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Research Article

On Adjacency Spectrum of Non-Zero Divisor Graph of $\mathbb{Z}_n$

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Abstract. In this article, utilizing the concept of the H-generalized join graph, we derive the adjacency spectrum of $\Phi(\mathbb{Z}_{p^k})$, where p be a prime and $k \geq 1$ is a positive integer.

Keywords. Non-zero divisor graph, Generalized join graph, Adjacency spectrum

Mathematics Subject Classification (2020). 05C50, 05C12, 15A18

1. Introduction

Let G be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. The graphs considered in this article are simple and finite. The *order* and *size* of graph G, is $|V(G)|$ and $|E(G)|$, respectively. If there exists an edge between vertices x and y, we say that x is *adjacent* to y, denoted by $x \sim y$. The neighborhood $N_G(v)$ of a vertex v is the set of vertices adjacent to v. The *degree* of v; denoted by $d_G(v)$, is the cardinality of $N(v)$. A graph G is called *regular* when every vertex has the same degree. If $d_G(v) = r$, $\forall\ v \in V(G)$, then G is called r-regular graph. K_n denotes the complete graph of n vertices. The complement of K_n is a null graph and it is denoted by $\bar{K}_n$. Clearly, K_n and $\bar{K}_n$ are r-regular graphs with $r = n-1$ and $r = 0$, respectively. Let $G = (V,E)$ be any graph, and let $S \subset V$ be any subset of vertices of G. Then the induced subgraph $G(S)$ is the graph whose vertex set is S and whose edge set consists of all of the edges in E that have both endpoints in S. The adjacency matrix of a graph G is the $n \times n$ real symmetric matrix $A(G) = [a_{ij}]$, where $a_{ij} = 1$ if the vertices v_i and v_j are adjacent and equal to 0, otherwise. The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of $A(G)$ are the eigenvalues of G. Furthermore, it should be noted that K_n has eigenvalues $n-1$ and -1 with multiplicity 1 and $n-1$, respectively and 0 is the only eigenvalue of $\bar{K}_n$ with multiplicity n. The *spectrum* of a graph G is the set of numbers which are eigenvalues of $A(G)$, together with their multiplicity. If the distinct eigenvalues of $A(G)$ are $\lambda_1 > \lambda_2 > \ldots > \lambda_d$, and their multiplicities are $m_1, m_2, \ldots, m_d$, then we shall write

$$\sigma(G) = \begin{Bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_d \\ m_1 & m_2 & \cdots & m_d \end{Bmatrix}.$$

We refer to Biggs [3] and Harary [6] for any undefined terminologies or notations.

Let R be a ring. A nonzero element $a \in R$ is called *unit* of R if $ab = 1$ for some $b \in R$. An element $a \in R$ is called a *zero-divisor* of R if there exists a non-zero element $b \in R$ such that $ab = 0$. The zero-divisor graphs of commutative rings were first introduced by Beck [2]. The author mainly focused on the coloring of commutative rings in this article. The definition of zero-divisor graph was later modified by Anderson and Livingston in [1]. For a commutative ring R, the set $Z(R)$ denote the set of zero-divisors of R and let $Z^*(R) = Z(R) \setminus \{0\}$ be the set of nonzero zero-divisors of R. The zero-divisor graph of R, is denoted by $\Gamma(R)$, and is a simple undirected graph whose vertex set is $Z^*(R)$ two distinct vertices v and u are adjacent if and only if $vu = 0$. The non-zero divisor graph $\Phi(R)$ of ring R is defined by Kadem *et al*. [7], with vertex set $V(R) = R \setminus \{0, 1, -1\}$, where distinct vertices $x, y \in V(R)$ are adjacent if and only if either $xy \neq 0$ or $yx \neq 0$. Any nonzero element a of $\mathbb{Z}_n$ is either a unit or a zero divisor. Therefore, the number of vertices in $\Phi(\mathbb{Z}_n)$ is $n-3$.

Motivated by work done by Chattopadhyay *et al*. [5], Magi *et al*. [8,9], Pirzada *et al*. [10], and Young [11], this article derives the adjacency spectrum of the non-zero divisor graph for rings $\mathbb{Z}_{p^{2m}}$ and $\mathbb{Z}_{p^{2m+1}}$, where p be a prime and $m \geq 1$ is a positive integer. In Section 2, we first identify and analyze the structural properties of $\Phi(\mathbb{Z}_n)$. In Section 3, we obtain the adjacency spectrum of $\Phi(\mathbb{Z}_{p^{2m}})$ and $\Phi(\mathbb{Z}_{p^{2m+1}})$.

