x_p^\star-x_q\|^2+\|x_{\hat{p}}-x_q\|^2-\|x_{\hat{p}}^\star-x_q\|^2=2\frac{1-\mu}{\mu}\|x_p-x_p^\star\|^2.
$$

Letting $(p_i,\hat{p}_i)$, $i\in[k]$ be interacting pairs with positive approaching rates μ_i, $i\in[k]$ at time t for the update at time $t+1$, it turns out that

$$
\begin{aligned}
(4) &= 2\Bigg[4\sum_{i\in[k]}\frac{1-\mu_i}{\mu_i}\|x_{p_i}-x_{p_i}^\star\|^2+4(k-1)\sum_{i\in[k]}\frac{1-\mu_i}{\mu_i}\|x_{p_i}-x_{p_i}^\star\|^2 \\
&\qquad +2(n-2k)\sum_{i\in[k]}\frac{1-\mu_i}{\mu_i}\|x_{p_i}-x_{p_i}^\star\|^2\Bigg] \\
&= 4n\sum_{i\in[k]}\frac{1-\mu_i}{\mu_i}\|x_{p_i}-x_{p_i}^\star\|^2 \\
&\geq 2n\sum_{i\in[n]}\|x_i-x_i^\star\|^2. \qquad \square
\end{aligned}
$$

Observe that

$$\sum_{i,j\in[n]}(\|x_i-x_j\|^2-\|x_i^\star-x_j^\star\|^2)\geq 2n\sum_{i\in[n]}\|x_i-x_i^\star\|^2 \tag{5}$$

if the update at time $t+1$ is under pair interaction.

Lemma 3.3. *If updates at times t and $t+1$ are under group interaction and $E(t)\subset E(t+1)$, then*

$$Z_t-Z_{t+1}\geq 4\sum_{i\in[n]}\|x_i(t)-x_i(t+1)\|^2.$$

Proof. Let $E=E(t)$ and $E^\star=E(t+1)$. If updates at times t and $t+1$ are under group interaction and $E\subset E^\star$, then

$$\begin{aligned}Z_t-Z_{t+1}&=Z_1(t)-Z_1(t+1)\\&\geq\sum_{i\in[n]}\sum_{j\in\mathcal{N}_i}(\|x_i-x_j\|^2-\|x_i^\star-x_j^\star\|^2)\\&\quad+\sum_{i\in[n]}\sum_{j\in\mathcal{N}_i^c}[\epsilon^2-(\|x_i^\star-x_j^\star\|^2\wedge\epsilon^2)\vee\epsilon^2\mathbb{1}\{(i,j)\notin E^\star\}]\\&\geq\sum_{i\in[n]}\sum_{j\in\mathcal{N}_i}(\|x_i-x_j\|^2-\|x_i^\star-x_j^\star\|^2)\\&\geq 4\sum_{i\in[n]}\|x_i-x_i^\star\|^2,\end{aligned}$$

where the last inequality follows (3). □

Lemma 3.4. *If updates at times t and $t+1$ are under pair interaction with equal approaching rates, then*

$$Z_t-Z_{t+1}\geq 2n\sum_{i\in[n]}\|x_i(t)-x_i(t+1)\|^2.$$

Proof. If updates at times t and $t+1$ are under pair interaction with equal approaching rates, then

$$\begin{aligned}Z_t-Z_{t+1}&=Z_2(t)-Z_2(t+1)\\&\geq 2n\sum_{i\in[n]}\|x_i-x_i^\star\|^2,\end{aligned}$$

where the inequality follows (5). □

Lemma 3.5. *Assume that all interacting pairs approach each other equally at their rate for update under pair interaction. Then, $x_i(t)$ is asymptotically stable for all $i\in[n]$ as $t\to\infty$ if the following conditions are met at each time t after some finite time:*

(i) *The social graph is complete at time t when updates at times t and $t+1$ occur under pair interaction with equal approaching rates and group interaction.*

