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Research Article

Continues Classical Quaternary Boundary Optimal Control Problem of Quaternary Linear Hyperbolic System

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Abstract. This work concerns with the study of the continuous classical quaternary boundary optimal control problem or for brief *quaternary boundary optimal control problem* (QBOCP) controlling by *quaternary linear hyperbolic system* (QLHS). The existence theorem for a unique *quaternary state vector solution* (QSVS) for the QLHS as well as for its *quaternary adjoint linear system* (QALS) is proved via the *method of Galerkin* (MG) with given *continuous boundary control quaternary vector* (CBCQV). The existence theorem of a *continuous boundary optimal control quaternary vector* (CBOCQV) controlling by the QLHS is demonstrated. The *directional derivative* (DDV) for the *objective functional* (OF) is derived. Lastly, the *necessity conditions for optimality* (NCO) of the problem is studied.

Keywords. Quaternary boundary optimal control, Quaternary linear hyperbolic system, Quaternary adjoint linear system, Directional derivative, Necessity conditions

Mathematics Subject Classification (2020). 93C95

1. Introduction

Various real-life applications are classified as ideal *optimal control problems* (OCPs). It has been used in many fields, like dynamic system (Kruse and Strack [9]), economic science (Barzegar *et al*. [2]), chemical reactor (Nurmagambetov [11]), engineering (Wang *et al*. [12]). In the field of applied mathematics OCPs usually are controlled by ODEs and PDEs and they were investigated by numerous research, e.g., Gerdts [6], and Manzoni *et al*. [10]. Other investigators, Casas and Yong [4], Gugat and Herty [7] and Kouri and Surowiec [8] interested about OCPs controlled by PDES of the kinds; elliptic, parabolic and hyperbolic respectively, whilst the interested about studding *boundary OCP* (BOCP) which are controlled by *couple of PDES* (CPDES) of the above three mentioned kinds, as well as the study of the *boundary OCP* (BOCP) controlled by *triple of PDES* (TPDES) of the three kinds all were achieved through the investigations by Al-Hawasy and Al-Ajeeli [1]. On the other hand, the investigation of the QBOCP controlling by *Quaternary PDES* (QPDES) of the kinds elliptic and parabolic by Diwan *et al*. [5]. All these investigations encourage us to aim about investigating QBOCP controlling by QLHS. This work is started with giving a description about the problem, the *weak formulation* (WFO) for the QLHS is formulated, and then the *method of Galerkin* (MG) is employed to demonstrate the theorem of existence of a unique QSVS for the WFO of the QLHS (of a unique vector solution of the WFO of the QAES associated the (QLHPDES)) when the CBCQV is fixed, also the state and demonstration for existence of a COBCQV controlling by QLHS is studied. Finally, the DDV for the OF is derived and the *necessary conditions for optimality* (NCO) of this OCP is studied.

In this paper, Section 2 deals with the description of the problem including the equations and their boundary conditions, then the WFO for the QLHS are found, at the end of this section some hypotheses are considered. Section 3 deals with the theorem of existence of unique solution for the resulting WFO through employing the MG under suitable hypos when the CBCQV is known. In Section 4, the existence of a CBCOQV is studied after the LC property from different spaces are proved. Then, the QAEs associated with the QSEs are formulate and the DD of the OF is derived. Finally, the NCO is proved.

2. Problem Description

Let $\Omega \subset R^2$, $x = (x_1, x_2)$, $Q = I \times \Omega$, $I = [0, T]$, $\Sigma = \partial Q = \partial\Omega \times I$, the QBOCP consists of the QLHS which is given in Q by:

$$y_{1tt} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{1ij}\frac{\partial y_1}{\partial x_j}\right) + a_1 y_1 - b_2 y_2 + b_3 y_3 - b_4 y_4 = f_1(x,t), \tag{1}$$

$$y_{2tt} - \sum_{i,j=1}^{2} \frac{\partial}{\delta x_i}\left(a_{2ij}\frac{\partial y_2}{\partial x_j}\right) + a_2 y_2 + b_2 y_1 - b_5 y_3 + b_6 y_4 = f_2(x,t), \tag{2}$$

$$y_{3tt} - \sum_{i,j=1}^{2} \frac{\partial}{\delta x_i}\left(a_{3ij}\frac{\partial y_3}{\partial x_j}\right) + a_3 y_3 - b_3 y_1 + b_5 y_2 + b_7 y_4 = f_3(x,t), \tag{3}$$

$$y_{4tt} - \sum_{i,j=1}^{2} \frac{\partial}{\delta x_i}\left(a_{4ij}\frac{\partial y_4}{\partial x_j}\right) + a_4 y_4 + b_4 y_1 - b_6 y_2 - b_7 y_3 = f_4(x,t), \tag{4}$$

$$y_l(x,0) = y_l^0(x),\ \ y_{lt}(x,0) = y_l^1(x), l = 1,2,3,4 \text{ on } \Omega, \tag{5}$$

$$\partial n_l y_l = \sum_{i,j=1}^{2} a_{lij},\ \partial x_j y_l \cos(n_l, x_j) = u_l(x,t),\ l = 1,2,3,4 \text{ on } \Sigma, \tag{6}$$

where n_l, for $l = 1,2,3,4$ is an outer normal vector on Σ, the angle between the x_j-axis and n_l and is referred by $(n_l, x_j)\vec{y} = (y_1, y_2, y_3, y_4) \in (H^2(Q))^4$ is the QSVS, $\vec{u} = (u_1, u_2, u_3, u_4) \in (L^2(\Sigma))^4 = L^2(\Sigma)$ is the QCBCV, $(f_1, f_2, f_3, f_4) \in (L^2(Q))^4 = L^2(Q)$, is given, $a_{lij} = a_{lij}(x,t) \in L^\infty(Q)$, $a_l = a_l(x,t) \in L^\infty(Q)$, $b_k = b_k(x,t) \in L^\infty(Q)$, $k = 2,3,4,5,7$.

The set of admissible QCBCV is $\vec{W} = \{\vec{u} \in L^2(\Sigma) : \vec{u} \in \vec{U}$ a.e. $\vec{U} \subset \mathcal{R}^4$, $\vec{U}$ convex$\}$ and the OF is

$$G_0(\vec{u}) = \frac{1}{2}\sum_{l=1}^{4} \|y_l - y_{ld}\|_Q^2 + \frac{\gamma}{2}\sum_{l=1}^{4} \|u_l\|_\Sigma^2, \quad \gamma \in \mathcal{R}^+. \tag{7}$$

Let $\vec{V} = V \times V \times V \times V = H^1(\Omega)$, $\vec{V} = \{\vec{v} : \vec{v} = (v_1, v_2, v_3, v_4) \in H^1(\Omega)\}$.

2.1 The Weak Formulation (WFO)

The WFO of the QLHS ((1)-(6)) is

$$\begin{aligned}&\langle y_{1tt}, v_1\rangle + r_1(t, y_1, v_1) + (a_1 y_1, v_1)_{L^2(\Omega)} - (b_2 y_2, v_1)_{L^2(\Omega)} + (b_3 y_3, v_1)_{L^2(\Omega)} - (b_4 y_4, v_1)_{L^2(\Omega)}\\ &\quad = (f_1, v_1)_{L^2(\Omega)} + (u_1, v_1)_{L^2(\delta\Omega)},\end{aligned} \tag{8}$$

$$\begin{aligned}&\langle y_{2tt}, v_2\rangle + r_2(t, y_2, v_2) + (a_2 y_2, v_2)_{L^2(\Omega)} + (b_2 y_1, v_2)_{L^2(\Omega)} - (b_5 y_3, v_2)_{L^2(\Omega)} + (b_6 y_4, v_2)_{L^2(\Omega)}\\ &\quad = (f_2, v_2)_{L^2(\Omega)} + (u_2, v_2)_{L^2(\delta\Omega)},\end{aligned} \tag{9}$$

$$\begin{aligned}&\langle y_{3tt}, v_3\rangle + r_3(t, y_3, v_3) + (a_3 y_3, v_3)_{L^2(\Omega)} - (b_3 y_1, v_3)_{L^2(\Omega)} + (b_5 y_2, v_3)_{L^2(\Omega)} + (b_7 y_4, v_3)_{L^2(\Omega)}\\ &\quad = (f_3, v_3)_{L^2(\Omega)} + (u_3, v_3)_{L^2(\delta\Omega)},\end{aligned} \tag{10}$$

$$\begin{aligned}&\langle y_{4tt}, v_4\rangle + r_4(t, y_4, v_4) + (a_4 y_4, v_4)_{L^2(\Omega)} + (b_4 y_1, v_4)_{L^2(\Omega)} - (b_6 y_2, v_4)_{L^2(\Omega)} - (b_7 y_3, v_4)_{L^2(\Omega)}\\ &\quad = (f_4, v_4)_{L^2(\Omega)} + (u_4, v_4)_{L^2(\delta\Omega)},\end{aligned} \tag{11}$$

$$(y_l^0, v_l)_\Omega = (y_l(0), v_1)_\Omega \text{ and } (y_l^1, v_l)_\Omega = (y_{lt}(0), v_1)_\Omega, \quad \forall\ v_l \in V_l,\ l = 1,2,3,4, \tag{12}$$

where $r_l(t, y_l, v_l) = \sum_{i,j=1}^{2} \iint_\Omega a_{lij} \frac{\partial y_l}{\partial x_i} \frac{\partial v_l}{\partial x_j} dx$, for $l = 1,2,3,4$.

2.2 Hypotheses (HYPOS)

$$\begin{aligned}s(t, \vec{y}, \vec{v}) &= r_1(t, y_1, v_1) + (a_1 y_1, v_1)_\Omega + r_2(t, y_2, v_2) + (a_2 y_2, v_2)_{L^2(\Omega)} + r_3 t, y_3, v_3)\\ &\qquad + (a_3 y_3, v_3)_{L^2(\Omega)} + r_4(t, y_4, v_4) + (a_4 y_4, v_4)_{L^2(\Omega)},\end{aligned}$$

$$|s(t, \vec{y}, \vec{v})| \le a\|\vec{y}\|_1 \|\vec{v}\|_1,$$

$$s(t, \vec{y}, \vec{y}) \ge \bar{a}\|\vec{y}\|_1^2,$$

$$|s_t(t, \vec{y}, \vec{v})| \le b\|\vec{y}\|_1 \|\vec{v}\|_1,$$

$$s_t(t, \vec{y}, \vec{y}) \ge \bar{b}\|\vec{y}\|_1^2,$$

where $s_t(t, \vec{y}, \vec{v}) = \iint_\Omega \sum_{i,j=1}^{2} \frac{\partial a_{lij}}{\partial t} \frac{\partial y_l}{\partial x_i} \frac{\partial v_l}{\partial x_j} dx$, and $a, \bar{a}, b, \bar{b} \in \mathcal{R}^+$.

In this work, the notations $\xrightarrow[L^2(Q)]{}$, $\xrightarrow[L^2(Q)]{}$, $(\xrightarrow[L^2(I,V)]{}, \xrightarrow[L^2(I,V)]{})$ will be referred to the convergent strongly in $L^2(Q)$, $L^2(Q)$, $(L^2(I,V), L^2(I,V))$, respectively, and $\xrightarrow[L^2(Q)]{}$, $\xrightarrow[L^2(Q)]{}$ $(\xrightarrow[L^2(I,V)]{}, \xrightarrow[L^2(I,V)]{})$ to the convergent weakly in $L^2(Q)$, $L^2(Q)$ $(L^2(I,V), L^2(I,V))$, respectively.