2. Structure of $\Phi(\mathbb{Z}_n)$

We begin with the definition of the non-zero divisor graph of rings.

Definition 2.1 ([7]). Let R be a ring. The *non-zero divisor graph* of R is denoted by $\Phi(R)$, and is a graph with vertices $V(R) = R \setminus \{0, 1, -1\}$, where distinct vertices $x, y \in V(R)$ are adjacent if and only if either $xy \neq 0$ or $yx \neq 0$.

Remember that when two graphs, G_1 and G_2, have disjoint vertex sets, the graph that results from union of G_1 and G_2 is called join $G_1 \vee G_2$, and it is created by adding new edges from every vertex in G_1 to every vertex in G_2. The following is a generalization of the definition of join graph.

Definition 2.2 ([4]). Let H be a graph with $V(H) = \{v_1, v_2, \ldots, v_t\}$ and let $G_1, G_2, \ldots, G_t$ be pairwise disjoint graphs. The H-generalized join graph $H[G_1, G_2, \ldots, G_t]$ of family of graphs $G_1, G_2, \ldots, G_t$ is the graph formed by replacing each vertex v_i of H by the graph G_i and then joining each vertex of G_i to every vertex of G_j whenever v_i is adjacent to v_j in H.

The greatest common divisor of two integers m and n, is denoted by (m, n). An integer d is called a divisor of n if d divides n, denoted by $d \mid n$, for $1 \leq d \leq n$. If d not divides n, then we write $d \nmid n$. Let $d_0 = 1, d_1, d_2, \dots, d_t$ be the distinct divisors of n, other than it self. For $0 \le i \le t$, consider the following:

$$S(d_i) = \{x \in \mathbb{Z}_n : (x,n) = d_i\}.$$

Clearly, the sets $S(d_0), S(d_1), S(d_2), \dots, S(d_t)$ are pairwise disjoint and the cardinality of $S(d_i)$ is given by the following lemma.

Lemma 2.1 ([11])**.** *For $0 \le i \le t$,*

$$|S(d_i)| = \phi\left(\frac{n}{d_i}\right).$$

Now, if we consider $A_{d_0} = S(d_0) \setminus \{1,-1\}$ and $A_{d_i} = S(d_i)$ for $1 \le i \le t$, then the sets $A_{d_0}, A_{d_1}, A_{d_2}, \dots, A_{d_t}$ are pairwise disjoint and partition the vertex set of $\Phi(\mathbb{Z}_n)$ as

$$V(\Phi(\mathbb{Z}_n)) = A_{d_0} \cup A_{d_1} \cup A_{d_2} \cup \dots \cup A_{d_t}$$

and the cardinality of A_{d_i} is given by the following:

Corollary 2.1. *For $0 \le i \le t$, the cardinality of A_{d_i} is as follows*

$$|A_{d_i}| = \begin{cases} \phi(n) - 2, & \text{if } i = 0, \\ \phi\left(\frac{n}{d_i}\right), & \text{if } 1 \le i \le t. \end{cases}$$

Proof. Follows from Lemma 2.1. □

Note that any element x of A_{d_i} can be written as $x = m_x d_i$ for some integer m_x with $0 < m_x < \frac{n}{d_i}$ and $\left(m_x, \frac{n}{d_i}\right) = 1$. The following lemma describes adjacency of vertices in $\Phi(\mathbb{Z}_n)$.

Lemma 2.2. *For $i, j \in \{0,1,2,\dots,t\}$, a vertex of A_{d_i} is adjacent to a vertex of A_{d_j} in $\Phi(\mathbb{Z}_n)$ if and only if $n \nmid d_i d_j$.*

Proof. Let $x \in A_{d_i}$ and $y \in A_{d_j}$. Then $x = m_x d_i$ and $y = m_y d_j$ for some integers m_x, m_y with $0 < m_x < \frac{n}{d_i}$, $0 < m_y < \frac{n}{d_j}$ and $\left(m_x, \frac{n}{d_i}\right) = 1 = \left(m_y, \frac{n}{d_j}\right)$. The vertices x and y are adjacent in $\Phi(\mathbb{Z}_n)$ if and only if n not divides xy, if and only if n not divides $m_x m_y d_i d_j$. Since $\left(m_x, \frac{n}{d_i}\right) = 1 = \left(m_y, \frac{n}{d_j}\right)$, we have

$$\begin{aligned} x \sim y &\iff n \nmid m_x m_y d_i d_j \\ &\iff \frac{n}{d_i} \nmid m_x m_y d_j \\ &\iff \frac{n}{d_i} \nmid m_y d_j \\ &\iff n \nmid m_y d_i d_j \\ &\iff \frac{n}{d_j} \nmid m_y d_i \\ &\iff \frac{n}{d_j} \nmid d_i \\ &\iff n \nmid d_i d_j \end{aligned}$$

□

We discuss the nature of the induced subgraph $\Phi(A_{d_i})$ of $\Phi(\mathbb{Z}_n)$ in the next corollary.