(ii) *The opinion graph is complete at time t when updates at times t and $t+1$ occur under group interaction and pair interaction with equal approaching rates.*

(iii) *$E(t)\subset E(t+1)$ when updates at times t and $t+1$ occur under group interaction.*

Proof. Without loss of generality, we assume that the assumption is satisfied at all times. If follows from Lemmas 3.1, 3.2, 3.3 and 3.4 that

$$Z_t - Z_{t+1} \geq 4 \sum_{i\in[n]} \|x_i(t) - x_i(t+1)\|^2, \quad \text{for all } t \geq 0$$

under the assumption of Lemma 3.5. By the fact that $\sqrt{n}\|x\|_2 \geq \|x\|_1$ for $x \in \mathbb{R}^n$, we get

$$\begin{aligned}
Z_t - Z_{t+k} &= \sum_{j=t}^{t+k-1} (Z_j - Z_{j+1}) \\
&\geq 4 \sum_{j=t}^{t+k-1} \sum_{i\in[n]} \|x_i(j) - x_i(j+1)\|^2 \\
&\geq \frac{4}{n} \sum_{j=t}^{t+k-1} \left(\sum_{i\in[n]} \|x_i(j) - x_i(j+1)\| \right)^2 \\
&\geq \frac{4}{nk} \left(\sum_{j=t}^{t+k-1} \sum_{i\in[n]} \|x_i(j) - x_i(j+1)\| \right)^2 \\
&= \frac{4}{nk} \left(\sum_{i\in[n]} \sum_{j=t}^{t+k-1} \|x_i(j) - x_i(j+1)\| \right)^2 \\
&\geq \frac{4}{nk} \left(\sum_{j\in[n]} \|x_i(t) - x_i(t+k)\| \right)^2.
\end{aligned}$$

Observe that $(Z_t)_{t\geq 0}$ is a nonnegative supermartingale. It follows from the martingale convergence theorem that Z_t converges to some random variable Z_∞ with finite expectation as t goes to ∞. It turns out that

$$Z_t - Z_{t+k} \to 0 \text{ as } t \to \infty, \text{ therefore, } \|x_i(t) - x_i(t+k)\| \to 0$$

as $t \to \infty$, for all $k \geq 0$ and $i \in [n]$. Thus, $(x_i(t))_{t\geq 0}$ is a Cauchy sequence for all $i \in [n]$, which implies $x_i(t) \to x_i$ as $t \to \infty$ for all $i \in [n]$. □

Lemma 3.6 ([9]). *If some component G in $\widetilde{G}(t) \cap \mathcal{G}(t)$ is δ-nontrivial and $\alpha_i(t) < 1$ for all $i \in V(G)$, then*

$$\sum_{i\in V(G)} \|x_i(t) - x_i(t+1)\|^2 > \frac{2\delta^2 \big(1 - \max_{i\in V(G)} \alpha_i(t)\big)^2}{|V(G)|^8}.$$

Proof of Theorem 2.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space for $\mathcal{F} \subset \mathcal{P}(\Omega)$ a σ-algebra and P a probability measure. If there is no consensus, it follows from Lemma 3.5 that there are individuals i and j with

$$x_i(t) \to x_i,\ x_j(t) \to x_j \text{ but } x_i \neq x_j \text{ as } t \to \infty \text{ on some } F \in \mathcal{F} \text{ with } P(F) > 0.$$

Thus, there are $s \geq 0$ and a random variable $\delta > 0$ such that $\|x_i(t) - x_j(t)\| > \delta$, for all $t \geq s$. By finiteness of the social graph, either of the statements true implies there are $(t_k)_{k\geq 0}$ increasing with $t_0 \geq s$ and a random variable $\gamma < 1$ such that $\alpha_i(t_k) < \gamma$ under some connected social graph G for all $i \in [n]$ and $k \geq 0$. By finiteness and connectedness of social graph G and the triangle inequality, there are $(t_k^{(1)})_{k\geq 0} \subset (t_k)_{k\geq 0}$ and edge $(p,q) \in E(G)$ with $\|x_p(t_k^{(1)}) - x_q(t_k^{(1)})\| > \delta/n$, for all $k \geq 0$. There are infinitely many times in $(t_k^{(1)})_{k\geq 0}$ for update under either pair interaction or group interaction.

