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Research Article

Upper Bound Estimates of Fourth Order Hankel and Toeplitz Determinants for Certain Analytic Functions Connected with Three Leaf Function

Rudrani Rangu*[1], Bharavi Sharma Rayaprolu[2] and Sambasiva Rao Siginam[3]

[1] *Department of Mathematics, Government Degree College (Autonomous), Narsampet, Warangal 506132, Telangana, India*
[2] *Department of Mathematics, Kakatiya University, Warangal 506009, Telangana, India*
[3] *Department of Humanities and Sciences, SVS Group of Institutions, Warangal 506015, Telangana, India*
***Corresponding author:** rangurudrani1981@gmail.com

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Abstract. The purpose of this paper is to compute upper bounds of Hankel and Toeplitz determinants up to fourth order for some normalized univalent functions defined on the open unit disk in the complex plane associated with three leaf function. Suitable examples are provided in support of proven results.

Keywords. Univalent functions, Starlike functions, Coefficient inequalities, Hankel determinants, Topelitz determinants, Three Leaf domain

Mathematics Subject Classification (2020). 30C45, 30C80

1. Introduction

1.1 Preliminaries

Let $\mathcal{A}$ represent the family of analytic functions f with normalization $f(0) = 0$ and $f'(0) = 1$ defined on the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. The Taylor series expansion of $f \in \mathcal{A}$ is of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad \text{for all } z \in \mathbb{D}, \text{ where } a_n = \frac{f^{(n)}(0)}{n!}. \tag{1.1}$$

The set of functions $f \in \mathcal{A}$ that are univalent is represented by $\mathcal{S}$. The set $\mathcal{B}$ of analytic functions $w(z)$ defined on $\mathbb{D}$ with the conditions $w(0)=0$ and $|w(z)|<1$, for all $z \in \mathbb{D}$ is refereed as the class of Schwarz functions (Goodman [7]).

A function $f \in \mathcal{A}$ is said to be subordinate to $g \in \mathcal{A}$ if there exists $w \in \mathcal{B}$ such that $f(z)=g(w(z))$, for all $z \in \mathbb{D}$. We denote it by $f \prec g$. In particular if g is univalent, then f is subordinate to g if, and only if, $g(0)=f(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. For more information regarding univalent function theory, we refer to Pommerenke [22].

The set of starlike and convex functions respectively $\mathcal{S}^*$, $\mathcal{C}$ which are subclasses of $\mathcal{S}$ and further extended by Ma and Minda [15] by utilizing the concept of subordination to $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$ where $\varphi \in \mathcal{A}$ satisying conditions $\Re\{\varphi(z))\}>0$, $\varphi'(0)>0$, $\varphi(\mathbb{D})$ is symmetric with respect to real axis and starlike with respect to $\varphi(0)=1$. Some Ma-Minda type subclasses of $\mathcal{S}$ were studied by choosing specific function φ in the recent past (Arif *et al*. [3], Gandhi [6], and Sharma *et al*. [27]).

The qth Hankel determinant of index $n \geq 1$ for a function $f \in \mathcal{A}$, with a series expansion given by (1.1), is denoted as $H_{q,n}(f)$ (or simply $H_q(n)$). It is defined as follows:

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \tag{1.2}$$

where $q \geq 2$ and $a_1=1$ (refer to Pommerenke [20, 21]).

Similarly, the qth symmetric Toeplitz determinant $T_q(n)$ for a function $f \in \mathcal{A}$ is defined as follows:

$$T_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix}, \tag{1.3}$$

where $q \geq 2$, $n \geq 1$, and $a_1=1$ (refer to Ali *et al*. [1]).

In 1960, Lawrence Zalcman conjecture that the coefficients of $f \in \mathcal{S}$ with series representation (1.1) satisfy the inequality (see, Libera and Ziotkiewicz [14])

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2, \quad \text{for } n \geq 2.$$

Ma [16] proposed generalized Zalcman conjecture for $f \in \mathcal{S}$ of the form (1.1) that

$$|a_n a_m - a_{n+m-1}| \leq (n-1)(m-1), \quad \text{for } n, m \geq 2$$

and proved this conjecture is true for starlike functions and univalent function with real coefficients.

1.2 Literature Review Concerning Hankel and Toeplitz Determinants in Geometric Function Theory

The pioneering works of Pommerenke [20, 21], Hayman [8], Babalola [5], Zaprawa [36], Kowalezyk *et al*. [13] on second and third Hankel determinants inspired Arif *et al*. [2] to estimate an upper bound of $|H_{4,1}(f)|$, for $f \in \mathcal{R}$. Subsequently, Srivastava *et al*. [31], Khan *et* *al.* [10,11], and Yakaiah *et al.* [34] computed upper bound for $|H_{4.1}(f)|$ for f being the member of subclasses of $\mathcal{S}$ subordinate to cardioid, sine, modified sigmoid and cosine functions, respectively.