3. Main Results

The Existence of a Unique Solution for the WFO

The next theorem deals with the existence of a unique QSVS for the WFO ((8)-(12)).

Theorem 3.1. *Let $\vec{u} \in L^2(Q)$ be a given QBOC, then the WFO* ((8)-(12)) *has a unique QSVS $\vec{y}$, with $\vec{y} \in (L^2(I,V))^4 = L^2(I,V)$ and $\vec{y}_t = (y_{1t}, y_{2t}, y_{3t}, y_{4t}) \in L^2(I,V^*)$.*

Proof. For every n, let the set of piecewise affine function in Ω be $\vec{V}_n = V_{1n} \times V_{2n} \times V_{3n} \times V_{4n} \subset \vec{V}$, $\{\vec{V}_n\}_{n=1}^{\infty}$ be a sequence of subspaces of $\vec{V}$, then by the MG $\forall\ \vec{v} \in \vec{V}$ there is a sequence $\{\vec{v}_n\}$, $\vec{v}_n \in \vec{V}_n$, $\forall\ n$ and $\vec{v}_n \xrightarrow[L^2(\vec{V})]{} \vec{v}$ (thus, $\vec{v}_n \xrightarrow[L^2(\Omega)]{} \vec{v}$).

Let $\{\vec{v}_j = (v_{1j}, v_{2j}, v_{3j}, v_{4j}) : j = 1,2,\dots,n\}$ spans $\vec{V}_n$ and the approximate Galerkin solution be

$$y_{ln} = \sum_{j=1}^{n} c_{lj}(t) v_{lj}(x), \tag{13}$$

$$z_{ln} = \sum_{j=1}^{n} d_{lj}(t) v_{lj}(x), \tag{14}$$

where $c_{lj}(t)$, $d_{lj}(t)$ are unknown functions of t, $\forall\ l = 1,2,3,4$, $j = 1,\dots,n$.

Utilizing y_{ln} (with $y_{lnt} = z_{ln}$), and $v_l \in V_n$, for $l = 1,2,3,4$ in ((8)-(12)), they yield to

$$\langle z_{1nt}, v_1\rangle + r_1(t, y_{1n}, v_1) + (a_1 y_{1n}, v_1)_{L^2(\Omega)} - (b_2 y_{2n}, v_1)_{L^2(\Omega)} + (b_3 y_{3n}, v_1)_{L^2(\Omega)} - (b_4 y_{4n}, v_1)_{L^2(\Omega)}$$
$$= (f_1, v_1)_{L^2(\Omega)} + (u_1, v_1)_{L^2(\partial\Omega)}, \tag{15}$$

$$\langle z_{2nt}, v_2\rangle + r_2(t, y_{2n}, v_2) + (a_2 y_{2n}, v_2)_{L^2(\Omega)} + (b_2 y_{1n}, v_2)_{L^2(\Omega)} - (b_5 y_{3n}, v_2)_{L^2(\Omega)} + (b_6 y_{4n}, v_2)_{L^2(\Omega)}$$
$$= (f_2, v_2)_{L^2(\Omega)} + (u_2, v_2)_{L^2(\partial\Omega)}, \tag{16}$$

$$\langle z_{3nt}, v_3\rangle + r_3(t, y_{3n}, v_3) + (a_3 y_{3n}, v_3)_{L^2(\Omega)} - (b_3 y_{1n}, v_3)_{L^2(\Omega)} + (b_5 y_{2n}, v_3)_{L^2(\Omega)} + (b_7 y_{4n}, v_3)_{L^2(\Omega)}$$
$$= (f_3, v_3)_{L^2(\Omega)} + (u_3, v_3)_{L^2(\partial\Omega)}, \tag{17}$$

$$\langle z_{4nt}, v_4\rangle + r_4(t, y_{4n}, v_4) + (a_4 y_{4n}, v_4)_{L^2(\Omega)} + (b_4 y_{1n}, v_4)_{L^2(\Omega)} - (b_6 y_{2n}, v_4)_{L^2(\Omega)} - (b_7 y_{3n}, v_4)_{L^2(\Omega)}$$
$$= (f_4, v_4)_{L^2(\Omega)} + (u_4, v_4)_{L^2(\partial\Omega)}, \tag{18}$$

$$(y_{ln}^0, v_l)_\Omega = (y_l^0, v_1)_\Omega \ \text{ and } \ (z_{ln}^1, v_l)_\Omega = (y_l^1, v_1)_\Omega, \quad \forall\ v_l \in V_l,\ l = 1,2,3,4, \tag{19}$$

where $y_{ln}^0 = y_{ln}^0(x) = y_{ln}(x,0) \in V_n$, $z_{ln}^0 = y_{ln}^1 = y_{ln}^1(x) = y_{lnt}(x,0) \in L^2(\Omega)$ be the projection of y_l^0 onto V of $y_l^1 = y_{lt}$ onto $L^2(\Omega)$, $\forall l = 1,2,3,4$, i.e.,

$$y_{ln}^0 \xrightarrow[L^2(V)]{} y_l^0, \ \text{ with } \|\vec{y}_n^{\,0}\|_1 \le b_0, \tag{20}$$

$$y_{ln}^1 \xrightarrow[L^2(\Omega)]{} y_l^1, \ \text{ with } \|\vec{y}_n^{\,1}\|_0 \le b_1. \tag{21}$$

Substituting (13) and (14) in ((15)-(19)) and setting $v_l = v_{li}$, $\forall\ i = 1,2,3,4,\dots,n$, then, the secured equations will be equivalent to the following 1st order ODEs of linear system with ICs

$$A_1 \acute{D}_1(t) + M_1 C_1(t) - F C_2(t) + G C_3(t) - H C_4(t) = d_1, \tag{22}$$

$$A_2 \acute{D}_2(t) + M_2 C_2(t) + F C_1(t) - O C_3(t) + R C_4(t) = d_2, \tag{23}$$

$$A_3 \acute{D}_3(t) + M_3 C_3(t) - G C_1(t) + O C_2(t) + P C_4(t) = d_3, \tag{24}$$

$$A_4 \acute{D}_4(t) + M_4 C_4(t) + H C_1(t) - R C_2(t) - P C_3(t) = d_4, \tag{25}$$

$$A_l C_l(0) = m_l^0 \ \text{ and } \ A_l D_l(0) = m_l^1, \tag{26}$$

where $A_l = (a_{lij})_{n\times n} = (v_{lj}, v_{li})$, $M_l = (m_{lij})_{n\times n} = [r_l(t, v_{lj}, v_{li}) + (a_l v_{lj}, v_{li})_{L^2(\Omega)}]$, $F =$

$(f_{ij})_{n\times n} = (b_2 v_{2j}, v_{2i})_{L^2(\Omega)}$, $G = (g_{ij})_{n\times n} = (b_3 v_{3j}, v_{1i})_{L^2(\Omega)}$, $C_l(0) = (c_{lj}(0))_{n\times 1}$, $H = (h_{ij})_{n\times n} = (b_4 v_{4j}, v_{1i})_{L^2(\Omega)}$, $O = (o_{ij})_{n\times n} = (b_5 v_{2j}, v_{3i})_{L^2(\Omega)}$, $C_l(t) = (c_{lj}(0))_{n\times 1}$, $P = (p_{ij})_{n\times n} = (b_7 v_{4j}, v_{3i})_{L^2(\Omega)}$, $R = (r_{ij})_{n\times n} = (b_6 v_{2j}, v_{4i})_{L^2(\Omega)}$, $D_l(0) = (d_{lj}(0))_{n\times 1}$, $d_{lj} = (f_l, v_{li})_{L^2(\Omega)} + (u_l, v_{li})_{L^2(\partial\Omega)}$, $m^0_{li} = (y^0_l, v_{li})$, $m^1_{li} = (y^1_l, v_{li})$, $D_l(t) = (d_{lj}(t))_{n\times 1}$, $\forall\ l = 1,2,3,4$, $j = 1,2,\dots,n$.
Since A_l^{-1} exists $\forall\ l = 1,2,3,4$, then ((22)-(26)) has a unique QSVS $\overrightarrow{y}_n$.

In the next steps, the norms $\|\overrightarrow{y}_n(t)\|_{L^2(\Omega)}$ and $\|\overrightarrow{y}_n(t)\|_{L^2(I,V)}$ are proved bounded.
Setting $v_l = y_{lnt}$, $\forall\ l = 1,2,3,4$ in ((12)-(15)) respectively, employing [12, Lemma 2.1] for the first two terms in the LHS for each expression, then gathering the outcome equations, to bring

$$\begin{aligned}
&\frac{d}{dt}[\|\overrightarrow{y}_{nt}(t)\|^2_{L^2(\Omega)} + s(t,\overrightarrow{y}_n,\overrightarrow{y}_n)] - s_t(t,\overrightarrow{y}_n,\overrightarrow{y}_n)\\
&= 2[(f_1, y_{1nt})_{L^2(\Omega)} + (u_1, y_{1nt})_{L^2(\partial\Omega)} + (b_2 y_{2n} + b_3 y_{3n} - b_4 y_{4n}, y_{1nt})_{L^2(\Omega)}\\
&\quad + (f_2, y_{2nt})_{L^2(\Omega)} + (u_2, y_{2nt})_{L^2(\partial\Omega)} + (b_2 y_{1n} - b_5 y_{3n} + b_6 y_{4n}, y_{2nt})_{L^2(\Omega)}\\
&\quad + (f_3, y_{3nt})_{L^2(\Omega)} + (u_3, y_{3nt})_{L^2(\partial\Omega)} + (b_5 y_{2n} - b_3 y_{1n} + b_7 y_{4n}, y_{3nt})_{L^2(\Omega)}\\
&\quad + (f_4, y_{4nt})_{L^2(\Omega)} + (u_4, y_{4nt})_{L^2(\partial\Omega)} + (b_4 y_{1n} - b_6 y_{2n} - b_7 y_{3n}, y_{4nt})_{L^2(\Omega)}].
\end{aligned} \tag{27}$$

Employing HYPOS, after take the absolute values of (27) it yields to:

$$\begin{aligned}
&\frac{d}{dt}[\|\overrightarrow{y}_{nt}(t)\|^2_{L^2(\Omega)} + \overline{a}\|\overrightarrow{y}_n\|^2_{H^1(\Omega)}]\\
&\le b\|\overrightarrow{y}_n\|^2_{H^1(\Omega)} + 2(|(b_2 y_{2n}, y_{1nt})_{L^2(\Omega)}| + |(b_3 y_{3n}, y_{1nt})_{L^2(\Omega)}| + |(b_4 y_{4n}, y_{1nt})_{L^2(\Omega)}|\\
&\quad + |(b_2 y_{1n}, y_{2nt})_{L^2(\Omega)}| + |(b_6 y_{4n}, y_{2nt})_{L^2(\Omega)}| + |(b_5 y_{3n}, y_{2nt})_{L^2(\Omega)}| + |(b_5 y_{2n}, y_{3nt})_{L^2(\Omega)}|\\
&\quad + |(b_7 y_{4n}, y_{3nt})_{L^2(\Omega)}| + |(b_3 y_{1n}, y_{3nt})_{L^2(\Omega)}| + |(b_4 y_{1n}, y_{4nt})_{L^2(\Omega)}| + |(b_6 y_{2n}, y_{4nt})_{L^2(\Omega)}|\\
&\quad + |(b_7 y_{3n}, y_{4nt})_{L^2(\Omega)}| + |(f_1, y_{1nt})_{L^2(\Omega)}| + |(f_2, y_{2nt})_{L^2(\Omega)}| + |(f_3, y_{3nt})_{L^2(\Omega)}|\\
&\quad + |(f_4, y_{4nt})_{L^2(\Omega)}| + |(u_1, y_{1nt})_{L^2(\partial\Omega)}| + |(u_2, y_{2nt})_{L^2(\partial\Omega)}| + |(u_3, y_{3nt})_{L^2(\partial\Omega)}|\\
&\quad + |(u_4, y_{4nt})_{L^2(\partial\Omega)}|.
\end{aligned} \tag{28}$$