If there are infinitely many times in $(t_k^{(1)})_{k\geq 0}$ for update under pair interaction, say $t_k^{(2)}$, $k\geq 0$ are the times in $(t_k^{(1)})_{k\geq 0}$ for update under pair interaction. Since $\bigcup_{a\in S} a\supset \binom{[n]}{2}$ for update under pair interaction with equal approaching rates and opinion graphs preserve completeness, edge (p,q) is an interacting pair with positive probability. Hence,

$$\|x_p(t_k^{(2)})-x_p(t_k^{(2)}+1)\|>\frac{(1-\gamma)\delta}{2n} \quad \text{and} \quad \|x_q(t_k^{(2)})-x_q(t_k^{(2)}+1)\|>\frac{(1-\gamma)\delta}{2n},$$

for all $k\geq 0$, contradicting $\|x_i(t)-x_i(t+1)\|\to 0$ as $t\to\infty$, for all $i\in[n]$.

If there are infinitely many times in $(t_k^{(1)})_{k\geq 0}$ for update under group interaction, say $t_k^{(2)}$, $k\geq 0$ are the times in $(t_k^{(1)})_{k\geq 0}$ for update under group interaction. Since $\bigcup_{a\in S} a\supset [n]$ for update under group interaction and opinion graphs preserve completeness, by Lemma 3.6 and conditional expectation, we get

$$\begin{aligned}\mathbf{E}_F[Z_{t_k^{(2)}}-Z_{t_k^{(2)}+1}] &\geq 4\min_{a\in S} P(U_{t_0^{(2)}}=a)\mathbf{E}_F\left[\sum_{i\in[n]}\|x_i(t)-x_i(t+1)\|^2\right]\\ &\geq 8\min_{a\in S} P(U_{t_0^{(2)}}=a)\mathbf{E}_F[\delta^2(1-\gamma)^2]/n^8>0,\end{aligned} \tag{6}$$

where $\mathbf{E}_F$ is the expectation on F. Via monotone convergence theorem, $\mathbf{E}_F[Z_t]\to\mathbf{E}_F[Z_\infty]$ as $t\to\infty$. As $k\to\infty$, (6) becomes

$$0\geq 8\min_{a\in S} P(U_{t_0^{(2)}}=a)\mathbf{E}_F[\delta^2(1-\gamma)^2]/n^8>0, \text{ a contradiction.}$$ □

4. Pair Interaction With Distinct Approaching Rates

In the Deffuant model, an interacting pair have the same approaching rate $\mu\in[0,1/2]$ at all times. In this section, we investigate the mixed HK model under pair interaction, where interacting pairs can approach each other at distinct rates. It turns out that the function W_t is nonincreasing if (2) holds.

Lemma 4.1. *We get*

$$W_t-W_{t+1}\geq \sum_{(i,j)\in\widetilde{E}(t)}(\alpha_j(t)-\alpha_i(t))(\|x_i(t)-c\|-\|x_j(t)-c\|)/2\geq 0$$

if (2) *holds.*

Proof. Let $x_i=x_i(t)$ and $\mu_i(t)=\mu_i$, for all $i\in[n]$. Assume that (p_i,q_i), $i\in[k]$ are interacting pairs at time t. Through the triangle inequality,