Ali *et al.* [1] studied the Toeplitz determinant $T_q(n)$ for the class $\mathcal{S}$ and certain of its subclasses. Zhang and Tang [37] studied the upper bounds of the fourth Toeplitz determinant for the class

$$\mathcal{S}^*(\sin) = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec 1 + \sin(z) \right\}.$$

Vijayalakshmi *et al.* [32] studied symmetric Toeplitz determinants for classes defined by post quantum operators subordinated to the limacon function. Mandal *et al.* [17] investigated Toeplitz determinants of logarithmic coefficients of inverse functions for certain classes of univalent function. For similar type of studies concerning Toeplitz determinant $|T_q(n)|$ for starlike, convex and bounded boundary rotation functions, we refer to Ali *et al.* [1] and Radhika *et al.* [23].

We now provide some more details of recent research done in this direction.

(1) Kaur and Singh [9] proved $|H_{4,1}(f)| \le \frac{4027899}{896000000}$ for the members of

$$\mathcal{R}_1 = \{ f \in \mathcal{A} : \Re\{f'(z) + zf''(z)\} > 0 \}.$$

(2) Yakaiah and Sharma [33] estimated $|H_{4,1}(f)| \le 0.136765285$, $|T_4(1)(f)| \le 0.0069$ and $|T_4(2)(f)| \le 0.00009$ for the members of

$$\mathcal{R}_1(\cos z) = \{ f \in \mathcal{A} : f'(z) + zf''(z) \prec cosz \}.$$

(3) Koride *et al.* [12] computed $|H_{4,1}(f)| \le 1.540436$, $|T_4(1)(f)| \le 2.363923$ and $|T_4(2)(f)| \le 0.140203$ for the members of

$$\mathcal{R}_1(1 + \sin z) = \{ f \in \mathcal{A} : f'(z) + zf''(z) \prec 1 + \sin z \}.$$

(4) Yakaiah *et al.* [34] computed $|H_{4,1}(f)| \le 1.24199978975$, $|T_4(1)(f)| \le 2.4375$ and $|T_4(2)(f)| \le 0.674461806$ for the members of

$$\mathcal{R}_1(1 + \tanh z) = \{ f \in \mathcal{A} : f'(z) + zf''(z) \prec 1 + \tanh z \}.$$

1.3 Identification of Research Problem

The Ma-Minda type function $\varphi_{3L}(z) = 1 + (4/5)z + (1/5)z^4$ maps $\mathbb{D}$ onto three leaf shaped domain and the class

$$\mathcal{S}^*_{3L} = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \varphi_{3L}(z) \right\},$$

have been studied by Gandhi *et al.* [6], Shi *et al.* [28,30]. Arif *et al.* [4] studied the bounds for third order Hankel determinant for two subfamilies of starlike and bounded turning functions associated with a three-leaf shaped domain. Raza *et al.* [26] estimated the second and third Hankel determinants for starlike and convex functions associated to three leaf function. Murugusundaramoorthy *et al.* [18] developed a new class of bi-starlike functions subordinate to a three leaf function induced by multiplicative calculus. Murugusundaramoorthy *et al.* [19] computed upper bound of $|H_3(1)|$, for the members of $\mathcal{R}_{3L}$. Shi *et al.* [29] studied the bounds of second Hankel determinanats and with a logarithmic coefficient as entry for the class of bounded turning functionsconnected with a three-leaf shaped domain.

Motivated by the earlier mentioned research works, in this paper, we compute the upper bounds of fourth Hankel and Toeplitz determinants for the members of

$$\mathcal{R}_1(\varphi_{3L}) = \left\{ f \in \mathcal{S} : f'(z) + zf''(z) \prec \varphi_{3L}(z) = 1 + \frac{4}{5}z + \frac{1}{5}z^4, \text{ for all } z \in \mathbb{D} \right\},$$

were computed in this paper.

2. A Set of Essential Lemmas

The class of analytic functions $p(z)$ defined on the unit disk $\mathbb{D}$ with $p(0) = 1$ and $\Re\{p(z)\} > 0$ is called the class of functions with positive real part, denoted by $\mathcal{P}$. For $p \in \mathcal{P}$, the Taylor series expansion of $p \in \mathcal{P}$ is

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad \text{for } z \in \mathbb{D}. \tag{2.1}$$

Unless otherwise stated throughout this paper, we assume the series representation of $p \in \mathcal{P}$ is of the form (2.1).

Lemma 2.1 ([22]). *Let $p \in \mathcal{P}$. Then $|c_n| \le 2$ for any positive integer n. The inequality is sharp for $p(z) = \frac{1+z}{1-z}$.*

Lemma 2.2 ([15]). *Let $p \in \mathcal{P}$ and $\eta \in \mathbb{C}$. Then $|c_2 - \eta c_1^2| \le 2\max\{1, |2\eta - 1|\}$. The inequality is sharp for $p(z) = \frac{1+z}{1-z}$ and $p(z) = \frac{1+z^2}{1-z^2}$.*

Lemma 2.3 ([3]). *Let $p \in \mathcal{P}$. Then for any real numbers A, B and C,*

$$|Ac_1^3 - Bc_1c_2 + Cc_3| \le 2|A| + 2|B - 2A| + 2|A - B + C|.$$

Lemma 2.4 ([25]). *Let $p \in \mathcal{P}$. Then for all $n, m \in \mathbb{N}$,*

$$|\eta c_n c_m - c_{n+m}| = \begin{cases} 2, & \text{if } 0 \le \eta \le 1, \\ 2|2\eta - 1|, & \text{otherwise.} \end{cases}$$

This inequality is sharp.