Corollary 2.2. *The following holds:*

(1) *For $i \in \{0,1,2,\dots,t\}$, the induced subgraph $\Phi(A_{d_i})$ of $\Phi(\mathbb{Z}_n)$ on the vertex set A_{d_i} either the complete graph $K_{|A_{d_i}|}$ or its complement graph $\bar{K}_{|A_{d_i}|}$. Also, $\Phi(A_{d_i})$ is $K_{|A_{d_i}|}$ if and only if $n \nmid d_i^2$.*

(2) *For $i,j \in \{1,2,\dots,k\}$ with $i \neq j$, a vertex of A_{d_i} is adjacent to either all or none of the vertices of A_{d_j} in $\Phi(\mathbb{Z}_n)$.*

Proof. (1) The subgraph $\Phi(A_{d_i})$ has $|A_{d_i}|$ vertices. Taking $i = j$ in Lemma 2.2, it follows that two distinct vertices of $\Phi(A_{d_i})$ are adjacent if and only if $n \nmid d_i^2$. This implies that $\Phi(A_{d_i})$ is the complete graph $K_{|A_{d_i}|}$ or its complement graph $\bar{K}_{|A_{d_i}|}$.

(2) Follows from Lemma 2.2. □

Let $d_1, d_2, \dots, d_t$ are the proper divisor of n. Then, the graph Υ_n is defined by Chattopadhyay *et al.* [5], and is the simple graph with vertex set $\{d_1, d_2, \dots, d_t\}$, in which two distinct vertices are connected by an edge if and only if $n \mid d_i d_j$.

Definition 2.3. Let $n \in \mathbb{N}$. The graph Ω_n is defined as a simple graph whose vertex set consists of the divisors of n excluding n itself, i.e., the vertices are $\{d_0, d_1, d_2, \dots, d_t\}$, where $1 \le d_i < n$ and each d_i divides n. Two distinct vertices d_i and d_j are adjacent in Ω_n if and only if $n \nmid d_i d_j$.

The number k of vertices of Ω_n can be calculated as follows: Let $n = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$ be the prime power factorization of n, where $r, n_1, n_2, \dots, n_r$ are positive integers and $p_1, p_2, \dots, p_r$ are distinct prime numbers. Every divisor of n is of the form $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ for some integers $\alpha_1, \alpha_2, \dots, \alpha_r$, where $0 \le \alpha_i \le n_i$ for each $i \in \{1,2,\dots,r\}$. Thus, the total number of divisors of n is $\prod_{i=1}^{r} (n_i + 1)$. Since we consider all the divisor of n excluding n itself, the number of vertices of Ω_n is given by:

$$k = |V(\Omega_n)| = \prod_{i=1}^{r} (n_i + 1) - 1. \tag{2.1}$$

The following lemma states that $\Phi(\mathbb{Z}_n)$ is a generalized join of certain complete graphs and null graphs.

Lemma 2.3. *Let $\Phi(A_{d_i})$ be the induced subgraph of $\Phi(\mathbb{Z}_n)$ on the vertex set A_{d_i}, for $0 \le i \le k-1$. Then $\Phi(\mathbb{Z}_n) = \Omega_n[\Phi(A_{d_0}), \Phi(A_{d_1}), \dots, \Phi(A_{d_{k-1}})]$.*

Proof. For $0 \le i \le k-1$, replace vertex d_i of Ω_n with $\Phi(A_{d_i})$. The result can be obtained by applying Lemma 2.2. □

3. Adjacency Spectrum of $\Phi(\mathbb{Z}_n)$

Usually it is difficult to obtain the adjacency spectrum of graphs in general. So, we attempt to get the adjacency spectrum of particular class of graphs. Through this section, let p be a prime number, and let $m \ge 1$ be a positive integer. First, we find the structure of $\Phi(\mathbb{Z}_{p^{2m}})$.