Applying the inequality of Cauchy for the RHS of (30), integrating on $(0,t)$, take in account that $\|y_{ln}\|_{L^2(\Omega)} \le \|y_{ln}\|_{H^1(\Omega)} \le \|\overrightarrow{y}_{ln}\|^{1(\Omega)}_{H}$, $\|y_{lnt}\|_{L^2(\Omega)} \le \|\overrightarrow{y}_{nt}\|_{L^2(\Omega)}$, $\|y_{lnt}\|_{L^2(\partial\Omega)} \le \overline{c}_l\|y_{lnt}\|_{H^1(\Omega)}$, $\|u_l\|_{L^2(\partial Q)} \le \overline{e}_l$, $\|f_l\|_{L^2(\partial Q)} \le e_l$, then utilizing the trace theorem, and HYPOS, to secure

$$\begin{aligned}
&\int_0^t \frac{d}{dt}[\|\overrightarrow{y}_{nt}(t)\|^2_{L^2(\Omega)} + \overline{a}\|\overrightarrow{y}_n\|^2_{H^1(\Omega)}]dt\\
&\le \int_0^t [\overline{h}_1\|\overrightarrow{y}_{nt}\|^2_{L^2(\Omega)} + \overline{h}_2\|\overrightarrow{y}_n\|^2_{H^1(\Omega)}]dt + \sum_{l=1}^4(\|f_l\|^2_Q + \|u_l\|^2_{L^2(\partial Q)})\\
&\le \overline{h}_4 + \overline{h}_3\int_0^t (|\overrightarrow{y}_{nt}\|^2_{L^2(\Omega)} + \|\overrightarrow{y}_{nt}\|^2_{H^1(\Omega)})dt,
\end{aligned} \tag{29}$$

where $|b_i| \le h_i$, $i = 2,3,4,5,6,7$, $h_8 = 3\max\limits_{2\le i\le 6} h_i$, $h_9 = \max\limits_{1\le l\le 4}\overline{c}_l$, $\overline{h}_1 = h_8 + h_9 + 1$, $\overline{h}_2 = h_8 + b$, $\overline{h}_3 = \max(\overline{h}_1, \overline{h}_2)$, $\overline{h}_4 = 4e_l + 4\overline{e}_l$.
Since $\|\overrightarrow{y}^0_n\|_{H^1(Q)} \le b_1$ and $\|\overrightarrow{y}^1_n\|_{L^2(\Omega)} \le b_0$ with $\overline{h}_5 = b_0 + \overline{a}b_1 + \overline{h}_4$, hence (29) turn into

$$\begin{aligned}
\overline{h}_6[\|\overrightarrow{y}_{nt}(t)\|^2_{L^2(\Omega)} + \|\overrightarrow{y}_n(t)\|^2_{H^1(Q)}] &\le \overline{h}_5 + \overline{h}_3\int_0^t [\|\overrightarrow{y}_{nt}\|^2_0 + \|\overrightarrow{y}_n\|^2_{H^1(Q)}]dt \qquad (\overline{h}_6 = \min(1,\overline{a}))\\
&\le \overline{h}_7 + \overline{h}_3\int_0^t [\|\overrightarrow{y}_{nt}\|^2_0 + \|\overrightarrow{y}_n\|^2_{H^1(Q)}]dt
\end{aligned}$$

with $\overline{h}_7 = \frac{\overline{h}_5}{\overline{h}_6}$, $\overline{h}_8 = \frac{\overline{h}_3}{\overline{h}_6}$.
Applying the inequality for Gronwall, it yields that $\forall\ t \in [0,T]$,

$$\|\vec{y}_{nt}(t)\|^2_{L^2(\Omega)} + \|\vec{y}_n(t)\|^2_{H^1(Q)} \le \overline{h}_7 e^{\overline{h}_3} = b^2(c)$$

$$\Rightarrow \quad \|\vec{y}_{nt}(t)\|^2_{L^2(\Omega)} \le b^2(c) \text{ and } \|\vec{y}_n\|^2_{H^1(Q)} \le b^2(c), \quad \forall\ t \in [0,T].$$

Therefore, $\|\vec{y}_{nt}(t)\|_{L^2(Q)} \le b_1(c)$ and $\|\vec{y}_n(t)\|_{L^2(I,V)} \le b(c)$.

The QSVS convergence: Assume $\vec{V}$ has a sequence of subspace of $\{\vec{V}_n\}_{n=1}^{\infty}$ s.t. $\forall\ \vec{v} \in \vec{V}$, there is a subsequence $\{\vec{V}_n\}$ with $\vec{v}_n \in \vec{V}_n$, $\forall\ n$ for which $\vec{v}_n \xrightarrow[V]{} \vec{v}$ and $\vec{v}_n \xrightarrow[L^2(Q)]{} \vec{v}$ but for any n, with $\vec{V}_n \subset \vec{V}$, problem ((8)-(12) has a unique QSVS $\vec{y}_n = (y_{1n}, y_{2n}, y_{3n}, y_{4n})$, hence corresponding to the sequence of spaces $\{\vec{V}_n\}_{n=1}^{\infty}$, there is a sequence of approximation problem of the from ((8)-(12), allow $\vec{v}_l = \vec{v}_{ln} = (v_{1n}, v_{2n}, v_{3n}, v_{4n})$ in them $\forall\ n = 1,2,\dots$,

$$\langle y_{1tt}, v_{1n}\rangle + r_1(t,y_1,v_{1n}) + (a_1y_1,v_{1n})_{L^2(\Omega)} - (b_2y_2,v_{1n})_{L^2(\Omega)} + (b_3y_3,v_{1n})_{L^2(\Omega)} - (b_4y_4,v_{1n})_{L^2(\Omega)}$$
$$= (f_1,v_{1n})_{L^2(\Omega)} + (u_1,v_{1n})_{L^2(\partial\Omega)}, \tag{30}$$

$$\langle y_{2tt}, v_{2n}\rangle + r_2(t,y_2,v_{2n}) + (a_2y_2,v_{2n})_{L^2(\Omega)} + (b_2y_1,v_{2n})_{L^2(\Omega)} - (b_5y_3,v_{2n})_{L^2(\Omega)} + (b_6y_4,v_{2n})_{L^2(\Omega)}$$
$$= (f_2,v_{2n})_{L^2(\Omega)} + (u_2,v_{2n})_{L^2(\partial\Omega)}, \tag{31}$$

$$\langle y_{3tt}, v_{3n}\rangle + r_3(t,y_3,v_{3n}) + (a_3y_3,v_{3n})_{L^2(\Omega)} - (b_3y_1,v_{3n})_{L^2(\Omega)} + (b_5y_2,v_{3n})_{L^2(\Omega)} + (b_7y_4,v_{3n})_{L^2(\Omega)}$$
$$= (f_3,v_{3n})_{L^2(\Omega)} + (u_3,v_{3n})_{L^2(\partial\Omega)}, \tag{32}$$

$$\langle y_{4tt}, v_{4n}\rangle + r_4(t,y_4,v_{4n}) + (a_4y_4,v_{4n})_{L^2(\Omega)} + (b_4y_1,v_{4n})_{L^2(\Omega)} - (b_6y_2,v_{4n})_{L^2(\Omega)} - (b_7y_3,v_{4n})_{L^2(\Omega)}$$
$$= (f_4,v_{4n})_{L^2(\Omega)} + (u_4,v_{4n})_{L^2(\partial\Omega)}, \tag{33}$$

$$(y_l^0, v_{ln})_\Omega = (y_l(0), v_{1n})_\Omega \text{ and } (y_l^1, v_{ln})_\Omega = (y_{lt}(0), v_{1n})_\Omega, \quad \forall\ v_{ln} \in V_{ln},\ l = 1,2,3,4. \tag{34}$$

Of course ((30)-(34)) has a sequence of QSVS $\{\vec{y}_n\}_{n=1}^{\infty}$ with $\|\vec{y}_n(t)\|_{L^2(Q)}$ and $\|\vec{y}_n(t)\|_{L^2(I,V)}$ are bounded. By applying theorem of Alaglou (Borthwick [3]) there is a subsequence of $\{\vec{y}_n\}_{n\in N}$, let for simplicity be $\{\vec{y}_n\}_{n\in N}$, s.t. $\vec{y}_n \xrightarrow[L^2(Q)]{} \vec{y}$ and $\vec{y}_n \xrightarrow[L^2(I,V)]{} \vec{y}$, and since $L^2(I,V) \subset L^2(Q) \cong (L^2(Q))^* \subset L^2(I,V^*)$, then by applying the theorem of Aubin (Borthwick [3]), there is a subsequence of $\{\vec{y}_n\}_{n\in N}$ say a gain $\{\vec{y}_n\}_{n\in N}$ s.t. $\vec{y}_n \xrightarrow[L^2(Q)]{} \vec{y}$.

Now, multiplying ((30)-(34)) by $\varphi_l(t) \in C^2[0,T]$, $\forall\ l = 1,2,3,4$, $\varphi_l(T) = \varphi_l'(T) = 0$, $\varphi_l(0) \ne 0$, $\varphi_l'(0) \ne 0$ integrating on $[0,T]$, then integrating by parts twice the 1st expression in each acquired inequalities, to bring

$$-\int_0^T \frac{d}{dt}(y_{1nt},v_{1n})\varphi_1'(t)dt + \int_0^T [\overline{r}_1(t,y_{1n},v_{1n}) + (b_3y_{3n} - b_2y_{2n} - b_4y_{4n},v_{1n})_{L^2(\Omega)}]\varphi_1(t)dt$$
$$= \int_0^T ((f_1,v_{1n})_{L^2(\Omega)} + (u_1,v_{1n})_{L^2(\partial\Omega)})\varphi_1(t)dt + (y_{1n}^1,v_{1n})_{L^2(\Omega)}\varphi_{1,0}, \tag{35}$$

$$\int_0^T (y_{1nt},v_{1n})\varphi''(t)dt + \int_0^T [\overline{r}_1(t,y_{1n},v_{1n}) + (b_3y_{3n} - b_2y_{2n} - b_4y_{4n},v_{1n})_{L^2(\Omega)}]\varphi_1(t)dt$$
$$= \int_0^T ((f_1,v_{1n})_{L^2(\Omega)} + (u_1,v_{1n})_{L^2(\partial\Omega)})\varphi_1(t)dt + (y_{1n}^1,v_{1n})_{L^2(\Omega)}\varphi_{1,0} - (y_{1n}^0,v_{1n})_{L^2(\Omega)}\varphi_{1,0}', \tag{36}$$