Lemma 2.5 ([25]). *Let $p \in \mathcal{P}$ and l, m, n and r be real numbers and if the inequalities $0 < m < 1$, $0 < r < 1$,*

$$8r(1-r)((mn - 2l)^2 + (m(r+m) - n)^2) + m(1-m)(n - 2rm)^2 \le 4m^2(1-m)^2 r(1-r) \tag{2.2}$$

hold, then

$$\left| lc_1^4 + rc_2^2 + 2mc_1c_3 - \frac{3n}{2}c_1^2c_2 - c_4 \right| \le 2. \tag{2.3}$$

Lemma 2.6 ([14]). *If the function $p \in \mathcal{P}$, then*

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{2.4}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z, \tag{2.5}$$

for some x, z with $|x| \le 1$ and $|z| \le 1$.

3. Main Results

Let $f \in \mathcal{R}_1(\varphi_{3L})$ be of the form (1.1). Then there exists $w \in \mathcal{B}$ such that

$$f'(z)+zf''(z)=\varphi_{3L}(w(z))=1+\frac{4}{5}w(z)+\frac{1}{5}w(z)^4, \quad \text{for all } z \in \mathbb{D}. \tag{3.1}$$

If we take $p(z)=\frac{1+w(z)}{1-w(z)}$, for all $z \in \mathbb{D}$ then $p \in \mathcal{P}$ and $w(z)=\frac{p(z)-1}{p(z)+1}$ so that

$$f'(z)+zf''(z)=1+\frac{4}{5}\left(\frac{p(z)-1}{p(z)+1}\right)+\frac{1}{5}\left(\frac{p(z)-1}{p(z)+1}\right)^4, \quad \text{for all } z \in \mathbb{D}. \tag{3.2}$$

Substituting (1.1) and (2.1) in (3.2) followed by comparing like coefficients on both sides of (3.2), we obtain

$$a_2=\frac{c_1}{10}, \tag{3.3}$$

$$a_3=\frac{2}{45}\left(c_2-\frac{c_1^2}{2}\right), \tag{3.4}$$

$$a_4=\frac{1}{40}\left(\frac{1}{4}c_1^3-c_1c_2+c_3\right), \tag{3.5}$$

$$a_5=\frac{2}{125}\left(c_4-c_1c_3-\frac{1}{2}c_2^2+\frac{3}{4}c_1^2c_2-\frac{3}{32}c_1^4\right), \tag{3.6}$$

$$a_6=\frac{2}{180}\left(c_5-c_1c_4-c_2c_3+\frac{3}{4}c_1^2c_3+\frac{3}{4}c_1c_2^2-\frac{3}{8}c_1^3c_2\right), \tag{3.7}$$

$$a_7=\frac{2}{245}\left(c_6-c_1c_5-c_2c_4-\frac{1}{2}c_3^2+\frac{3}{4}c_1^2c_4+\frac{3}{2}c_1c_2c_3+\frac{1}{4}c_2^3-\frac{3}{8}c_1^3c_3-\frac{9}{16}c_1^2c_2^2+\frac{3}{64}c_1^6\right). \tag{3.8}$$

Example 3.1. By taking the Schwarz functions $w(z)=z$, $w(z)=z^2$, $w(z)=z^3$ and $w(z)=z^4$ in (3.1) followed by integrating on both sides and utilizing the fact $f(0)=0$, $f'(0)=1$, we get

(1) $f_1(z)=z+\frac{1}{5}z^2+\frac{1}{125}z^5$,
(2) $f_2(z)=z+\frac{4}{45}z^3+\frac{1}{405}z^9$,
(3) $f_3(z)=z+\frac{1}{20}z^4+\frac{1}{845}z^{13}$,
(4) $f_4(z)=z+\frac{4}{125}z^5+\frac{1}{1445}z^{17}$.

We can observe that all the above functions are in $\mathcal{R}_1(\varphi_{3L})$.

We now estimate initial coefficient bounds for the functions in $\mathcal{R}_1(\varphi_{3L})$.

Theorem 3.1. *If $f \in \mathcal{R}_1(\varphi_{3L})$ is given by* (1.1)*. Then $|a_2| \le \frac{1}{5}$, $|a_3| \le \frac{4}{45}$, $|a_4| \le \frac{1}{20}$, $|a_5| \le \frac{4}{125}$, $|a_6| \le \frac{1}{10}$ and $|a_7| \le \frac{38}{245}$. The functions f_1, f_2, f_3, f_4 as in Example* 3.1 *are extremal functions for first four inequalities, respectively.*