Theorem 3.1. *Let $\Phi(\mathbb{Z}_n)$ be the non-zero divisor graph of ring of integers modulo n, where $n = p^{2m}$. Then*

$$\Phi(\mathbb{Z}_{p^{2m}}) = \Omega_{p^{2m}}[K_{\phi(p^{2m})-2}, K_{\phi(p^{2m-1})}, \dots, K_{\phi(p^{m+1})}, \bar{K}_{\phi(p^m)}, \dots, \bar{K}_{\phi(p^2)}, \bar{K}_{\phi(p)}].$$

Proof. Let $n = p^{2m}$. Then by Definition 2.3 and equation (2.1), we have

$$V(\Omega_{p^{2m}}) = \{1 = p^0, p, p^2, \dots, p^{m-1}, p^m, p^{m+1}, \dots, p^{2m-2}, p^{2m-1}\} \text{ and } |V(\Omega_{p^{2m}})| = 2m.$$

Also,

$$\begin{aligned}
&1 = p^0 \sim 1, p, \dots, p^{m-1}, p^m, p^{m+1}, \dots, p^{2m-1},\\
&p^1 \sim 1, p, \dots, p^{m-1}, p^m, p^{m+1}, \dots, p^{2m-2},\\
&p^2 \sim 1, p, \dots, p^{m-1}, p^m, p^{m+1}, \dots, p^{2m-3},\\
&\quad\vdots\\
&p^{m-1} \sim 1, p, \dots, p^{m-1}, p^m,\\
&p^m \sim 1, p, \dots, p^{m-1},\\
&p^{m+1} \sim 1, p, \dots, p^{m-2},\\
&\quad\vdots\\
&p^{2m-1} \sim 1
\end{aligned}$$

We observed that, $p^i \sim p^j$ if and only if $i + j < 2m$, for $i = 0, 1, \dots 2m-1$.
Now, by Corollary 2.1,

$$|A_{p^i}| = \begin{cases} \phi(p^{2m}) - 2, & \text{if } i = 0,\\ \phi(p^{2m-i}), & \text{if } 1 \le i \le 2m-1. \end{cases}$$

Also, by Corollary 2.2, we have

$$G_i = \begin{cases} \Phi(A_{p^i}) = K_{\phi(p^{2m})-2}, & \text{if } i = 0,\\ \Phi(A_{p^i}) = K_{\phi(p^{2m-i})}, & \text{if } 1 \le i \le m-1,\\ \Phi(A_{p^i}) = \bar{K}_{\phi(p^{2m-i})}, & \text{if } m \le i \le 2m-1. \end{cases}$$

By using Lemma 2.3, the generalized join of the non-zero divisor graph $\Phi(\mathbb{Z}_{p^{2m}})$ is given by

$$\Phi(\mathbb{Z}_{p^{2m}}) = \Omega_{p^{2m}}[K_{\phi(p^{2m})-2}, K_{\phi(p^{2m-1})}, \dots, K_{\phi(p^{m+1})}, \bar{K}_{\phi(p^m)}, \dots, \bar{K}_{\phi(p^2)}, \bar{K}_{\phi(p)}]. \qquad \square$$

Cardoso *et al.* [4, Theorem 5] demonstrated the following theorem, which states that the adjacency spectrum of a generalized join graph $H[G_1, G_2, \dots, G_k]$ is represented in terms of the spectrum of the $n \times n$ matrix $C_A(G)$ and the adjacency spectrum of the r_i-regular graphs G_i of order n_i.

Theorem 3.2 ([4])**.** *Let H be a graph with $V(H) = \{1, 2, \dots, k\}$ and let G_i, $1 \le i \le k$, be k pairwise disjoint r_i-regular graphs of order n_i, respectively. Then, the adjacency spectrum of $G = H[G_1, G_2, \dots, G_k]$ is given by*

$$\sigma(G) = \left(\bigcup_{i=1}^{k} (\sigma(G_i) \setminus \{r_i\})\right) \cup \sigma(C_A(H)),$$

where

$$C_A(H) = [c_{ij}]_{k\times k} = \begin{cases} r_i, & \textit{if } i = j, \\ \sqrt{n_i n_j}, & \textit{if } ij \in E(H), \\ 0, & \textit{otherwise}. \end{cases}$$

The following theorem describes the adjacency spectrum of $\Phi(\mathbb{Z}_{p^{2m}})$ in detail, combining simple eigenvalues with specified multiplicities and a more complex matrix-based representation for the remaining eigenvalues.