$$-\int_0^T \frac{d}{dt}(y_{2nt},v_{2n})\varphi_2'(t)dt+\int_0^T[\overline{r}_2(t,y_{2n},v_{2n})+(b_2y_{1n}-b_5y_{3n}+b_6y_{4n},v_{2n})_{L^2(\Omega)}]\varphi_2(t)dt$$
$$=\int_0^T((f_2,v_{2n})_{L^2(\Omega)}+(u_2,v_{2n})_{L^2(\partial\Omega)})\varphi_2(t)dt+(y_{2n}^1,v_{2n})_{L^2(\Omega)}\varphi_{2,0}, \tag{37}$$

$$\int_0^T(y_{2nt},v_{2n})\varphi_2''(t)dt+\int_0^T[\overline{r}_2(t,y_{2n},v_{2n})+(b_2y_{1n}-b_5y_{3n}+b_6y_{4n},v_{2n})_{L^2(\Omega)}]\varphi_2(t)dt$$
$$=\int_0^T((f_2,v_{2n})_{L^2(\Omega)}+(u_2,v_{2n})_{L^2(\partial\Omega)})\varphi_2(t)dt+(y_{2n}^1,v_{2n})_{L^2(\Omega}\varphi_{2,0}-(y_{2n}^0,v_{2n})_{L^2(\Omega)}\varphi_{2,0}', \tag{38}$$

$$-\int_0^T \frac{d}{dt}(y_{3nt},v_{3n})\varphi_3'(t)dt+\int_0^T[\overline{r}_3(t,y_{3n},v_{3n})+(b_5y_2-b_3y_{1n}+b_7y_4,v_{3n})_{L^2(\Omega)}]\varphi_3(t)dt$$
$$=\int_0^T((f_3,v_{3n})_{L^2(\Omega)}+(u_3,v_{3n})_{L^2(\partial\Omega)})\varphi_3(t)dt+(y_{3n}^1,v_{3n})_{L^2(\Omega)}\varphi_{3,0}, \tag{39}$$

$$\int_0^T(y_{3nt},v_{3n})\varphi''{}_3(t)dt+\int_0^T[\overline{r}_3(t,y_{3n},v_{3n})+(b_5y_{2n}-b_3y_{1n}+b_7y_{4n},v_{3n})_{L^2(\Omega)}]\varphi_3(t)dt$$
$$=\int_0^T((f_3,v_{3n})_{L^2(\Omega)}+(u_3,v_{3n})_{L^2(\partial\Omega)})\varphi_3(t)dt+(y_{3n}^1,v_{3n})_{L^2(\Omega)}\varphi_{3,0}-(y_{3n}^0,v_{3n})_{L^2(\Omega)}\varphi_{3,0}', \tag{40}$$

$$-\int_0^T \frac{d}{dt}(y_{4nt},v_{4n})\varphi_4'(t)dt+\int_0^T[\overline{r}_4(t,y_{4n},v_{4n})+(b_4y_{1n}-b_6y_{2n-}b_7y_{3n},v_{4n})_{(L^2(\Omega)}]\varphi_4(t)dt$$
$$=\int_0^T((f_4,v_{4n})_{L^2(\Omega)}+(u_4,v_{4n})_{L^2(\partial\Omega)})\varphi_4(t)dt+(y_{4n}^1,v_{4n})_{(L^2(\Omega)}\varphi_{4,0}, \tag{41}$$

$$\int_0^T(y_{4nt},v_{4n})\varphi_4''(t)dt+\int_0^T[\overline{r}_4(t,y_{4n},v_{4n})+(b_4y_{1n}-b_6y_{2n}-b_7y_{3n},v_{4n})_{L^2(\Omega)}]\varphi_4(t)dt$$
$$=\int_0^T((f_4,v_{4n})_{L^2(\Omega)}+(u_4,v_{4n})_{L^2(\partial\Omega)})\varphi_4(t)dt+(y_{4n}^1,v_{4n})_{L^2(\Omega)}\varphi_{4,0}-(y_{4n}^0,v_{4n})_{L^2(\Omega)}\varphi_{4,0}' \tag{42}$$

with $\overline{r}_l(t,y_{ln},v_{ln})=r_l(t,y_{ln},v_{ln})+(a_ly_{ln},v_{ln})_{L^2(\Omega)}$, $\varphi_{l,0}=\varphi_l(0)$ and $\varphi_{l,0}'=\varphi_l'(0)$, $l=1,2,3,4$.
First, since $\forall\ l=1,2,3,4$,

$$v_{ln}\underset{V}{\rightarrow}v_l\Rightarrow\begin{cases}\begin{cases}v_{ln}\varphi_l(t)\xrightarrow[L^2(I,\vec{V})]{}v_l\varphi_l(t),\\ v_{ln}\varphi_l'(t)\rightarrow[L^2(I,\vec{V})]v_l\varphi_l'(t),\end{cases}\\ v_{ln}\varphi_l(0)\xrightarrow[L^2(\Omega)]{}v_l\varphi_l(0)\end{cases}$$

and

$$v_{ln}\xrightarrow[L^2(\Omega)]{}v_l\Rightarrow\begin{cases}\begin{cases}v_{ln}\varphi_l'(t)\xrightarrow[L^2(Q)]{}v_l\varphi_l'(t),\\ v_{ln}\varphi_l''(t)\xrightarrow[L^2(Q)]{}v_l\varphi_l''(t),\end{cases}\\ v_{ln}\varphi_l'(0)\xrightarrow[L^2(\Omega)]{}v_l\varphi_l'(0)\end{cases}$$

Second, $y_{lnt}\xrightarrow[L^2(Q)]{}y_{lt}$ and $y_{ln}\xrightarrow[L^2(I,V)]{}y_l$ and $y_{ln}\xrightarrow[L^2(Q)]{}y_l$.
Third, since $v_{ln}\varphi_l\xrightarrow[L^2(Q)]{}v_l\varphi_l$, then $\forall\ l=1,2,3,4$, then

$$\int_0^T((f_l,v_{ln})_{L^2(\Omega)}+(u_l,v_{ln})_{L^2(\partial\Omega)})\varphi_l(t)dt$$

$$\Rightarrow \int_0^T ((f_l, v_l)_{L^2(\Omega)} + (u_l, v_l)_{L^2(\partial\Omega)})\varphi_l(t)dt.$$

These above three points helps to passage the limits in ((35)-(42)), to turn out

$$-\int_0^T \frac{d}{dt}(y_{1t}, v_1)\varphi_1'(t)dt + \int_0^T [r_1(t, y_1, v_1) + (a_1y_1 - b_2y_2 + b_3y_3 - b_4y_4, v_1)_{L^2(\Omega)}]\varphi_1(t)dt$$
$$= \int_0^T ((f_1, v_1)_{L^2(\Omega)} + (u_1, v_1)_{L^2(\partial\Omega)})\varphi_1(t)dt + (y_1^1, v_1)_{L^2(\Omega)}\varphi_{1,0}, \tag{43}$$

$$\int_0^T (y_1, v_1)\varphi''(t)dt + \int_0^T [r_1(t, y_1, v_1) + (a_1y_1 - b_2y_2 + b_3y_3 - b_4y_4, v_1)_{L^2(\Omega)}]\varphi_1(t)dt$$
$$= \int_0^T ((f_1, v_1)_{L^2(\Omega)} + (u_1, v_1)_{L^2(\partial\Omega)})\varphi_1(t)dt + (y_1^1, v_1)_{L^2(\Omega)}\varphi_{1,0} - (y_1^0, v_1)_{L^2(\Omega)}\varphi_{1,0}', \tag{44}$$

$$-\int_0^T \frac{d}{dt}(y_{2t}, v_2)\varphi_2'(t)dt + \int_0^T [r_2(t, y_2, v_2) + (a_2y_2 + b_2y_1 - b_5y_3 + b_6y_4, v_2)_{L^2(\Omega)}]\varphi_2(t)dt$$
$$= \int_0^T ((f_2, v_2)_{L^2(\Omega)} + (u_2, v_2)_{L^2(\partial\Omega)})\varphi_2(t)dt + (y_2^1, v_2)_{L^2(\Omega)}\varphi_{2,0}, \tag{45}$$

$$\int_0^T (y_2, v_2)\varphi_2''(t)dt + \int_0^T [r_2(t, y_2, v_2) + (a_2y_2 + b_2y_1 - b_5y_3 + b_6y_4, v_2)_{L^2(\Omega)}]\varphi_2(t)dt$$
$$= \int_0^T ((f_2, v_2)_{L^2(\Omega)} + (u_2, v_2)_{L^2(\partial\Omega)})\varphi_2(t)dt + (y_2^1, v_2)_{L^2(\Omega}\varphi_{2,0} - (y_2^0, v_2)_{L^2(\Omega)}\varphi_{2,0}', \tag{46}$$

$$-\int_0^T \frac{d}{dt}(y_{3t}, v_3)\varphi_3'(t)dt + \int_0^T [r_3(t, y_3, v_3) + (a_3y_3 - b_3y_1 + b_5y_2 + b_7y_4, v_3)_{L^2(\Omega)}]\varphi_3(t)dt$$
$$= \int_0^T ((f_3, v_3)_{L^2(\Omega)} + (u_3, v_3)_{L^2(\partial\Omega)})\varphi_3(t)dt + (y_3^1, v_3)_{L^2(\Omega)}\varphi_{3,0}, \tag{47}$$

$$\int_0^T (y_3, v_3)\varphi_3''(t)dt + \int_0^T [r_3(t, y_{3n}, v_{3n}) + (a_3y_{3n} - b_3y_{1n} + b_5y_2 + b_7y_4, v_{3n})_{L^2(\Omega)}]\varphi_3(t)dt$$
$$= \int_0^T ((f_3, v_3)_{L^2(\Omega)} + (u_3, v_3)_{L^2(\partial\Omega)})\varphi_3(t)dt + (y_3^1, v_3)_{L^2(\Omega)}\varphi_{3,0} - (y_3^0, v_3)_{L^2(\Omega)}\varphi_{3,0}', \tag{48}$$

$$-\int_0^T \frac{d}{dt}(y_{4t}, v_4)\varphi_4'(t)dt + \int_0^T [r_4(t, y_4, v_4) + (a_4y_4 + b_4y_1 - b_6y_2 - b_7y_3, v_4)_{L^2(\Omega)}]\varphi_4(t)dt$$
$$= \int_0^T ((f_4, v_4)_{L^2(\Omega)} + (u_4, v_4)_{L^2(\partial\Omega)})\varphi_4(t)dt + (y_4^1, v_4)_{(L^2(\Omega)}\varphi_{4,0}, \tag{49}$$

$$\int_0^T (y_4, v_4)\varphi_4''(t)dt + \int_0^T [r_4(t, y_4, v_4) + (a_4y_4 + b_4y_1 - b_6y_2 - b_7y_3, v_4)_{L^2(\Omega)}]\varphi_4(t)dt$$
$$= \int_0^T ((f_4, v_4)_{L^2(\Omega)} + (u_4, v_4)_{L^2(\partial\Omega)})\varphi_4(t)dt + (y_4^1, v_4)_{L^2(\Omega)}\varphi_{4,0} - (y_4^0, v_4)_{L^2(\Omega)}\varphi_{4,0}'. \tag{50}$$

