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Existence of a Non-Oscillating Solution for a Second Order Nonlinear ODE

Research Article

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Abstract. In this paper we have considered the following nonlinear ordinary differential equation.

$$y''(x) + F(x, y(x)) = 0 \tag{0.1}$$

where $F(t, x(t))$ is real valued function on $[0,\infty) \times R$, $x \geq 0$. We have given sufficient conditions for the existence of a non oscillating solution for equation (0.1). These conditions are generalized with respect to the nonlinear function F and are in the spirit of the classical result by Atkinson [1].

Keywords. Nonlinear coupled ordinary differential equations; Fixed-point Theorem; Non-oscillation

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1. Introduction

Lot of work was done on the study of oscillation and non-oscillation theory of ordinary differential equations. The study of second order ordinary differential equations of the type (0.1) has a lot of importance. In fact the study of oscillation theory was started first with such equations by F.V. Atkinson in his classic paper [1]. After this paper most of the results in this field as in [5, 6, 7, 8] were centered on establishing necessary and sufficient conditions for the oscillations (or non-oscillations) of all solutions of the nonlinear ODEs where the nonlinear function is of the variables separable form and falls under the superlinear case. A simpler and unified presentation of these results that eliminates the need for specific techniques in each of the classical cases (linear, sublinear and superlinear) is presented by Dube in [4]. In [4] general criteria for the existence of a non-oscillating solution for (0.1) are found using the fixed point theorem technique.In the current paper we would like to slightly weaken the conditions specified in [4]. The following is the main result proved in [4].

Theorem 1.1. *Let $X = \{u \in C[0,\infty)| : 0 \le u(t) \le M$, for all $t \ge 0\}$, where $M > 0$ is given but fixed. Assume that $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ and that for any $u \in X$,*

$$\int_0^\infty tF(t,u(t))dt \le M \tag{1.1}$$

and that there exists a function $k : \mathbb{R}^+ \to \mathbb{R}^+$ such that k is continuous and

$$\int_0^\infty tk(t)dt < 1, \tag{1.2}$$

such that for any $u, v \in \mathbb{R}^+$, we also have

$$|F(t,u) - F(t,v)| \le k(t)|u - v|, \quad t \ge 0. \tag{1.3}$$

Then (0.1) has a positive (and so non-oscillatory) monotone solution on $(0,\infty)$ such that $y(x) \to M$ as $x \to \infty$.

The goal of [4] was providing a condition for the existence of asymptotically constant solution to the differential equation (0.1). To this end, conditions (1.1), (1.2), (1.3), seem to be strong with regards to the non-oscillation of the solutions of the differential equation (0.1), but this was not given weight as the goal was much more than non-oscillation.

In [9], Hempel has given necessary and sufficient conditions for the solutions of (0.1) to be positive asymptotically. These conditions are not of the Lipschitzian type but are of a more general nature than those given by Dube in [4].

Our aim in this paper is to present some weak conditions with respect to those given in [4] which guarantee the existence of a non-oscillatory solution, which is also asymptotically constant. We have also shown the application of these results on a type of first order system of Nonlinear ODEs. However, presenting a proof for more generalized and weaker conditions like the one give by Hempel in [9] is yet to be explored. There is a lot of work done on asymptotic representation of solutions to (0.1). More recently in [11], [10], the authors have presented necessary and sufficient conditions for the solutions of (0.1) to asymptotically looking like first degree polynomials. In [10], Corollary 3.1 gives us the conditions

$$|f(t,x)| \le h_1(t)g\left(\frac{|x|}{t}\right) \tag{1.4}$$

and

$$\int_0^\infty sh_i(s)ds < \infty, \quad i = 1,2 \tag{1.5}$$

for a differential equation $x''(t) = f(t,x)$ to have a solution of the type $x(t) = at + b + o(1)$, but there is no mention about nonoscillatory property of such a solution. We can say that the conditions that we presented in our paper in Corollary 3.2 are comparable with the ones presented above from [10]. Though our conditions seem to be weaker, they guarantee that solution not only asymptotically tends to be a constant but also is positive and thereby nonoscillatory. Similar comments hold when we compare our conditions to those given in [11].

2. Mathematical Preliminaries

Definition 2.1. A solution of a real second order nonlinear differential equation is said to be oscillatory on a half axis provided it has an infinite number of zeros on that semi-axis.