Theorem 3.3. *The adjacency spectrum of $\Phi(\mathbb{Z}_{p^{2m}})$ consists of the eigenvalue $0, -1$ with multiplicity $p^m - m - 1$ and $p^{2m} - p^m - m - 2$ respectively, and the remaining adjacency eigenvalues of $\Phi(\mathbb{Z}_{p^{2m}})$ are the eigenvalues of matrix $C_A(\Omega_{p^{2m}})$.*

Proof. By using Theorem 3.1 and Theorem 3.2, we observe that the adjacency eigenvalues of $\Phi(\mathbb{Z}_{p^{2m}})$ is 0 with multiplicity

$$\sum_{i=m}^{2m-1} (\phi(p^{2m-i}) - 1) = p^m - m - 1$$

and -1 with multiplicity

$$(\phi(p^{2m}) - 2) - 1 + \sum_{i=1}^{m-1} (\phi(p^{2m-i}) - 1) = p^{2m} - p^m - m - 2.$$

Therefore, we have

$$\sigma(\Phi(\mathbb{Z}_{p^{2m}})) = \begin{Bmatrix} 0 & -1 \\ p^m - m - 1 & p^{2m} - p^m - m - 2 \end{Bmatrix} \cup \sigma(C_A(\Omega_{p^{2m}})),$$

where the matrix $C_A(\Omega_{p^{2m}})$ is of the form

$$\left(\begin{array}{ccccc|cccc} r_0 & \beta_{0,1} & \cdots & \beta_{0,m-2} & \beta_{0,m-1} & \beta_{0,m} & \beta_{0,m+1} & \cdots & \beta_{0,2m-1} \\ \beta_{1,0} & r_1 & \cdots & \beta_{1,m-2} & \beta_{1,m-1} & \beta_{1,m} & \beta_{1,m+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{m-2,0} & \beta_{m-2,1} & \cdots & r_{m-2} & \beta_{m-2,m-1} & \beta_{m-2,m} & \beta_{m-2,m+1} & \cdots & 0 \\ \beta_{m-1,0} & \beta_{m-1,1} & \cdots & \beta_{m-1,m-2} & r_{m-1} & \beta_{m-1,m} & 0 & \cdots & 0 \\ \hline \beta_{m,0} & \beta_{m,1} & \cdots & \beta_{m,m-2} & \beta_{m,m-1} & 0 & 0 & \cdots & 0 \\ \beta_{m+1,0} & \beta_{m+1,1} & \cdots & \beta_{m+1,m-2} & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{2m-2,0} & \beta_{2m-2,1} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \beta_{2m-1,0} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{array}\right),$$

where

$$\beta_{i,j} = \beta_{j,i} = \begin{cases} \sqrt{(\phi(p^{2m}) - 2)\phi(p^{2m-j})}, & \text{if } i = 0 \text{ and } 1 \le j \le 2m-1, \\ \sqrt{\phi(p^{2m-i})\phi(p^{2m-j})}, & \text{if } 1 \le i, j \le 2m-1 \end{cases}$$

and

$$r_i = \begin{cases} \phi(p^{2m}) - 3, & \text{if } i = 0, \\ \phi(p^{2m-i}) - 1, & \text{if } 1 \le i \le m-1. \end{cases}$$

□

In Theorem 3.3, taking $m = 1$, we have the following consequence.

Corollary 3.1. *The adjacency spectrum of $\Phi(\mathbb{Z}_{p^2})$ is*

$$\sigma(\Phi(\mathbb{Z}_{p^2})) = \begin{Bmatrix} 0 & -1 \\ p-2 & p^2-p-3 \end{Bmatrix} \cup \textit{roots of } \lambda^2 - \lambda(p^2-p-3)-(p^3-2p^2-p+2).$$

Proof. By using Theorem 3.1 and Theorem 3.3, we have

$$\Phi(\mathbb{Z}_{p^2}) = \Omega_{p^2}[K_{\Phi(p^2)-2}, \bar{K}_{\phi(p)}] = \Omega_{p^2}[K_{p^2-p-2}, \bar{K}_{p-1}]$$

and

$$\sigma(\Phi(\mathbb{Z}_{p^2})) = \begin{Bmatrix} 0 & -1 \\ p-2 & p^2-p-3 \end{Bmatrix} \cup \sigma(C_A(\Omega_{p^2})),$$

where

$$C_A(\Omega_{p^2}) = \begin{pmatrix} p^2-p-3 & \sqrt{(p^2-p-2)(p-1)} \\ \sqrt{(p^2-p-2)(p-1)} & 0 \end{pmatrix}$$

Therefore, the other adjacency eigenvalues of $\Phi(\mathbb{Z}_{p^2})$ are the roots of characteristics polynomial

$$\lambda^2 - \lambda(p^2-p-3)-(p^3-2p^2-p+2). \qquad \square$$

If we set $m=4$ in Theorem 3.3, the resulting consequence provides the adjacency spectrum of $\Phi(\mathbb{Z}_{p^4})$.