Case 1: Pick out $\varphi_l \in C^2[0,T]$ ($\forall\ l = 1,2,3,4$), in (44), (46), (48) and (49) with $\varphi_l(0) = \varphi_l'(0) = 0$, $\varphi_l'(T) = \varphi_l(T) = 0$, then integrating twice the first expression in their LHS, they yield to

$$\int_0^T (y_{1tt}, v_1)\varphi(t)dt + \int_0^T (r_1(t, y_1, v_1) + (a_1y_1 - b_2y_2 + b_3y_3 - b_4y_4, v_1)_{L^2(\Omega)})\varphi_1(t)dt$$
$$= \int_0^T ((f_1, v_1)_{L^2(\Omega)} + (u_1, v_1)_{L^2(\partial\Omega)})\varphi_1(t)dt, \tag{51}$$

$$\int_0^T (y_{2tt}, v_2)\varphi_2(t)dt + \int_0^T (r_2(t, y_2, v_2) + (a_2 y_2 + b_2 y_1 - b_5 y_3 + b_6 y_4, v_2)_{L^2(\Omega)})\varphi_2(t)dt$$
$$= \int_0^T ((f_2, v_2)_{L^2(\Omega)} + (u_2, v_2)_{L^2(\partial\Omega)})\varphi_2(t)dt, \tag{52}$$
$$\int_0^T (y_{3tt}, v_3)\varphi_3(t)dt + \int_0^T (r_3(t, y_{3n}, v_{3n}) + (a_3 y_{3n} - b_3 y_{1n} + b_5 y_2 + b_7 y_4, v_{3n})_{L^2(\Omega)})\varphi_3(t)dt$$
$$= \int_0^T ((f_3, v_3)_{L^2(\Omega)} + (u_3, v_3)_{L^2(\partial\Omega)})\varphi_3(t)dt, \tag{53}$$
$$\int_0^T (y_{4tt}, v_4)\varphi_4(t)dt + \int_0^T (r_4(t, y_4, v_4) + (a_4 y_4 + b_4 y_1 - b_6 y_2 - b_7 y_3, v_4)_{L^2(\Omega)})\varphi_4(t)dt$$
$$= \int_0^T ((f_4, v_4)_{L^2(\Omega)} + (u_4, v_4)_{L^2(\partial\Omega)})\varphi_4(t)dt. \tag{54}$$

Hence $\vec{y}$ is a QSVS of ((8)-(11)) a.e. on I.

Case 2: Pick out $\varphi_l \in C^2[0,T]$, $\forall\ l = 1,2,3,4$, s.t. $\varphi_l(t) = \varphi_l'(0) = \varphi_l'(T) = 0$ and $\varphi_l(0) \neq 0$, MBS ((8), (9), (10), (11) by $\varphi_1(t)$, $\varphi_2(t)$, $\varphi_3(t)$ and $\varphi_4(t)$ respectively, then integrate on $[0,T]$, and then integrating the 1st expression in LHS in each acquired inequalities, then subtracting each equality from the each corresponding one of ((43), (45), (47) and (48)) respectively, to acquire

$$(y_{lt}(0), v_l)\varphi_l(0) = (y_l^1, v_i)\varphi_l(0), \quad \forall\ l = 1,2,3,4.$$

Case 3: Pick out $\varphi_l \in C^2[0,T]$, $\forall\ l = 1,2,3,4$, with $\varphi_l(0) = \varphi_l(T) = \varphi_l'(T) = 0$, $\varphi_l'(0) \neq 0$, multiplying of ((8), (9), (10) and (11)) respectively, then integrate on $[0,T]$, and then integrating by parts twice the 1st expression in LHS in each acquired inequalities, then subtracting each equality from the each corresponding one of ((44), (46), (48) and (49)) respectively, to acquire

$$(y_l^0, v_l)\varphi_l'(0) = (y_l(0), v_l)\varphi_l'(0), \quad \forall\ l = 1,2,3,4.$$

From the above two previous cases, one acquires the ICs equation (12).

To prove $\vec{y}_n \xrightarrow[L^2(I,V)]{} \vec{y}$, integrate (27) on $[0,T]$, to acquire that

$$\|\vec{y}_{nt}(T)\|_Q^2 - \|\vec{y}_{nt}(0)\|_Q^2 + s(t, \vec{y}_n, \vec{y}_n)(T) - s(t, \vec{y}_n, \vec{y}_n)(0) - \int_0^T s_t(t, \vec{y}_n, \vec{y}_{nt})dt$$
$$= \int_0^T ((55\text{a}) + (55\text{b}))dt, \tag{55}$$

where

$$(55\text{a}) = 2((-b_2 y_{2n} + b_3 y_{3n} - b_4 y_{4n}, y_{1nt})_{L^2(\Omega)} + (b_2 y_{1n} - b_5 y_{3n} + b_6 y_{4n}, y_{2nt})_{L^2(\Omega)}$$
$$+ (-b_3 y_{1n} + b_5 y_{2n} + b_7 y_{4n}, y_{3nt})_{L^2(\Omega)} + (b_4 y_{1n} - b_6 y_{2n} - b_7 y_{3n}, y_{4nt})_{L^2(\Omega)}, \tag{55a}$$
$$(55\text{b}) = 2((f_1, y_{1nt})_{L^2(\Omega)} + (u_1, y_{1nt})_{L^2(\delta\Omega)} + (f_2, y_{2nt})_{L^2(\Omega)} + (u_2, y_{2nt})_{L^2(\delta\Omega)}$$
$$+ (f_3, y_{3nt})_{L^2(\Omega)} + (u_3, y_{3nt})_{L^2(\delta\Omega)} + (f_4, y_{4nt})_{L^2(\Omega)} + (u_4, y_{4nt})_{L^2(\delta\Omega)}). \tag{55b}$$

Now, replace $y_{ln} = y_l$, $\forall\ l = 1,2,3,4$, in (33), integrate the resulting equality on $[0,T]$ to acquire

$$\|\vec{y}_t(T)\|_Q^2 - \|\vec{y}_t(0)\|_Q^2 + s(t, \vec{y}, \vec{y})(T) - s(t, \vec{y}, \vec{y})(0) - \int_0^T s_t(t, \vec{y}, \vec{y}_t)dt$$
$$= \int_0^T ((56\text{a}) + (56\text{b}))dt, \tag{56}$$

where

$$(56\text{a}) = 2((-b_2y_2 + b_3y_3 - b_4y_4, y_{1t})_{L^2(\Omega)} + (b_2y_1 - b_5y_3 + b_6y_4, y_{2t})_{L^2(\Omega)} + (-b_3y_1 + b_5y_2 + b_7y_4, y_{3t})_{L^2(\Omega)} + (b_4y_1 - b_6y_2 - b_7y_3, y_{4t})_{L^2(\Omega)}, \tag{56a}$$

$$(56\text{b}) = 2((f_1, y_{1t})_{L^2(\Omega)} + (u_1, y_{1t})_{L^2(\delta\Omega)} + (f_2, y_{2t})_{L^2(\Omega)} + (u_2, y_{2t})_{L^2(\delta\Omega)} + (f_3, y_{3t})_{L^2(\Omega)} + (u_3, y_{3t})_{L^2(\delta\Omega)} + (f_4, y_{4t})_{L^2(\Omega)} + (u_4, y_{4t})_{L^2(\delta\Omega)}). \tag{56b}$$

Since

$$\|\vec{y}_{nt}(T) - \vec{y}_t(T)\|^2_{L^2(\Omega)} - \|\vec{y}_{nt}(0) - \vec{y}_t(0)\|^2_{L^2(\Omega)} + s(t, \vec{y}_n - \vec{y}, \vec{y}_n - \vec{y})(T) - s(t, \vec{y}_n - \vec{y}, \vec{y}_n - \vec{y})(0) - \int_0^T s_t(t, \vec{y}_n - \vec{y}, \vec{y}_{nt} - \vec{y}_t)dt = ((57\text{a}) - (57\text{b}) - (57\text{c})), \tag{57}$$

where

$$(57\text{a}) = \|\vec{y}_{nt}(T)\|^2_{L^2(Q)} - \|\vec{y}_n(0)\|^2_{L^2(\Omega)} + s(t, \vec{y}_n, \vec{y}_n)(T) - s(t, \vec{y}_n, \vec{y}_n)(0) - \int_0^T s_t(t, \vec{y}_n, \vec{y}_{nt})dt, \tag{57a}$$

$$(57\text{b}) = (\vec{y}_{nt}(T), \vec{y}_t(T)) - (\vec{y}_{nt}(0), \vec{y}_t(0)) + s(t, \vec{y}_n, \vec{y})(T) - s(t, \vec{y}_n, \vec{y})(0) - \int_0^T s_t(t, \vec{y}_n, \vec{y}_t)dt \tag{57b}$$

$$(57\text{c}) = (\vec{y}_t(T), \vec{y}_{nt} - \vec{y}_t(T)) - (\vec{y}_t(0), \vec{y}_{nt}(0) - \vec{y}_t(0)) + s(t, \vec{y}_n, \vec{y}_n - \vec{y})(T) - s(t, \vec{y}_n, \vec{y}_n - \vec{y})(0) - \int_0^T s_t(t, \vec{y}_n, \vec{y}_{nt} - \vec{y}_t)dt, \tag{57c}$$

thus,

$$(57\text{a}) = \text{L.H.S. of } (55) = \int_0^T ((55\text{a}) + (55\text{b}))dt \Rightarrow \int_0^T ((56\text{a}) + (56\text{b}))dt.$$

A similar manner which utilized to get (21), utilizes also here to obtain

$$\vec{y}_{nt}(T) \xrightarrow[L^2(\Omega)]{} \vec{y}_t(T). \tag{58}$$

On the other hand, since $\vec{y}_n \xrightarrow[L^2(I,V)]{} \vec{y}$, then utilizing (21) and (58) in (57b), it yields that (57b) → L.H.S. (56) = $\int_0^T$ (56a) + (56b)dt.

Beside these convergences, all the expressions in (57c) imply to zero, in addition the first two expressions in LHS of (57); hence (57) gives

$$\int_0^T \|\vec{y}_n(t) - \vec{y}(t)\|^2_{H^1(Q)} dt \to 0 \ \text{ as } \ n \to \infty$$

thus, we get

$$\vec{y}_n \xrightarrow[L^2(I,V)]{} \vec{y}.$$

Uniqueness of the QSVS: Assume that $\vec{y}$ and $\vec{\vec{y}}$ are two QSVS of the WFO ((8)-(12)), then subtracting each equality form its conforming one and letting $\vec{v} = \vec{y} - \vec{\vec{y}}$, utilizing [12, Lemma 2.1] for the 1st expression in each equality and then gathering all the above equalities, utilizing HYPOS and integrating from 0 to t, to acquire

$$\int_0^t \frac{d}{dt}[\|(\vec{y}-\vec{\bar{y}})_t\|^2_{L^2(Q)} + \bar{a}\|(\vec{y}-\vec{\bar{y}})\|^2_{H^1(\Omega)}]dt \le \int_0^t b\|(\vec{y}-\vec{\bar{y}})\|^2_{H^1(\Omega)}dt.$$

Utilizing the ICs,

$$\|(\vec{y}-\vec{\bar{y}})_t(t)\|^2_{L^2(Q)} + \|(\vec{y}-\vec{\bar{y}})(t)\|^2_{H^1(\Omega)} \le h_3\int_0^t [\|(\vec{y}-\vec{\bar{y}})_t\|^2_{L^2(\Omega)} + b\|(\vec{y}-\vec{\bar{y}})\|^2_{H^1(\Omega)}]dt,$$

where $h_1 = \min(1,\bar{a})$, $h_2 = \max(1,b)$, $h_3 = h_2/h_1$.