Proof. Let $f \in \mathcal{R}_1(\varphi_{3L})$ be given by (1.1). By applying Lemmas 2.1, 2.2 and 2.3 to (3.3), (3.4) and (3.5) respectively, we obtain

$$|a_2|=\frac{|c_1|}{10}\le\frac{1}{5},$$

$$|a_3|=\left|\frac{2}{45}\left(c_2-\frac{c_1^2}{2}\right)\right|\le\frac{4}{45},$$

$$|a_4| = \frac{1}{160}|c_1^3 - 4c_1c_2 + 4c_3| \le \frac{2(1+2+1)}{160} = \frac{1}{20}.$$

Now,

$$\begin{aligned}|a_5| &= \frac{2}{125}\left|c_4 - c_1c_3 - \frac{1}{2}c_2^2 + \frac{3}{4}c_1^2c_2 - \frac{3}{32}c_1^4\right| \\ &= \frac{2}{125}\left|lc_1^4 + rc_2^2 + 2mc_1c_3 - \frac{3n}{2}c_1^2c_2 - c_4\right|,\end{aligned}$$

where $l = \frac{3}{32}$, $r = \frac{1}{2}$, $m = \frac{1}{2}$ and $n = \frac{1}{2}$.

These values of l, m, n and r satisfy the hypothesis of Lemma 2.5 and hence $|a_5| \le \frac{4}{125}$.

By Lemmas 2.1, 2.4 and 2.5, we have $|c_1| \le 2$, $|c_2| \le 2$, $|c_5 - c_2c_3| \le 2$ and $\left|\frac{3}{8}c_2^2 + \frac{3}{4}c_1c_3 - \frac{3}{8}c_1^2c_2 - c_4\right| \le 2$. Consequently, we obtain the following after a simple computation

$$\begin{aligned}|a_6| &= \frac{1}{90}\left|c_5 - c_1c_4 - c_2c_3 + \frac{3}{4}c_1^2c_3 + \frac{3}{4}c_1c_2^2 - \frac{3}{8}c_1^3c_2\right| \\ &\le \frac{1}{90}\left(|c_5 - c_2c_3| + |c_1|\left|\frac{3}{8}c_2^2 + \frac{3}{4}c_1c_3 - \frac{3}{8}c_1^2c_2 - c_4\right| + \frac{3}{8}|c_1||c_2^2|\right) \\ &\le \frac{1}{10}.\end{aligned}$$

Further, in view of Lemma 2.3, we have $\left|-\frac{3}{8}c_1^3 + c_1c_2 - \frac{1}{2}c_3\right| \le \frac{3}{2}$ and in view of Lemma 2.4, we have $|c_6 - c_1c_5| \le 2$, $|c_4 - \frac{1}{2}c_1c_3| \le 2$, $|c_2 - \frac{9}{4}c_1^2| \le 7$ and using the fact $|c_n| \le 2$ for $n \ge 1$, we obtain

$$\begin{aligned}|a_7| &= \frac{2}{245}\left|c_6 - c_1c_5 - c_2c_4 - \frac{1}{2}c_3^2 + \frac{3}{4}c_1^2c_4 + \frac{3}{2}c_1c_2c_3 + \frac{1}{4}c_2^3 - \frac{3}{8}c_1^3c_3 - \frac{9}{16}c_1^2c_2^2 + \frac{3}{64}c_1^6\right| \\ &\le \frac{2}{245}\left(|c_6 - c_1c_5| + \frac{|c_2|^2}{4}\left|c_2 - \frac{9}{4}c_1^2\right| + |c_2|\left|c_4 - \frac{1}{2}c_1c_3\right| + |c_3|\left|-\frac{3}{8}c_1^3 + c_1c_2 - \frac{1}{2}c_3\right| + \frac{3}{64}|c_1|^6\right) \\ &\le \frac{38}{245}.\end{aligned}$$

We now obtain upper bound for Fekete-Szego functional of the class $\mathcal{R}_1(\varphi_{3L})$.

Theorem 3.2. *If $f \in \mathcal{R}_1(\varphi_{3L})$ is given by* (1.1) *and $H_{2,1}^{\eta}(f) = a_3 - \eta a_2^2$ for $\eta \in \mathbb{C}$, then*

$$|H_{2,1}^{\eta}(f)| \le \frac{4}{45}\max\left\{1, \frac{9}{20}|\eta|\right\} \tag{3.9}$$

and this inequality is sharp.

Proof. Let $f \in \mathcal{R}_1(\varphi_{3L})$. Then in view of Lemma 2.2, we obtain

$$\begin{aligned}|H_{2,1}^{\eta}(f)| &= |a_3 - \eta a_2^2| \\ &= \frac{2}{45}\left|c_2 - \left(\frac{20+9\eta}{40}\right)c_1^2\right| \\ &\le \frac{4}{45}\max\left\{1, \left|2\left(\frac{20+9\eta}{40}\right) - 1\right|\right\} \\ &= \frac{4}{45}\max\left\{1, \frac{9}{20}|\eta|\right\}.\end{aligned}$$

Sharpness is evident from the fact that if $|\eta| \le \frac{20}{9}$, then $|H_{2,1}^{\eta}(f_2)| = \frac{4}{45}$ and if $|\eta| > \frac{20}{9}$, then $|H_{2,1}^{\eta}(f_1)| = \frac{|\eta|}{25}$, where f_1 and f_2 are as in Example 3.1. □

We now estimate an upper bound for second Hankel determinants for the class $\mathcal{R}_1(\varphi_{3L})$.