Corollary 3.2. *The adjacency spectrum of $\Phi(\mathbb{Z}_{p^4})$ is*

$$\sigma(\Phi(\mathbb{Z}_{p^4})) = \begin{Bmatrix} 0 & -1 \\ p^2-3 & p^4-p^2-4 \end{Bmatrix} \cup \sigma(C_A(\Omega_{p^4})).$$

Proof. Followed by Theorem 3.1 and Theorem 3.3, we have

$$\Phi(\mathbb{Z}_{p^4}) = \Omega_{p^4}[K_{\Phi(p^4)-2}, K_{\phi(p^3)}, \bar{K}_{\phi(p^2)}\bar{K}_{\phi(p)}] = \Omega_{p^4}[K_{p^2-p-2}, K_{p^3-p^2}, \bar{K}_{p^2-p}\bar{K}_{p-1}]$$

and

$$\sigma(\Phi(\mathbb{Z}_{p^4})) = \begin{Bmatrix} 0 & -1 \\ p^2-3 & p^4-p^2-4 \end{Bmatrix} \cup \sigma(C_A(\Omega_{p^4})),$$

where $C_A(\Omega_{p^4})$ is the following matrix

$$\begin{pmatrix} p^4-p^3-3 & \sqrt{(p^4-p^3-2)(p^3-p^2)} & \sqrt{(p^4-p^3-2)(p^2-p)} & \sqrt{(p^4-p^2-2)(p-1)} \\ \sqrt{(p^4-p^3-2)(p^3-p^2)} & p^3-p^2-1 & \sqrt{(p^3-p^2)(p^2-p)} & 0 \\ \sqrt{(p^4-p^3-2)(p^2-p)} & \sqrt{(p^3-p^2)(p^2-p)} & 0 & 0 \\ \sqrt{(p^4-p^2-2)(p-1)} & 0 & 0 & 0 \end{pmatrix}. \qquad \square$$

Now, we will find the structure of $\Phi(\mathbb{Z}_{p^{2m+1}})$.

Theorem 3.4. *Let $\Phi(\mathbb{Z}_n)$ be the non-zero divisor graph of ring of integers modulo n, where $n = p^{2m+1}$. Then*

$$\Phi(\mathbb{Z}_{p^{2m+1}}) = \Omega_{p^{2m+1}}[K_{\phi(p^{2m+1})-2}, K_{\phi(p^{2m})}, \dots, K_{\phi(p^{m+1})}, \bar{K}_{\phi(p^m)}, \dots, \bar{K}_{\phi(p^2)}, \bar{K}_{\phi(p)}].$$

Proof. Let $n = p^{2m+1}$. Then by using Definition 2.3 and equation (2.1), the vertex set of $\Phi(\mathbb{Z}_{p^{2m+1}})$ as follows:

$$V(\Omega_{p^{2m+1}}) = \{1 = p^0, p, p^2, \dots, p^{m-1}, p^m, p^{m+1}, p^{m+2}, \dots, p^{2m-1}, p^{2m}\}$$

and the dimension of $\Phi(\mathbb{Z}_{p^{2m+1}})$ is

$$|V(\Omega_{p^{2m+1}})| = 2m+1.$$

Also,

$$\begin{aligned}
&1 = p^0 \sim 1, p, \dots, p^{m-1}, p^m, p^{m+1}, \dots, p^{2m},\\
&p^1 \sim 1, p, \dots, p^{m-1}, p^m, p^{m+1}, \dots, p^{2m-1},\\
&p^2 \sim 1, p, \dots, p^{m-1}, p^m, p^{m+1}, \dots, p^{2m-2},\\
&\quad \vdots\\
&p^{m-1} \sim 1, p, \dots, p^m, p^{m+1},\\
&p^m \sim 1, p, \dots, p^m,\\
&p^{m+1} \sim 1, p, \dots, p^{m-1},\\
&\quad \vdots\\
&p^{2m} \sim 1
\end{aligned}$$

we observed that, $p^i \sim p^j$ if and only if $i+j < 2m+1$, for $i = 0, 1, \dots 2m$.
Now, by Corollary 2.1,

$$|A_{p^i}| = \begin{cases} \phi(p^{2m+1}) - 2, & \text{if } i = 0,\\ \phi(p^{2m+1-i}), & \text{if } 1 \le i \le 2m. \end{cases}$$

Also, by Corollary 2.2, we have

$$G_i = \begin{cases} \Phi(A_{p^i}) = K_{\phi(p^{2m+1})-2}, & \text{if } i = 0,\\ \Phi(A_{p^i}) = K_{\phi(p^{2m+1-i})}, & \text{if } 1 \le i \le m,\\ \Phi(A_{p^i}) = \bar{K}_{\phi(p^{2m+1-i})}, & \text{if } m+1 \le i \le 2m. \end{cases}$$