Employing the inequality of Gronwall, to acquire the uniqueness of the QSVS as follow

$$\|(\vec{y}-\vec{\bar{y}})(t)\|^2_{H^1(\Omega)} = 0, \quad \forall\, t \in I$$

$$\Longrightarrow \quad \|(\vec{y}-\vec{\bar{y}})(t)\|^2_{L^2(I,V)} = 0$$

$$\Longrightarrow \quad \vec{y} = \vec{\bar{y}}.$$

4. Existence of a Boundary Quaternary Control Vector

The following lemma is very useful in the study of the existence of a CQBCV.

Lemma 4.1. *Besides to HYPOS, consider $\vec{y}$ and $\vec{y}+\delta\vec{y}$ are the QSVS conforming corresponding to the bounded CBCQV $\vec{u}$ and $\vec{u}+\overrightarrow{\delta u}$ respectively, then fro $K \in \mathcal{R}^+$.*

(i) $\|\delta\vec{y}\|_{L^\infty(I,L^2(\Omega))} \le K\|\delta\vec{u}\|_{L^2(\Sigma)}$, $\|\delta\vec{y}\|_{L^2(Q)} \le K\|\delta\vec{u}\|_{L^2(\Sigma)}$, $\|\delta\vec{y}\|_{L^2(I,V)} \le K\|\delta\vec{u}\|_{L^2(\Sigma)}$.

(ii) *The operator $\vec{u} \longrightarrow \vec{y}_{\vec{u}}$ holds the Lipschitz continuity (LC) property from $L^2(Q)$ in to $L^\infty(I,L^2(\Omega))$ (or in to $L^2(I,V)$ or into $L^2(Q)$).*

Proof. (i) Assume $\vec{u},\vec{\bar{u}} \in L^2(\partial Q)$, then $\vec{u}_\sigma = \vec{u}+\sigma\delta\vec{u} \in L^2(Q)$, with $\overrightarrow{\delta u} = \vec{\bar{u}}-\vec{u}$ and $\sigma > 0$, then by Theorem 3.1, $\vec{y} = \vec{y}_{\vec{u}}$, $\vec{y}_\sigma = \vec{y}_{\vec{u}_\sigma}$ are their QSVS, thus ((8)-(12)) for $\vec{y} = \vec{y}_\sigma$, give

$$\langle y_{1\sigma tt}, v_1\rangle + r_1(t,y_{1\sigma},v_1) + (a_1y_{1\sigma},v_1)_{L^2(\Omega)} - (b_2y_{2\sigma},v_1)_{L^2(\Omega)} + (b_3y_{3\sigma},v_1)_{L^2(\Omega)} - (b_4y_{4\sigma},v_1)_{L^2(\Omega)}$$
$$= (f_1,v_1)_{L^2(\Omega)} + (u_{1\sigma},v_1)_{L^2(\Omega)}, \tag{59}$$

$$\langle y_{2\sigma tt}, v_2\rangle + r_2(t,y_{2\sigma},v_2) + (a_2y_{2\sigma},v_2)_{L^2(\Omega)} + (b_2y_{1\sigma},v_2)_{L^2(\Omega)} - (b_5y_{3\sigma},v_2)_{L^2(\Omega)} + (b_6y_{4\sigma},v_2)_{L^2(\Omega)}$$
$$= (f_2,v_2)_{L^2(\Omega)} + (u_{2\sigma},v_2)_{L^2(\delta\Omega)}, \tag{60}$$

$$\langle y_{3\sigma tt}, v_3\rangle + r_3(t,y_{3\sigma},v_3) + (a_3y_{3\sigma},v_3)_{L^2(\Omega)} - (b_3y_{1\sigma},v_3)_{L^2(\Omega)} + (b_5y_{2\sigma},v_3)_{L^2(\Omega)} + (b_7y_{4\sigma},v_3)_{L^2(\Omega)}$$
$$= (f_3,v_3)_{L^2(\Omega)} + (u_{3\sigma},v_3)_{L^2(\delta\Omega)}, \tag{61}$$

$$\langle y_{4\sigma tt}, v_4\rangle + r_4(t,y_{4\sigma},v_4) + (a_4y_{4\sigma},v_4)_{L^2(\Omega)} + (b_4y_{1\sigma},v_4)_{L^2(\Omega)} - (b_6y_{2\sigma},v_4)_{L^2(\Omega)} - (b_7y_{3\sigma},v_4)_{L^2(\Omega)}$$
$$= (f_4,v_4)_{L^2(\Omega)} + (u_{4\sigma},v_4)_{L^2(\delta\Omega)} \tag{62}$$

$$(y^0_{l\sigma},v_l)_{L^2(\Omega)} = (y_l(0),v_1)_{L^2(\Omega)} \text{ and } (y^1_{l\sigma},v_l)_{L^2(\Omega)} = (y_{lt}(0),v_1)_{L^2(\Omega)}, \quad \forall\, v_l \in V_l,\ l = 1,2,3,4. \tag{63}$$

Subtracting ((8)-(12)) from ((59)-(63)), putting $\vec{y}_\sigma - \vec{y} = \delta\vec{y}_\sigma$, they acquire

$$\langle \delta y_{1\sigma tt}, v_1\rangle + r_1(t,\delta y_{1\sigma},v_1) + (a_1\delta y_{1\sigma},v_1)_{L^2(\Omega)} - (b_2\delta y_{2\sigma},v_1)_{L^2(\Omega)} + (b_3\delta y_{3\sigma},v_1)_{L^2(\Omega)} - (b_4\delta y_{4\sigma},v_1)_{L^2(\Omega)}$$
$$= (\sigma\delta u_{1\sigma},v_1)_{L^2(\delta\Omega)}, \tag{64}$$

$$(b_6\delta y_{4\sigma},v_2)_{L^2(\Omega)} = (\sigma\delta u_{2\sigma},v_2)_{L^2(\delta\Omega)}, \tag{65}$$

$$\langle \delta y_{3\sigma tt}, v_3\rangle + r_3(t,\delta y_{3\sigma},v_3) + (a_3\delta y_{3\sigma},v_3)_{L^2(\Omega)} - (b_3\delta y_{1\sigma},v_3)_{L^2(\Omega)} + (b_5\delta y_{2\sigma},v_3)_{L^2(\Omega)} + (b_7\delta y_{4\sigma},v_3)_{L^2(\Omega)}$$
$$= (\sigma\delta u_{3\sigma},v_3)_{L^2(\delta\Omega)}, \tag{66}$$

$$\langle \delta y_{4\sigma tt}, v_4\rangle + r_4(t,\delta y_{4\sigma}, v_4) + (a_4\delta y_{4\sigma}, v_4)_{L^2(\Omega)} + (b_4\delta y_{1\sigma}, v_4)_{L^2(\Omega)} - (b_6\delta y_{2\sigma}, v_4)_{L^2(\Omega)} - (b_7\delta y_{3\sigma}, v_4)_{L^2(\Omega)} = (\sigma\delta u_{4\sigma}, v_4)_{L^2(\delta\Omega)}, \tag{67}$$

$$(\delta y^0_{l\sigma}, v_l)_{L^2(\Omega)} = 0 \text{ and } (\delta y^1_{l\sigma}, v_l)_{L^2(\Omega)} = 0, \quad \forall\, v_l \in V_l,\ l = 1,2,3,4. \tag{68}$$

By letting $v_l = \delta y_{l\sigma t}$, for $l = 1,2,3,4$ in ((64)-(68)), then applying [12, Lemma 2.1] for the 1st term in the LHS of each resulting equality, then utilizing the same manner which utilized to acquire (26), a similar equality will be gotten but within position of $\vec{y}_n$, then integrating on $[0,t]$, applying the Cauchy inequality, with setting $|b_i| \le c_i > 0$, $i = 2,\dots,7$ to acquire

$$\begin{aligned}
&\|\overrightarrow{\delta y}_{\sigma t}(t)\|^2_{L^2(\Omega)} + \bar{a}\|\overrightarrow{\delta y}_{\sigma}(t)\|^2_{H^1(\Omega)}\\
&\le b\int_0^t \|\overrightarrow{\delta y}_{\sigma}\|^2_{H^1(\Omega)}dt\\
&\quad + 2\int_0^t [c_2\|\delta y_{2\sigma}\|_{L^2(\Omega)} + c_3\|\delta y_{3\sigma}\|_{L^2(\Omega)} + c_4\|\delta y_{4\sigma}\|_{L^2(\Omega)} + \|\sigma\delta u_1\|_{L^2(\partial\Omega)}]\|\delta y_{1\sigma t}\|_{L^2(\partial\Omega)}dt\\
&\quad + 2\int_0^t [c_2\|\delta y_{1\sigma}\|_{L^2(\Omega)} + c_5\|\delta y_{3\sigma}\|_{L^2(\Omega)} + c_6\|\delta y_{4\sigma}\|_{L^2(\Omega)} + \|\sigma\delta u_2\|_{L^2(\partial\Omega)}]\|\delta y_{2\sigma t}\|_{L^2(\partial\Omega)}dt\\
&\quad + 2\int_0^t [c_3\|\delta y_{1\sigma}\|_{L^2(\Omega)} + c_5\|\delta y_{2\sigma}\|_{L^2(\Omega)} + c_7\|\delta y_{4\sigma}\|_{L^2(\Omega)} + \|\sigma\delta u_3\|_{L^2(\partial\Omega)}]\|\delta y_{3\sigma t}\|_{L^2(\partial\Omega)}dt\\
&\quad + 2\int_0^t [c_4\|\delta y_{1\sigma}\|_{L^2(\Omega)} + c_6\|\delta y_{2\sigma}\|_{L^2(\Omega)} + c_7\|\delta y_{3\sigma}\|_{L^2(\Omega)} + \|\sigma\delta u_4\|_{L^2(\partial\Omega)}]\|\delta y_{4\sigma t}\|_{L^2(\partial\Omega))}dt.
\end{aligned}$$