$$\begin{aligned}W_t-W_{t+1} &= \sum_{i\in[k]}(\|x_{p_i}-c\|+\|x_{q_i}-c\|-\|x_{p_i}^\star-c\|-\|x_{q_i}^\star-c\|)\\ &\geq \sum_{i\in[k]}(\|x_{p_i}-c\|+\|x_{q_i}-c\|-(1-\mu_{p_i})\|x_{p_i}-c\|-\mu_{p_i}\|x_{q_i}-c\|\\ &\quad -(1-\mu_{q_i})\|x_{q_i}-c\|-\mu_{q_i}\|x_{p_i}-c\|)\\ &= \sum_{i\in[k]}(\mu_{p_i}-\mu_{q_i})(\|x_{p_i}-c\|-\|x_{q_i}-c\|)\\ &= \sum_{i\in[k]}(\alpha_{q_i}-\alpha_{p_i})(\|x_{p_i}-c\|-\|x_{q_i}-c\|)/2\geq 0 \text{ if (2) holds.}\end{aligned}$$ □

Lemma 4.2. *Asymptotic stability holds in* (1) *under pair interaction if*

$$\inf_{t\geq 0}\min_{i,j\in[n];i\neq j}|\alpha_i(t)-\alpha_j(t)|>0 \text{ and (2) holds.}$$

Proof. Assume that x_i is not asymptotically stable for some $i\in[n]$. By finiteness of the social graph, there are $j\in[n]$, $\delta>0$ and $(t_k)_{k\geq 0}\subset\mathbb{N}$ increasing such that the interacting pair (i,j) satisfies $\|x_i(t_k)-x_j(t_k)\|>\delta$, for all $k\geq 0$. Without loss of generality, say $\mu_i\geq\mu_j$. Since the convex hull generated by $(x_i(t))_{i\in[n]}$, $Cv(\{x_i(t)\}_{i\in[n]})$, is bounded by $Cv(\{x_i(0)\}_{i\in[n]})$, which is compact. Thus, $Cv(\{x_i(0)\}_{i\in[n]})$ can be covered by finitely many cubes of length $\delta/(4\sqrt{n})$. One of the cubes contains infinitely many $x_j(t_k)$, saying $x_j(t_k^{(1)})$, $k\geq 0$ with $(t_k^{(1)})\subset(t_k)$. Picking c in that cube, via the triangle inequality,

$$\|x_i(t_k^{(1)})-c\|-\|x_j(t_k^{(1)})-c\|\geq\|x_i(t_k^{(1)})-x_j(t_k^{(1)})\|-2\|x_j(t_k^{(1)})-c\|>\delta/2.$$

By Lemma 4.1, W_t is a nonnegative supermartingale. It follows from the martingale convergence theorem that W_t converges to some random variable W_∞ with finite expectation as $t\to\infty$,

$$W_t-W_{t+1}\geq\inf_{t\geq 0}\min_{i,j\in[n];i\neq j}|\alpha_i(t)-\alpha_j(t)|\delta/4$$

implies

$$0\geq\inf_{t\geq 0}\min_{i,j\in[n];i\neq j}|\alpha_i(t)-\alpha_j(t)|\delta/4>0 \text{ as } t\to\infty, \text{ a contradiction.} \qquad\square$$

Proof of Theorem 2.2. By finiteness of the social graph, the social graph connected infinitely many times implies there is a social graph G connected infinitely many times, saying t_k, $k\geq 0$ the times. Since opinion graphs preserve completeness and $\bigcup_{a\in S}a\supset\binom{[n]}{2}$, all edges (i,j) in $E(G)$ are interacting pairs with positive probabilities at times in $(t_k)_{k\geq 0}$. It follows from Lemma 4.2 that $x_i(t)\to x_i$ as $t\to\infty$, for all $i\in[n]$. If there is an edge $(p,q)\in E(G)$ with $x_p\neq x_q$, there are $\delta>0$ and $(s_k)\subset(t_k)$ with $\|x_p(s_k)-x_q(s_k)\|>\delta$, for all $k\geq 0$. Following the same method in the proof of Lemma 4.2, we get a contradiction. This completes the proof. $\square$

Acknowledgement

The author is partially supported by NSTC grant.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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