By using Lemma 2.3, the generalized join of the non-zero divisor graph $\Phi(\mathbb{Z}_{p^{2m+1}})$ is given by

$$\Phi(\mathbb{Z}_{p^{2m+1}}) = \Omega_{p^{2m+1}}[K_{\phi(p^{2m+1})-2}, K_{\phi(p^{2m})}, \dots, K_{\phi(p^{m+1})}, \bar{K}_{\phi(p^m)}, \dots, \bar{K}_{\phi(p^2)}, \bar{K}_{\phi(p)}].$$ □

The graph $\Phi(\mathbb{Z}_{p^{2m+1}})$ is associated with the ring of integers modulo p^{2m+1}, and its structure is described in Theorem 3.4. The following theorem presents the adjacency spectrum of this graph.

Theorem 3.5. *The adjacency spectrum of $\Phi(\mathbb{Z}_{p^{2m+1}})$ consists of the eigenvalue $0, -1$ with multiplicity $p^m - m - 1$ and $p^{2m+1} - p^m - m - 3$ respectively, and the remaining adjacency eigenvalues of $\Phi(\mathbb{Z}_{p^{2m+1}})$ are the eigenvalues of matrix $C_A(\Omega_{p^{2m+1}})$.*

Proof. By using Theorem 3.2 and Theorem 3.4, the adjacency eigenvalues of $\Phi(\mathbb{Z}_n)$ are 0 and -1 with multiplicities as follows:

$$\sum_{i=m+1}^{2m} (\phi(p^{2m+1-i}) - 1) = p^m - m - 1$$

and

$$(\phi(p^{2m+1}) - 2) - 1 + \sum_{i=1}^{m} (\phi(p^{2m+1-i}) - 1) = p^{2m+1} - p^m - m - 3,$$

respectively. Therefore, we have

$$\sigma(\Phi(\mathbb{Z}_{p^{2m+1}})) = \begin{Bmatrix} 0 & -1 \\ p^m - m - 1 & p^{2m+1} - p^m - m - 3 \end{Bmatrix} \cup \sigma(C_A(\Omega_{p^{2m+1}})),$$

where the matrix $C_A(\Omega_{p^{2m+1}}) =$

$$\left(\begin{array}{ccccc|cccc}
r_0 & \beta_{0,1} & \cdots & \beta_{0,m-1} & \beta_{0,m} & \beta_{0,m+1} & \beta_{0,m+2} & \cdots & \beta_{0,2m} \\
\beta_{1,0} & r_1 & \cdots & \beta_{1,m-1} & \beta_{1,m} & \beta_{1,m+1} & \beta_{1,m+2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\beta_{m-1,0} & \beta_{m-1,1} & \cdots & r_{m-1} & \beta_{m-1,m} & \beta_{m-1,m+1} & 0 & \cdots & 0 \\
\beta_{m,0} & \beta_{m,1} & \cdots & \beta_{m,m-1} & r_m & 0 & 0 & \cdots & 0 \\
\hline
\beta_{m+1,0} & \beta_{m+1,1} & \cdots & \beta_{m+1,m-1} & 0 & 0 & 0 & \cdots & 0 \\
\beta_{m+2,0} & \beta_{m+2,1} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\beta_{2m-1,0} & \beta_{2m-1,1} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\beta_{2m,0} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0
\end{array}\right),$$

where

$$\beta_{i,j} = \beta_{j,i} = \begin{cases} \sqrt{(\phi(p^{2m+1}) - 2)\phi(p^{2m+1-j})}, & \text{if } i = 0 \text{ and } 1 \le j \le 2m, \\ \sqrt{\phi(p^{2m+1-i})\phi(p^{2m+1-j})}, & \text{if } 1 \le i,\ j \le 2m \end{cases}$$

and

$$r_i = \begin{cases} \phi(p^{2m+1}) - 3, & \text{if } i = 0, \\ \phi(p^{2m+1-i}) - 1, & \text{if } 1 \le i \le m. \end{cases}$$ □

If we take $m = 1$ in Theorem 3.5, then the following consequence gives the adjacency spectrum of $\Phi(\mathbb{Z}_{p^3})$.