Utilizing the trace operator, Young's inequality for products, the relations between the norms and then gathering the same terms, it yields to

$$\begin{aligned}
&\bar{c}_8[\|\overrightarrow{\delta y}_{\sigma t}(t)\|^2_{L^2(\Omega)} + \|\overrightarrow{\delta y}_{\sigma}(t)\|^2_{H^1(\Omega)}]\\
&\le \int_0^t [b\|\overrightarrow{\delta y}_{\sigma}\|^2_{H^1(\Omega)} + \bar{c}_5\|\delta y_{1\sigma t}\|^2_{L^2(\Omega)} + \bar{c}_6\|\delta y_{2\sigma t}\|^2_{L^2(\Omega)} + \bar{c}_7\|\delta y_{3\sigma t}\|^2_{L^2(\Omega)} + \bar{c}_8\|\delta y_{4\sigma t}\|^2_{L^2(\Omega)}]dt\\
&\quad + \int_0^t [c_2\|\delta y_{2\sigma}\|^2_{L^2(\Omega)} + c_3\|\delta y_{3\sigma}\|^2_{L^2(\Omega)} + c_4\|\delta y_{4\sigma}\|^2_{L^2(\Omega)} + \sigma\|\delta u_1\|^2_{L^2(\partial\Omega)}]dt\\
&\quad + \int_0^t [c_2\|\delta y_{1\sigma}\|^2_{L^2(\Omega)} + c_5\|\delta y_{3\sigma}\|^2_{L^2(\Omega)} + c_6\|\delta y_{4\sigma}\|^2_{L^2(\Omega)} + \sigma\|\delta u_2\|^2_{L^2(\partial\Omega)}]dt\\
&\quad + \int_0^t [c_3\|\delta y_{1\sigma}\|^2_{L^2(\Omega)} + c_5\|\delta y_{2\sigma}\|^2_{L^2(\Omega)} + c_7\|\delta y_{4\sigma}\|^2_{L^2(\Omega)} + \sigma\|\delta u_3\|^2_{L^2(\partial\Omega)}]dt\\
&\quad + \int_0^t [c_4\|\delta y_{1\sigma}\|^2_{L^2(\Omega)} + c_6\|\delta y_{2\sigma}\|^2_{L^2(\Omega)} + c_7\|\delta y_{3\sigma}\|^2_{L^2(\Omega)} + \sigma\|\delta u_4\|^2_{L^2(\partial\Omega)}]dt\\
&\le \int_0^t [b\|\overrightarrow{\delta y}_{\sigma}\|^2_{H^1(\Omega)} + c_8\|\overrightarrow{\delta y}_{\Sigma t}\|^2_{L^2(\Omega)}]dt + \int_0^t [c_9\|\overrightarrow{\delta y}_{\sigma}\|^2_{H^1(\Omega)} + \sigma\|\overrightarrow{\delta u}\|^2_{L^2(\partial\Omega)}]dt,
\end{aligned}$$

where $\bar{c}_1 = c_2 + c_3 + c_4$, $\bar{c}_2 = c_2 + c_5 + c_6$, $\bar{c}_3 = c_3 + c_5 + c_7$, $\bar{c}_4 = c_4 + c_6 + c_7$, $\bar{c}_5 = \bar{c}_1 + \sigma$, $\bar{c}_6 = \bar{c}_2 + \sigma$, $\bar{c}_7 = \bar{c}_3 + \sigma$, $\bar{c}_8 = \bar{c}_4 + \sigma$, $c_8 = \max(\bar{c}_5, \bar{c}_6, \bar{c}_7, \bar{c}_8)$, $c_9 = \max(\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4)$.

Hence

$$\|\overrightarrow{\delta y}_{\Sigma t}(t)\|^2_{L^2(\Omega)} + \|\overrightarrow{\delta y}_{\Sigma}(t)\|^2_{H^1(\Omega)} \le c_{11}\|\overrightarrow{\delta u}(t)\|^2_{\Sigma} + c_{10}\int_0^t (\|\overrightarrow{\delta y}_{\Sigma t}\|^2_{L^2(\Omega)} + \|\overrightarrow{\delta y}_{\Sigma}\|^2_{H^1(\Omega)})dt,$$

where $\bar{c}_8 = \min(1, \bar{a})$, $\bar{c}_9 = \max(b, c_9, c_8)$, $c_{10} = \bar{c}_9/\bar{c}_8$, $c_{11} = \sigma/\bar{c}_8$.

By employing the inequality of Gronwall, with $k^2 = c_{11}e^{Tc_{10}} > 0$, it yields

$$\|\overrightarrow{\delta y}_{\Sigma t}(t)\|^2_{L^2(\Omega)} + \|\overrightarrow{\delta y}_{\Sigma}(t)\|^2_{H^1(\Omega)} \le k^2\|\overrightarrow{\delta u}(t)\|^2_{\Sigma}, \quad \forall\, t \in I$$

$\Rightarrow \quad \|\overrightarrow{\delta y}_\Sigma(t)\|^2_{H^1(\Omega)} \le k^2\|\overrightarrow{\delta u}(t)\|^2_\Sigma$ and $\|\overrightarrow{\delta y}_{\Sigma t}(t)\|^2_{L^2(\Omega)} \le k^2\|\overrightarrow{\delta u}(t)\|^2_\Sigma, \quad \forall\, t \in I.$

The other results acquire immediately.

(ii) If $\vec{u}_\sigma \xrightarrow[L^2(Q)]{} \vec{u}$ then $\vec{y}_\sigma \xrightarrow[L^\infty(I,L^2(\Omega))]{} \vec{y}$ then from part (i), $\|\vec{y}_\sigma - \vec{y}\|_{L^\infty(I,L^2)} \le K\|\vec{u}_\sigma - \vec{u}\|_Q$ it acquires immediately that $\vec{u} \longrightarrow \vec{y}_{\vec{u}}$ is LC from $L^2(\Sigma)$ in to $L^\infty(I,L^2(\Omega))$. The other two results are come by the same manner. □

Lemma 4.2. *The OF defines in* (7) *is WLSC.*

Proof. Since $\|\vec{u}\|_\Sigma$ is *weakly lower semi continuous* (WLSC), now when $\vec{u}_k \xrightarrow[L^2(\Sigma)]{} \vec{u}$, then $\vec{y}_k \xrightarrow[L^2(Q)]{} \vec{y}$ and then $\|\vec{y} - \vec{y}_d\| \le \lim_{k\to\infty} \inf_{y_k \in v_k} \|\vec{y}_k - \vec{y}_d\|$. Thus, $G_0(\vec{u})$ is WLSC. □

Theorem 4.1. *The problem under consideration has a CBOCQV if the $G_0(\vec{u})$ is coercive.*

Proof. From the coercive property of $G_0(\vec{u})$, with $G_0(\vec{u}) \ge 0$ then there exists a minimum sequence $\{\vec{u}_k\} \in \vec{W}$, $\forall\, k$, s.t. $\lim_{n\to\infty} G_0(\vec{u}_k) = \inf_{\vec{u}_k \in \vec{W}} G_0(\vec{\overline{u}})$, and $\|\vec{u}_k\| \le c$.

Then by theorem of Alaglou the sequence $\{\vec{u}_k\}$, has a subsequence let be $\{\vec{u}_k\}$ again, s.t. $\vec{u}_k \xrightarrow[L^2(\Sigma)]{} \vec{u}$, as $k \to \infty$. From Theorem 3.1, the sequence of the CBCQV $\{\vec{u}_k\}$, has conforming sequence of the unique QSVS $\{\vec{y}_k = \vec{y}_{\vec{u}_k}\}$ with $\|\vec{y}_k\|_{L^2(I,V)}$, $\|\vec{y}_{kt}\|_{L^2(Q)}$ are bounded, and then by theorem of Alaglou the sequence of QSVS $\{\vec{y}_k\}$ has a subsequence, let be $\{\vec{y}_k\}$, s.t. $\vec{y}_{kt} \xrightarrow[L^2(Q)]{} \vec{y}$, $\vec{y}_k \xrightarrow[L^2(I,V)]{} \vec{y}$.

Now, for any k, the QSVS $\vec{y}_k$ satisfies the WFO ((15)-(19)), multiplying every equality by $\varphi_l \in C^2[0,T]$, $\forall\, l = 1,2,3,4$, with $\varphi_l(T) = \varphi'_l(T) = 0$, $\varphi_l(0) \ne 0$, $\varphi'_l(0) \ne 0$, respectively, integrating on $[0,T]$, then integrating twice the 1st resulting expression terms, one obtains same inequalities like ((35)-(42)) with setting $v_{ln} = v_l$ and the RHS of each equality will be,

$$\int_0^T ((f_l, v_l)_{L^2(\Omega)} + (u_{lk}, v_l)_{L^2(\partial\Omega)})\varphi_l(t)dt, \quad \forall\, l = 1,2,3,4, \text{ and } \forall\, k. \tag{69}$$

To passage the limit as $k \to \infty$, in the above four mentioned equality, the same technique which employed in the proof of Theorem 3.1, will also be employ her and the convergence obtain in the both sides of each equality (to avoid any repetition in the steps of proof)except the new expression in (69), and since $u_{lk} \xrightarrow[L^2(\Sigma)]{} u_l$, then it is convergent to

$$\int_0^T ((f_l, v_l)_{L^2(\Omega)} + (u_{lk}, v_l)_{L^2(\partial\Omega)})\varphi_l(t)dt, \quad \forall\, l = 1,2,3,4, \ \forall\, k.$$

This mean that the QSVS satisfies the WFO ((8)-(11)) also, by the same steps that utilized in the proof of *Case* 2 and *Case* 3 in Theorem 3.1, can also be utilized here to get the ICs is held. and thus the limit point $\vec{y} = (y_1, y_2, y_3, y_4)$ is a solution of the QSE.

Lastly, since $G_0(\vec{u})$ is WLSC and $\vec{u}_k \xrightarrow[L^2(\Omega)]{} \vec{u}$ then from Lemma 4.1,

$$G_0(\vec{u}) \le \lim_{k\to\infty} \inf_{\vec{u}_k \in \vec{W}} G_0(\vec{u}_k) = \lim_{n\to\infty} G_0(\vec{u}_k) = \inf_{\vec{u}_k \in \vec{W}} G_0(\vec{\overline{u}}) \Rightarrow G_0(\vec{u}) = \min_{\vec{u}_k \in \vec{W}} G_0(\vec{\overline{u}}).$$

Therefore $\vec{u}$ is a CQBOCV.

The Adjiont Equations and the Directional Derivative: The QAEs associated with the QSEs are formulate and the DD of the OF in (7) is derived in the following:

Theorem 4.2. *The QAEs of the QSEs* ((1)-(5)) *are formulated as*

$$z_{1tt}-\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(a_{1ij}\frac{\partial z_1}{\partial x_j}\right)+a_1z_1+b_2z_2-b_3z_3+b_4z_4=y_1-y_{1d}, \tag{70}$$

$$z_{2tt}-\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(a_{2ij}\frac{\partial z_2}{\partial x_j}\right)+a_2z_2-b_2z_1+b_5z_3-b_6z_4=y_2-y_{2d}, \tag{71}$$

$$z_{3tt}-\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(a_{3ij}\frac{\partial z_3}{\partial x_j}\right)+a_3z_3+b_3z_1-b_5z_2-b_7z_4=y_3-y_{3d}, \tag{72}$$

$$z_{4tt}-\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(a_{4ij}\frac{\partial z_4}{\partial x_j}\right)+a_4z_4-b_4z_1+b_6z_2+b_7z_3=y_4-y_{4d}, \tag{73}$$

$$z_l(x,t)=0,\ z_{lt}(x,t)=0,\quad l=1,2,3,4\ on\ \Omega, \tag{74}$$

$$\partial n_l z_l=0,\quad l=1,2,3,4\ \ on\ \ \Sigma. \tag{75}$$

Then for $\overrightarrow{u}\in\overrightarrow{W}$, the DD of G_0 is

$$DG(\overrightarrow{u},\overrightarrow{\overline{u}}-\overrightarrow{u})=\lim_{\sigma\to 0}\frac{G(\overrightarrow{u}+\sigma\overrightarrow{\delta u})-G(\overrightarrow{u})}{\sigma}=\int_{\partial Q}H_{\overrightarrow{u}}(x,t,\overrightarrow{y},\overrightarrow{u},\overrightarrow{z})\cdot(\overrightarrow{\overline{u}}-\overrightarrow{u})dq.$$