If the equation has at least one non-trivial solution with a finite number of zeros it is termed non-oscillatory.

The Schauder Theorem. *Let E be a Banach space and X any nonempty convex and closed subset of E. If S is a continuous mapping of X into itself and SX is relatively compact, then the mapping S has at least one fixed point (i.e. there exists an $x \in X$ with $x = Sx$).*

Let $E = B([0,\infty))$, where $B([0,\infty))$ is the Banach space of all continuous and bounded real valued functions on the interval $[0,\infty]$, endowed with the sup-norm $\|.\|$:

$$\|h\| = \sup_{t\geq 0}|h(t)| \quad \textit{for } h \in B([0,\infty)).$$

We need the following compactness criterion for subsets of $B([0,\infty))$ which is a corollary of the Arzela-Ascoli theorem. This compactness criterion is an adaptation of a lemma due to Avramescu [2].

Compactness Criterion. *Let H be an equicontinuous and uniformly bounded subset of the Banach space $B([0,\infty))$. If H is equiconvergent at ∞, it is also relatively compact.*

Note that a set H of real-valued functions defined on the interval $[0,\infty)$ is called equiconvergent at ∞ if all functions in H are convergent in R at the point ∞ and, in addition, for every $\epsilon > 0$ there exists a $T \geq 0$ such that, for all functions $h \in H$, it holds $|h(t) - \lim_{s\to\infty} h(s)| < \epsilon$ for all $t \geq T$.

3. Main Result

Theorem 3.1. *Let $X = \{u \in B([0,\infty)) : 0 \leq u(x) < \infty$, for all $x \geq 0\}$. Assume that $F(x,u(x)) : R^+ \times R \to R^+$ is a positive real valued function on $[0,\infty) \times R$, and that for any $u \in X$*

$$\int_0^\infty xF(x,u(x))dx < \infty, \tag{3.1}$$

$$\int_T^\infty xF(x,u(x))dx < M, \tag{3.2}$$

for some $M > 0$ and $T \geq 0$, then (0.1) has a positive (therefore non-oscillating) monotone solution on $[T,\infty)$ such that $y(x) \to M$ as $x \to \infty$.

Proof. Let us define an operator S on the set X as follows

$$(Sy)(x) = M - \int_x^\infty (t-x)F(t,y(t))dt \quad \text{for all } y \in X \tag{3.3}$$

Since

$$0 \leq \int_x^\infty (t-x)F(t,y(t))dt \leq \int_0^\infty tF(t,y(t))dt < \infty \tag{3.4}$$

This happens because $x \geq 0$, $F(x, y(x)) \geq 0$ and the indefinite integral in the inequality is non-increasing function of x. We can say that $SX \subseteq X$. So SX is uniformly bounded. Now we will show that SX is equicontinuous. For every $y(x) \in X$ and $x_1, x_2 > T$ such that $x_2 > x_1$, for some large T, consider

$$\begin{aligned}
|(Sy)(x_2) - (Sy)(x_1)| &= \left| \int_{x_2}^{\infty} (t - x_2) F(t, y(t)) dt - \int_{x_1}^{\infty} (t - x_1) F(t, y(t)) dt \right| \\
&= \left| \int_{x_2}^{\infty} \left[\int_r^{\infty} F(t, y(t)) dt \right] dr - \int_{x_1}^{\infty} \left[\int_r^{\infty} F(t, y(t)) dt \right] dr \right| \\
&= \left| \int_{x_2}^{x_1} \left[\int_r^{\infty} F(t, y(t)) dt \right] dr \right| \\
&\leq \int_{x_2}^{x_1} \left[\int_r^{\infty} |F(t, y(t))| dt \right] dr \\
&\leq M |x_2 - x_1|.
\end{aligned}$$

So by choosing suitable width for $|x_2 - x_1|$, we can show that SX is equicontinuous. Now let us show that it is equiconvergent also. To this end consider for all $x \geq T'$ for some $T' > T$,

$$\begin{aligned}
\left| (Sy)(x) - \lim_{x \to \infty} (Sy)(x) \right| &= |(Sy)(x) - M| \\
&= \left| \int_x^{\infty} (t - x) F(t, y(t)) dt \right|
\end{aligned}$$