Corollary 3.3. *The adjacency spectrum of $\Phi(\mathbb{Z}_{p^3})$ is*

$$\sigma(\Phi(\mathbb{Z}_{p^3})) = \begin{Bmatrix} 0 & -1 \\ p-2 & p^3 - p - 4 \end{Bmatrix} \cup C_A(\Omega_{p^3}).$$

Proof. The divisors of p^3 others than p^3 are $1 = p^0, p^1, p^2$. Therefore by Theorem 3.4, we have

$$\Phi(\mathbb{Z}_{p^3}) = \Omega_{p^3}[K_{\Phi(p^3)-2}, K_{\phi(p^2)}, \bar{K}_{\phi(p)}] = \Omega_{p^3}[K_{p^3-p^2-2}, K_{p^2-p}, \bar{K}_{p-1}]$$

By using Theorem 3.5,

$$\sigma(\Phi(\mathbb{Z}_{p^3})) = \begin{Bmatrix} 0 & -1 \\ p-2 & p^3 - p - 4 \end{Bmatrix} \cup \sigma(C_A(\Omega_{p^3}))$$

and other adjacency eigenvalues of $\Phi(\mathbb{Z}_{p^3})$ are the eigenvalues of the following matrix $C_A(\Omega_{p^3})$,

$$\begin{pmatrix} p^3 - p^2 - 3 & \sqrt{(p^3 - p^2 - 2)(p^2 - p)} & \sqrt{(p^3 - p^2 - 2)(p-1)} \\ \sqrt{(p^3 - p^2 - 2)(p^2 - p)} & p^2 - p - 1 & 0 \\ \sqrt{(p^3 - p^2 - 2)(p-1)} & 0 & 0 \end{pmatrix}.$$ □

4. Conclusion

In this study, we explored the adjacency spectrum of the non-zero divisor graph $\Phi(Z_n)$ associated with the ring of integers modulo n. Leveraging structural properties and generalized join graph techniques, we provided explicit formulations for the adjacency spectrum of $\Phi(Z_{p^{2m}})$ and $\Phi(Z_{p^{2m+1}})$, where p is a prime and $m \geq 1$. The results offer new insights into the spectral characteristics of these graphs, particularly regarding their eigenvalues and their multiplicities.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, *Journal of Algebra* **217**(2) (1999), 434 – 447, DOI: 10.1006/jabr.1998.7840.

[2] I. Beck, Coloring of commutative rings, *Journal of Algebra* **116**(1) (1988), 208 – 226, DOI: 10.1016/0021-8693(88)90202-5.

[3] N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, England, 205 pages (1974).

[4] D. M. Cardoso, M. A. A. de Freitas, E. A. Martins and M. Robbiano, Spectra of graphs obtained by a generalization of the join graph operation, *Discrete Mathematics* **313**(5) (2013), 733 – 741, DOI: 10.1016/j.disc.2012.10.016.

[5] S. Chattopadhyay, K. L. Patra and B. K. Sahoo, Laplacian eigenvalues of the zero-divisor graph of the ring $\mathbb{Z}_n$, *Linear Algebra and its Applications* **584** (2020), 267 – 286, DOI: 10.1016/j.laa.2019.08.015.

[6] F. Harary, *Graph Theory*, Addison-Wesley Publishing Company, Reading – Massachusetts, vi + 274 pages (1969).

[7] S. Kadem, A. Aubad and A. H. Majeed, The non-zero divisor graph a ring, *Italian Journal of Pure and Applied Mathematics* **43** (2020), 975 – 983, URL: https://ijpam.uniud.it/online_issue/202043/82%20AliAubad-SameerKadem-AbdulrahmanMajeed.pdf.

[8] P. M. Magi, S. M. Jose and A. Kishore, Adjacency matrix and eigenvalues of the zero divisor graph $\Gamma(Z)_n$, *Journal of Mathematical and Computational Science* **10**(4) (2020), 1285 – 1297, URL: https://scik.org/index.php/jmcs/article/view/4590.

[9] P. M. Magi, S. M. Jose and A. Kishore, Spectrum of the zero-divisor graph on the ring of integers modulo n, *Journal of Mathematical and Computational Science* **10**(5) (2020), 1643 – 1666, DOI: 10.28919/jmcs/4719.

[10] S. Pirzada, B. A. Wani and A. Somasundaram, On the eigenvalues of zero-divisor graph associated to finite commutative ring $\mathbb{Z}_{p^M q^N}$, *AKCE International Journal of Graphs and Combinatorics* **18**(1) (2021), 1 – 6, DOI: 10.1080/09728600.2021.1873060.

[11] M. Young, Adjacency matrices of zero-divisor graphs of integer modulo n, *Involve* **8**(5) (2015), 753 – 761, DOI: 10.2140/involve.2015.8.753.