Theorem 3.3. *If $f \in \mathcal{R}_1(\varphi_{3L})$ is given by* (1.1)*, then* $|H_{2,2}(f)| = |a_2a_4 - a_3^2| \le \frac{16}{2025}$. *Sharpness is obtained for the function f_2 as in Example* 3.1.

Proof. Let $f \in \mathcal{R}_1(\varphi_{3L})$ be given by (1.1). By utilizing (3.3), (3.4) and (3.5), we have

$$\begin{aligned}|H_{2,2}(f)| &= \frac{1}{25}\left|\frac{c_1}{5184}(17c_1^3 - 68c_1c_2 + 324c_3) - \frac{4}{81}c_2^2\right| \\ &= \frac{1}{129600}|324c_1c_3 - 68c_1^2c_2 + 17c_1^4 - 256c_2^2|.\end{aligned}$$

Applying Lemma 2.6, we obtain

$$\begin{aligned}|H_{2,2}(f)| &= \frac{1}{129600}|(81-34+17-64)c_1^4 + (162-34-128)c_1^2x(4-c_1^2) \\ &\qquad - 81c_1^2x^2(4-c_1^2) - 64x^2(4-c_1^2)^2 + 162(4-c_1^2)c_1(1-|x|^2)z| \\ &= \frac{1}{129600}|-81c_1^2x^2(4-c_1^2) - 64x^2(4-c_1^2)^2 + 162(4-c_1^2)c_1(1-|x|^2)z| \\ &\le \frac{1}{129600}(81c^2t^2(4-c^2) + 64t^2(4-c^2)^2 + 162(4-c^2)c(1-t^2)z) \\ &:= F(c,t),\end{aligned}$$

where $c = |c_1| \in [0,2]$, $t = |x| \in [0,1]$ and $0 \le z \le 1$. Thus,

$$\begin{aligned}\frac{\partial F}{\partial t} &= \frac{1}{64800}(4-c^2)(17c^2 - 162c + 256)t \\ &= \frac{1}{64800}(4-c^2)(2-c)(128-17c)t \\ &\ge 0.\end{aligned}$$

Therefore, $F(c,t)$ is an increasing function in variable t and hence

$$\begin{aligned}|H_{2,2}(f)| &\le F(c,1) \\ &= \frac{1}{129600}(81c^2(4-c^2) + 64(4-c^2)^2) \\ &:= \psi(c).\end{aligned}$$

It is clear that $\psi'(c) = 0$ implies $c = 0$ and $c = \sqrt{\frac{-376}{68}}$. Since $0 \le c \le 2$, we take $c = 0$. Further, $\psi''(0) = -376 < 0$. Therefore, ψ attains its maximum at $c = 0$. Hence,

$$\begin{aligned}|H_{2,2}(f)| &\le \psi(0) \\ &= \frac{1024}{129600} \\ &= \frac{16}{2025}.\end{aligned}$$

□

Theorem 3.4. *If $f \in \mathcal{R}_1(\varphi_{3L})$, then* $|H_{2,3}(f)| \le \frac{3373}{360000}$.

Proof. Let $f \in \mathcal{R}_1(\varphi_{3L})$ be given by (1.1). By using (3.4), (3.5) and (3.6), we obtain

$$|H_{2,3}(f)| = \frac{1}{5760000}|4096c_2c_4 - 2048c_2^3 + 3014c_1c_2c_3 + 496c_1^2c_2^2 - 120c_1^4c_2 - 2048c_1^2c_4$$

$$+248c_1^3c_3-33c_1^6+248c_1^3c_3-33c_1^6-3600c_3^2|$$

$$\leq \frac{1}{5760000}\left(4096|c_2|\left|\frac{15}{512}c_1^4+\frac{1}{2}c_2^2+\frac{293}{1024}c_1c_3-\frac{31}{256}c_1^2c_2-c_4\right|+3600|c_3||c_3-c_1c_2|\right.$$
$$\left.+2048|c_1|^2\left|c_4-\frac{31}{256}c_1c_3\right|+586|c_1||c_2||c_3|+33|c_1|^6\right).$$

An application of Lemma 2.5 shows that

$$\left|\frac{15}{512}c_1^4+\frac{1}{2}c_2^2+\frac{293}{1024}c_1c_3-\frac{31}{256}c_1^2c_2-c_4\right|\leq 2.$$