To show that $\left| \int_x^{\infty} (t - x) F(t, y(t)) dt \right| < \epsilon$, it suffices to show that $\left| \int_{T'}^{\infty} (t - x) F(t, y(t)) dt \right| < \epsilon$, because F is non-negative, x is defined on R^+ and $x \geq T'$. so,

$$\begin{aligned}
\left| \int_{T'}^{\infty} (t - x) F(t, y(t)) dt \right| &= \left| \int_T^{\infty} (t - x) F(t, y(t)) dt \right| - \left| \int_T^{T'} (t - x) F(t, y(t)) dt \right| \\
&= \left| \int_T^{\infty} (t - x) F(t, y(t)) dt \right| - \left| \int_T^{T'} \left[\int_r^{\infty} F(s, y(s)) dr \right] dt \right| \\
&< \int_T^{\infty} (t - x) |F(t, y(t))| dt - \int_T^{T'} \left[\int_r^{\infty} |F(s, y(s))| dr \right] dt \\
&< M(1 - |T - T'|)
\end{aligned}$$

Thus, by choosing T' in such a way that $|T - T'| < 1 - \frac{\epsilon}{M}$ we can easily verify that SX is equiconvergent at ∞.

By using the given compactness criterion, we can now say that SX is relatively compact.

Invoking Schauder's fixed point theorem on S we get that S has a fixed point $y(x)$. We will now show that this fixed point will also satisfy the second order differential equation (0.1). Let $y(x)$ be a fixed point for the integral equation (3.3), so

$$y(x) = M - \int_x^{\infty} (t - x) F(t, y(t)) dt$$

This can be rewritten as

$$y(x) = M + \int_T^x (t-x)F(t,y(t))dt + \int_T^\infty (t-x)F(t,y(t))dt \quad \text{for some } T.$$

Differentiating this equation twice using the fundamental theorem of calculus we can easily verify that

$$y''(x) + F(x,y(x)) = 0$$

And so the fixed point of the integral equation (3.3) is also a solution to (0.1). This solution has to be monotone because all the quantities in the operator S are non-negative. It is definitely positive, continuous and bounded because it is found in the space X. □

To give point wise criterion for the same differential equation we state the following corollary of the previous theorem.

Corollary 3.2. *Let* $X = \{u \in B([0,\infty)) : 0 \le u(x) < \infty$, *for all* $t \ge 0\}$. *Assume that* $F(x,u(x))$ *is a real valued function on* $[0,\infty) \times R$, $x \ge 0$ *and let* $f(t)$ *be a function such that*

$$\int_0^\infty t f(t)dt \le 1 \tag{3.5}$$

and

$$F(t,u(t)) \le f(t)g(u) \tag{3.6}$$

and for some non-negative real-valued function $g : [T,M] \to [T,M]$, *for some* $T \ge 0, M > T$. *Then* (0.1) *has a positive (therefore non-oscillating) monotone solution on* $[T,\infty)$.

Proof. The proof for this corollary similar to that of the main theorem, except for, we use (3.5) and (3.6) in lieu of (3.1) and (3.2) wherever necessary. □

4. Application

Consider the following system of ODEs

$$y_1' = y_2, \quad y_2' = b(t)y_1^n \tag{4.1}$$

This system can be easily converted to the second order differential equation

$$y'' = b(t)y^n \tag{4.2}$$

where n determines the nature of non-linearity(Linear, sub-linear or super-linear). Let us consider the case where $n = \frac{1}{2}$. We can clearly see that the function $y^{\frac{1}{2}}$ is a self-map from $[0,1]$ to $[0,1]$, but it is not Lipschitzian. So the results given in [4] can not be invoked here. Nevertheless, by choosing the function $b(t)$ such that

$$\int_0^\infty t b(t)dt \le 1 \tag{4.3}$$

all the conditions in our corollary are met and so we can guarantee now that the ODE and thereby the system of first order ODEs have a non-oscillating positive monotone solution.

5. Conclusion and Future Scope

We have given necessary and sufficient condition for a second order nonlinear ordinary differential equation to possess non oscillating solutions which asymptotically tend to a constant. Similar work can be extended to a more general functional type non-linear ordinary differential equations and also to higher order systems. The conditions that are posed can also be further weakened by using advanced fixed point techniques. This could be the future scope in this area.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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