Proof. The WFO of the QAEs $\forall\ v_1,v_2,v_3,v_4\in V$ (a.e. on I) is

$$\langle z_{1tt},v_1\rangle+\overline{r}_1(t,z_1,v_1)+(b_2z_2,v_1)-(b_3z_3,v_1)+(b_4z_4,v_1)=(y_1-y_{1d},v_1), \tag{76}$$

$$\langle z_{2tt},v_2\rangle-\overline{r}_2(t,z_2,v_2)-(b_2z_1,v_2)+(b_5z_3,v_2)-(b_6z_4,v_2)=(y_2-y_{2d},v_2), \tag{77}$$

$$\langle z_{3tt},v_3\rangle-\overline{r}_3(t,z_3,v_3)+(b_3z_1,v_3)-(b_5z_2,v_3)-(b_7z_4,v_3)=(y_3-y_{3d},v_3), \tag{78}$$

$$\langle z_{4tt},v_4\rangle-\overline{r}_4(t,z_4,z_4)-(b_4z_1,v_4)+(b_6z_2,v_4)+(b_7z_3,v_4)=(y_4-y_{4d},v_4), \tag{79}$$

$$(z_l(T),v_4)=(z_{4t}(T),v_4)=0,\quad l=1,2,3,4, \tag{80}$$

where $\overline{r}_l(t,z_l,v_l)=r_l(t,z_l,v_l)+(a_lz_l,v_l)_{L^2(\Omega)}$, $\forall\ l=1,2,3,4$.
Replacing $v_l=\delta y_{l\sigma}$, $\forall\ l=1,2,3,4$ in ((76)-(79)) respectively and integrating on $[0,T]$, to acquire

$$\begin{aligned}&\int_0^T(z_{1tt},\delta y_{1\sigma})dt+\int_0^T(r_1(t,z_1,\delta y_{1\sigma})+(a_1z_1,\delta y_{1\sigma})_{L^2(\Omega)}+(b_2z_2,\delta y_{1\sigma})_{L^2(\Omega)}\\&-(b_3z_3,\delta y_{1\sigma})_{L^2(\Omega)}+(b_4z_4,\delta y_{1\sigma})_{L^2(\Omega)})dt=\int_0^T((y_1-y_{1d},\delta y_{1\sigma})_{L^2(\Omega)})dt,\end{aligned} \tag{81}$$

$$\begin{aligned}&\int_0^T(z_{2tt},\delta y_{2\sigma})dt+\int_0^T(r_2(t,z_2,\delta y_{2\sigma})+(a_2z_2,\delta y_{2\sigma})_{L^2(\Omega)}-(b_2z_1,\delta y_{2\sigma})_{L^2(\Omega)}\\&+(b_5z_3,\delta y_{2\sigma})_{L^2(\Omega)}-(b_6z_4,\delta y_{2\sigma})_{L^2(\Omega)})dt=\int_0^T((y_2-y_{2d},\delta y_{2\sigma})_{L^2(\Omega)})dt,\end{aligned} \tag{82}$$

$$\begin{aligned}&\int_0^T(z_{3tt},\delta y_{3\sigma})dt+\int_0^T(r_3(t,z_3,\delta y_3)+(a_3z_3,\delta y_{3\sigma})_{L^2(\Omega)}+(b_3z_1,\delta y_{3\sigma})_{L^2(\Omega)}\\&-(b_5z_2,\delta y_{3\sigma})_{L^2(\Omega)}-(b_7z_4,\delta y_{3\sigma})_{L^2(\Omega)})dt=\int_0^T((y_3-y_{3d},\delta y_{3\sigma})_{L^2(\Omega)})dt,\end{aligned} \tag{83}$$

$$\int_0^T (z_{4tt},\delta y_{4\sigma})dt+\int_0^T (r_4(t,z_4,\delta y_4)+(a_4z_4,\delta y_{4\sigma})_{L^2(\Omega)}-(b_4z_1,\delta y_{4\sigma})_{L^2(\Omega)}$$
$$+(b_6z_2,\delta y_{4\sigma})_{L^2(\Omega)}+(b_7z_3,\delta y_{4\sigma})_{L^2(\Omega)})dt=\int_0^T ((y_4-y_{4d},\delta y_{4\sigma})_{L^2(\Omega)})dt. \tag{84}$$

Now, let $\overrightarrow{\overline{u}},\vec{u}\in(L^2(Q))^4$, $\delta\vec{u}=\overrightarrow{\overline{u}}-\vec{u}$, $\vec{u}_\sigma=\vec{u}+\sigma\delta\vec{u}\in(L^2(Q))^4$ with for $\sigma>0$ then by Theorem 3.1, $\vec{y}=\vec{y}_{\vec{u}}$, $\vec{y}_\sigma=\vec{y}_{\vec{u}_\sigma}$ are QSVSs conforming to the CBCQV $\vec{u}$, $\vec{u}_\sigma$. Setting $\delta\vec{y}_\sigma=\vec{y}_\sigma-\vec{y}$, then replacing $v_l=z_l$, $\forall\ l=1,2,3,4$ in ((59)-(62)) respectively, integrating on $[0,T]$, then integrating by parts twice the 1st expression in the LHS of each equality, they yield

$$\int_0^T (\delta y_{1\sigma},z_{1tt})dt+\int_0^T (r_1(t,\delta y_{1\sigma},z_1)+(a_2\delta y_{1\sigma},z_1)_{L^2(\Omega)}-(b_2\delta y_{2\sigma},z_1)_{L^2(\Omega)}$$
$$+(b_3\delta y_{3\sigma},z_1)_{L^2(\Omega)}-(b_4\delta y_{4\sigma},z_1)_{L^2(\Omega)})dt=\int_0^T (\sigma\delta u_{1\sigma},z_1)_{L^2(\partial\Omega)}dt, \tag{85}$$
$$\int_0^T (\delta y_{2\sigma},z_{2tt})dt+\int_0^T (r_2(t,\delta y_{2\sigma},z_2)+(a_2\delta y_{2\sigma},z_2)_{L^2(\Omega)}+(b_2\delta y_{1\sigma},z_2)_{L^2(\Omega)}$$
$$-(b_5\delta y_{3\sigma},z_2)_{L^2(\Omega)}+(b_6\delta y_{4\sigma},z_2)_{L^2(\Omega)})dt=\int_0^T (\sigma\delta u_{2\sigma},z_2)_{L^2(\partial\Omega)}dt, \tag{86}$$
$$\int_0^T (\delta y_{3\sigma},z_{3tt})dt+\int_0^T (r_3(t,\delta y_{3\sigma},z_3)+(a_3\delta y_{3\sigma},z_3)_{L^2(\Omega)}-(b_3\delta y_{1\sigma},z_3)_{L^2(\Omega)}$$
$$+(b_5\delta y_{2\sigma},z_3)_{L^2(\Omega)}+(b_7\delta y_{4\sigma},z_3)_{L^2(\Omega)})dt=\int_0^T (\sigma\delta u_{3\sigma},z_3)_{L^2(\partial\Omega)}dt, \tag{87}$$
$$\int_0^T (\delta y_{4\sigma},z_{4tt})dt+\int_0^T (r_4(t,\delta y_{4\sigma},z_4)+(a_4\delta y_{4\sigma},z_4)_{L^2(\Omega)}+(b_4\delta y_{1\sigma},z_4)_{L^2(\Omega)}$$
$$-(b_6\delta y_{2\sigma},z_4)-(b_7\delta y_{3\sigma},z_4)_{L^2(\Omega)})dt=\int_0^T (\sigma\delta u_{4\sigma},z_4)_{L^2(\partial\Omega)}dt. \tag{88}$$

Subtract ((85)-(88)) from ((81)-(84)), then gathering all the outcome equality, yield to

$$\sigma\sum_{l=1}^{4}\int_0^T ((\delta u_l,z_l))_{L^2(\partial\Omega)}dt=\sum_{l=1}^{4}\int_0^T (y_l-y_{ld},\delta y_{l\sigma})_{L^2(\Omega)}dt. \tag{89}$$

Beside this, one has

$$G_0(\vec{u}_\sigma)-G_0(\vec{u})=\sum_{l=1}^{4}\left[\int_0^T (y_l-y_{ld},\delta y_{l\sigma})_{L^2(\Omega)}dt+\sigma\gamma\int_0^T ((\delta u_l,u_l))_{L^2(\partial\Omega)}dt\right]+O_1(\Sigma), \tag{90}$$

where $O_1(\Sigma)=\frac{1}{2}\|\overrightarrow{\delta y}_\sigma\|^2_{L^2(Q)}+\frac{\gamma}{2}\sigma^2\|\overrightarrow{\delta u}\|^2_{L^2(Q)})\to 0$, as $\sigma\to 0$.

Applying (89) in the RHS of (90), it yields

$$G_0(\vec{u}_\sigma)-G_0(\vec{u})=\sigma\int_{\partial Q}((\overrightarrow{\delta u},\vec{z})+(\gamma\vec{u},\overrightarrow{\delta u}))dq+O_1(\sigma),\ \text{with } O_1(\sigma)\to 0 \text{ as } \sigma\to 0.$$

The DD of G_0 is obtained after dividing both sides by σ and taking $\sigma\to 0$,

$$DG_0(\vec{u},\overrightarrow{\overline{u}}-\vec{u})=\int_{\partial Q}(\vec{z}+\gamma\vec{u})\cdot\delta\vec{u}\,dq,$$
$$(\vec{z}+\gamma\vec{u})=(z_1+\gamma u_1,z_2+\gamma u_2,z_3+\gamma u_3,z_4+\gamma u_4)^T.$$

□

Theorem 4.3. *The CBOCQV of the above problem is* $\vec{y}=\vec{y}_{\vec{u}}$ *and* $\vec{z}=\vec{z}_{\vec{u}}$.

Proof. If $\vec{u}$ is a CBOCQV of the QBOCP, then

$$G_0(\vec{\bar{u}}) = \min_{\vec{u} \in \vec{W}} G_0(\vec{u}), \quad \forall \ \vec{u} \in (L^2(Q))^4,$$

i.e.,

$$DG_0(\vec{u}, \vec{\bar{u}} - \vec{u}) = 0 \implies \vec{z} + \gamma \vec{\bar{u}} = 0, \quad \delta \vec{u} = \vec{w} - \vec{\bar{u}}.$$

The NCO is $(\vec{z} + \gamma \vec{\bar{u}}, \delta \vec{u}) \geq 0$ or $(\vec{z} + \gamma \vec{\bar{u}}, \vec{w}) \geq (\vec{z} + \gamma \vec{\bar{u}}, \vec{u})$, $\forall \ \vec{w} \in (L^2(Q))^4$. □

5. Conclusions

From the study of the QBOCP controlling by QLHS. The QSVS of the WFO for the QLHS was proved existence a unique through employing the MG under suitable hypos when the CBCQV is known. The continuity of the LC between the QSVS and the conforming QBOCP is proved. The existence of CBCOQV for the problem was proved under suitable hypos. The QALS associated with the QLHS was formulated and studied. The DD for the OF was obtained. The theorem of the necessary conditions for optimality was studied.

List of Abbreviations

QBOCP	Quaternary Boundary Optimal Control Problem
WFO	Weak Formulation
QLHS	Quaternary Linear Hyperbolic System
QSVS	Quaternary State Vector Solution
QALS	Quaternary Adjoint Linear System
MG	Method of Galerkin
CBCQV	Continuous Boundary Control Quaternary Vector
DDV	Directional Derivative
NCO	Necessity Conditions for Optimality
OCPs	Optimal Control Problems
WLSC	Weakly Lower Semi Continuous
OF	Objective Functional
CBOCQV	Continuous Boundary Optimal Control Quaternary Vector
HYPOS	Hypotheses

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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