Further, it is clear to see that an application of Lemma 2.4 yields $|c_3-c_1c_2|\leq 2$, $|c_4-\frac{31}{256}c_1c_3|\leq 2$. In view of these inequalities along with $|c_n|\leq 2$ for all $n\geq 1$, we finally obtain

$$|H_{2,3}(f)|\leq \frac{3373}{360000}$$
$$\approxeq 0.009369444.$$
□

3.1 Bounds of Zalcman Functionals, $|H_{3,1}(f)|$, $|H_{3,2}(f)|$ for the class $\mathcal{R}_1(\varphi_{3L})$

Theorem 3.5. *If $f\in\mathcal{R}_1(\varphi_{3L})$, then $|a_4-a_2a_3|\leq\frac{1}{20}$. The sharpness is obtained for*

$$f_3(z)=z+\frac{1}{20}z^4+\frac{z^{13}}{845}\in\mathcal{R}_1(\varphi_{3L}).$$

Proof. Let $f\in\mathcal{R}_1(\varphi_{3L})$ be given by (1.1). Utilizing (3.3), (3.4) and (3.5) and applying Lemma 2.3, we obtain

$$|a_4-a_2a_3|=\frac{1}{7200}|77c_1^3-212c_1c_2+180c_3|$$
$$\leq\frac{2}{7200}(77+|212-154|+|77-212+180|)$$
$$=\frac{1}{20}.$$
□

Theorem 3.6. *Let $f\in\mathcal{R}_1(\varphi_{3L})$ be given by (1.1). Then $|a_3-a_2^2|\leq\frac{4}{45}$, $|a_5-a_3^2|\leq\frac{4}{125}$ and $|a_7-a_4^2|\leq\frac{3089}{19600}$. Further, $f_2(z)=z+\frac{4}{45}z^3+\frac{1}{405}z^9$, $f_4(z)=z+\frac{4}{125}z^5+\frac{1}{1445}z^{17}$ as given in Example* 3.1 *are extremal functions for first two inequalities, respectively.*

Proof. Taking $\mu=1$ in eq. (3.9) yields $|a_3-a_2^2|\leq\frac{4}{45}$. Using (3.6) and (3.4), we obtain

$$|a_5-a_3^2|=\left|\frac{2}{125}\left(c_4-c_1c_3-\frac{1}{2}c_2^2+\frac{3}{4}c_1^2c_2-\frac{3}{32}c_1^4\right)-\frac{4}{2025}\left(c_2-\frac{1}{2}c_1^2\right)^2\right|$$
$$=\frac{2}{125}\left|\frac{37}{288}c_1^4+\frac{101}{162}c_2^2+c_1c_3-\frac{283}{324}c_1^2c_2-c_4\right|$$
$$=\frac{2}{125}\left|lc_1^4+rc_2^2+2mc_1c_3-\frac{3n}{2}c_1^2c_2-c_4\right|,$$

where $l=\frac{37}{288}$, $r=\frac{101}{162}$, $m=\frac{1}{2}$, $n=\frac{283}{486}$. These values of l,r,m,n satisfy the hypothesis of Lemma 2.4 as it is evident from the facts that $0<r<1$, $0<m<1$, $4m^2(1-m)^2r(1-r)=0.0586896$ and

$$8r(1-r)((mn-2l)^2+(m(r+m)-n)^2)+m(1-m)(n-2rm)^2=0.00341618.$$

Therefore, by Lemma 2.4,

$$\left|\frac{37}{288}c_1^4+\frac{101}{162}c_2^2+c_1c_3-\frac{283}{324}c_1^2c_2-c_4\right|\le 2.$$

Hence $|a_5-a_3^2|\le\frac{4}{125}$. On similar lines, utilizing the inequalities $|a_7|\le\frac{38}{245}$ and $|a_4|\le\frac{1}{20}$ as proved in Theorem 3.1, we obtain $|a_7-a_4^2|\le|a_7|+|a_4|^2\le\frac{3084}{19600}$. □

We now estimate an upper bound of $|H_{3,1}(f)|$ for $f\in\mathcal{R}_1(\varphi_{3L})$.

Theorem 3.7. *If $f\in\mathcal{R}_1(\varphi_{3L})$, then $|H_{3,1}(f)\le\frac{481}{90000}$.*

Proof. Let $f\in\mathcal{R}_1(\varphi_{3L})$ be given by (1.1). Then in view of Theorem 3.5 and Theorem 3.6, we have $|a_4-a_2a_3|\le\frac{1}{20}$, $|a_3-a_2^2|\le\frac{4}{45}$ and $|a_5-a_3^2|\le\frac{4}{125}$.
It is found in [24] that

$$|H_{3,1}(f)|\le|a_5-a_3^2||a_3-a_2^2|+|a_4-a_2a_3|^2.$$

Hence, $|H_{3,1}(f)|\le\left(\frac{4}{125}\right)\left(\frac{4}{45}\right)+\left(\frac{1}{20}\right)^2=\frac{481}{90000}\approxeq 0.00534444.$ □

Theorem 3.8. *If $f\in\mathcal{R}_1(\varphi_{3L})$ is given by* (1.1)*, then $|H_{3,2}(f)|\le\frac{2978957}{1620000000}$.*

Proof. Let $f\in\mathcal{R}_1(\varphi_{3L})$ be given by (1.1). By using (3.3)-(3.6), we obtain

$$\begin{aligned}|a_2a_5-a_3a_4|&=\frac{1}{11250}\left|18c_1\left(c_4-\frac{11}{36}c_1c_3\right)-2c_1^3\left(c_2-\frac{23}{32}c_1^2\right)-\frac{1}{4}c_2(13c_1^3-64c_1c_2+100c_3)\right|\\&\le\frac{1}{11250}\left(18|c_1|\left|c_4-\frac{11}{36}c_1c_3\right|+2|c_1|^3\left|c_2-\frac{23}{32}c_1^2\right|\frac{1}{4}|c_2||13c_1^3-64c_1c_2+100c_3|\right)\\&\le\frac{1}{11250}(72+32+100)\\&=\frac{34}{1875}.\end{aligned}$$

By using the fact that $|a_2a_5-a_3a_4|\le\frac{34}{1875}$ and utilizing the bounds proven in Theorem 3.1, Theorem 3.3 and Theorem 3.4, we obtain

$$\begin{aligned}|H_{3,2}(f)|&\le|a_6||a_2a_4-a_3^2|+|a_5||a_2a_5-a_3a_4|+|a_4||a_3a_5-a_4^2|\\&\le\frac{8}{10125}+\frac{136}{234375}+\frac{3373}{7200000}\\&=\frac{2978957}{1620000000}\\&\approxeq 0.00183886.\end{aligned}$$

□

3.2 Upper Bounds of $|H_{4,1}(f)|$, $|T_4(1)|$, $|T_4(2)|$ for $f\in\mathcal{R}_1(\varphi_{3L})$

Theorem 3.9. *If $f\in\mathcal{R}_1(\varphi_{3L})$ is given by* (1.1)*, then $|a_5-a_2a_4|\le\frac{4}{125}$. This inequality is sharp.*

Proof. Let $f\in\mathcal{R}_1(\varphi_{3L})$ be given by (1.1). By using (3.6), (3.3) and (3.5),

$$\begin{aligned}|a_5-a_2a_4|&=\frac{2}{125}\left|\frac{17}{128}c_1^4+\frac{1}{2}c_2^2+\frac{37}{32}c_1c_3-\frac{29}{32}c_1^2c_2-c_4\right|\\&=\frac{2}{125}\left|lc_1^4+rc_2^2+2mc_1c_3-\frac{3n}{2}c_1^2c_2-c_4\right|,\end{aligned}$$

where $l=\frac{17}{128}$, $r=\frac{1}{2}$, $m=\frac{37}{64}$ and $n=\frac{29}{48}$.
The values of l,r,m,n satisfy the hypothesis of Lemma 2.5 and hence

$$|a_5-a_2a_4|=\frac{2}{125}\left|lc_1^4+rc_2^2+2mc_1c_3-\frac{3n}{2}c_1^2c_2-c_4\right|\le\frac{4}{125}.$$

The equality hold in $|a_5-a_2a_3|\le\frac{4}{125}$, for $f_4(z)=z+\frac{4}{125}z^5+\frac{1}{1445}z^{17}\in\mathcal{R}_1(\varphi_{3L})$. □

Theorem 3.10. *If* $f\in\mathcal{R}_1(\varphi_{3L})$ *is given by* (1.1)*, then* $|a_6-a_3a_4|\le\frac{77}{900}$.

Proof. Let $f\in\mathcal{R}_1(\varphi_{3L})$ be given by (1.1). By using (3.7), (3.5) and (3.4) leads to

$$\begin{aligned}|a_6-a_3a_4|&=\frac{1}{900}\left|10c_5-11c_2c_3+\frac{17}{4}c_1c_2^2+10c_1\left(\frac{1}{80}c_1^4+\frac{17}{40}c_2^2+\frac{4}{5}c_1c_3-\frac{9}{20}c_1^2c_2-c_4\right)\right|\\&\le\frac{1}{900}\left(10\left|c_5-\frac{27}{40}c_2c_3\right|+\frac{17}{4}|c_2||c_3-c_1c_2|+10|c_1|\left|lc_1^4+rc_2^2+2mc_1c_3-\frac{3n}{2}c_1^2c_2-c_4\right|\right),\end{aligned}$$

where $l=\frac{1}{80}$, $\frac{17}{40}$, $m=\frac{2}{5}$ and $n=\frac{3}{10}$.
These values of l,r,m,n satisfy the hypothesis of Lemma 2.5 and hence

$$\left|\frac{1}{80}c_1^4+\frac{17}{40}c_2^2+\frac{4}{5}c_1c_3-\frac{9}{20}c_1^2c_2-c_4\right|\le 2.$$

In view of Lemma 2.4, we have $\left|c_5-\frac{27}{40}c_2c_3\right|\le 2$, $|c_3-c_1c_2|\le 2$. Further from Lemma 2.1, we have $|c_1|\le 2$, $|c_2|\le 2$. Utilising these inequalities, we obtain that $|a_6-a_3a_4|\le\frac{77}{900}$. □

Remark 3.1. It is clear from Theorem 3.6 that the Zalcman conjecture is true for $n=2,3,4$ for $f\in\mathcal{R}_1(\varphi_{3L})$. Further, generalized Zalcman conjecture for certain initial values of n, m in view of Theorems 3.5, 3.9, 3.10.

Theorem 3.11. *If* $f\in\mathcal{R}_1(\varphi_{3L})$*, then* $|H_{4,1}(f)|\le 0.00179948585$.

Proof. Let $f\in\mathcal{R}_1(\varphi_{3L})$ be given by (1.1). It is found in [24] that if $f\in\mathcal{S}$ of the form (1.1), then

$$\begin{aligned}|H_{4,1}(f)|&\le|a_7-a_4^2||H_{3,1}(f)|+|a_6-a_3a_4|^2|a_3-a_2^2|\\&\quad+|a_5-a_2a_4|^2|a_5-a_3^2|+2|a_6-a_3a_4||a_5-a_2a_4||a_4-a_2a_3|.\end{aligned}\tag{3.10}$$

Thus, the required upper bound follows by utilizing the bounds obtained in Theorem 3.2, Theorem 3.5, Theorem 3.6, Theorem 3.7, Theorem 3.9 and Theorem 3.10 in (3.10). □

Theorem 3.12. *If* $f\in\mathcal{R}_1(\varphi_{3L})$*, then* $|T_4(1)|\le 1.149169$.

Proof. Let $f\in\mathcal{R}_1(\varphi_{3L})$ be given by (1.1). Then by using the bounds of initial coefficients of $f\in\mathcal{R}_1(\varphi_{3L})$ proved in Theorem 3.1, we obtain

$$\begin{aligned}|a_3-a_2a_4|&\le|a_3|+|a_2||a_4|\le\frac{4}{45}+\frac{1}{5}\left(\frac{1}{20}\right)=\frac{89}{900},\\|a_2-a_2a_3|&\le|a_2|(1+|a_3|)\le\frac{1}{5}\left(1+\frac{4}{45}\right)=\frac{49}{225},\\|1-a_2^2|&\le 1+|a_2|^2\le 1+\frac{1}{25}=\frac{26}{25}.\end{aligned}$$

Also, $|a_3 - a_2^2| \leq \frac{4}{45}$, $|a_2a_3 - a_4| \leq \frac{1}{20}$ and $|H_{2,2}(f)| \leq \frac{16}{2025}$ are as proved in Theorems 3.6, 3.5 and 3.3, respectively. In view of [37], we have

$$\begin{aligned}|T_4(1)| &= |(1-a_2^2)^2 - (a_2a_3 - a_4)^2 + (a_3^2 - a_2a_4)^2 - (a_2 - a_2a_3)^2 + 2(a_2^2 - a_3)(a_3 - a_2a_4)| \\ &\leq |1-a_2^2|^2 + |a_2a_3 - a_4|^2 + |a_3^2 - a_2a_4|^2 + |a_2 - a_2a_3|^2 + 2|a_2^2 - a_3|\,|a_3 - a_2a_4| \\ &\leq \left(\frac{26}{25}\right)^2 + \frac{1}{400} + \left(\frac{16}{2025}\right)^2 + \left(\frac{49}{225}\right)^2 + 2\left(\frac{4}{45}\right)\left(\frac{89}{900}\right) \\ &\approxeq 1.149169.\end{aligned}$$

□

Theorem 3.13. *If $f \in \mathcal{R}_1(\varphi_{3L})$, then $|T_4(2)| \leq 0.00340790418$.*

Proof. Since $f \in \mathcal{R}_1(\varphi_{3L})$, in view of Theorems 3.4, 3.3 and 3.8, we have $|H_{2,3}(f)| \leq \frac{3373}{360000}$, $|H_{2,2}(f)| \leq \frac{16}{2025}$ and $|a_3a_4 - a_2a_5| \leq \frac{34}{1875}$. Further

$$\begin{aligned}&|T_2(2)| = |a_2^2 - a_3^2| \leq |a_2^2| + |a_3^2| \leq \frac{97}{2025}, \\ &|a_2a_3 - a_3a_4| \leq |a_3|(|a_2| + |a_4|) \leq \frac{1}{45}, \\ &|a_2a_4 - a_3a_5| \leq |a_2||a_4| + |a_3||a_5| \leq \frac{1}{100} + \frac{16}{5625} = \frac{289}{22500}.\end{aligned}$$

Following the inequality found in [37], we obtain

$$\begin{aligned}|T_4(2)| &\leq |T_2(2)|^2 + |a_3a_4 - a_2a_5|^2 + |H_{2,3}(f)|^2 + |a_2a_3 - a_3a_4|^2 + 2|H_{2,2}(f)|\,|a_2a_4 - a_3a_5| \\ &\leq \frac{9409}{4100625} + \frac{1156}{3515625} + \frac{11377129}{129600000000} + \frac{1}{2025} + 2\left(\frac{16}{2025}\right)\left(\frac{289}{22500}\right) \\ &= 0.00340790418.\end{aligned}$$

□

4. Concluding Remarks

The upper bounds of Hankel and Toeplitz determinants of order four have been established, shown that the Zalcman conjecture is true for cases $n = 2,3,4$. Additionally, the generalized Zalcman conjecture has been validated for initial values of n and m within the class $\mathcal{R}_1(\varphi_{3L})$ in this paper. One can investigate the upper bounds of $|H_{5,1}(f)|$, $|T_5(1)|$ for $\mathcal{R}_1(\varphi_{3L})$ as continuation of the work.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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