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Research Article

Investigating Weakly Connected 2-Domination in the Complementary Prism of Graphs and in Some Unary Graph Operations

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Abstract. This paper dealt with the concepts of weakly connected 2-domination in the complementary prisms of graphs and graphs obtained by reducing their edges and vertices. In particular, bounds and exact values of the weakly connected 2-domination number in the complementary prism of graphs, graphs resulting from deleting an edge and vertex and line graphs are presented. In addition, properties of the graphs with weakly connected 2-domination number of complementary prism equal to 2 and 3, are provided.

Keywords. Weakly connected 2-domination, Complementary prism, Edge deletion, Vertex deletion, Unary operations, Line graphs

Mathematics Subject Classification (2020). 05C69, 05C76

1. Introduction

The concept of weakly connected domination in graphs is one of the most interesting variations of standard domination in a connected graph which occur by weakening the condition of the connectedness of a set of vertices in graphs. This domination parameter was studied by Dunbar *et al.* [3]. Another interesting variation on standard domination is the concept of 2-domination which is a particular case of k-domination in graphs which arises when it is necessary to augment the level of domination of each vertex so that even if an edge fails, the given set is still a dominating set. The notion of dominating a vertex multiple times was introduced by Fink and Jacobson [4]. The two previously mentioned domination parameters were combined to form another parameter called weakly connected 2-domination which was investigated by Militante and Eballe [7].

The complementary prism of a graph is a binary operation in graphs, denoted by $G\bar{G}$ which is formed from the disjoint union of a graph G and its complement $\bar{G}$ by adding a perfect matching between corresponding vertices. Several graph theoretic properties of complementary prisms, such as roman domination and k-independence were investigated by Alhashim *et al.* [1] and Otávio and Cappelle [8], respectively. Edge and vertex deletions are graph unary operations obtained by removing some edges and vertices. Lemańska [5] shows that removing an edge from a graph can increase the weakly connected domination number by at most one. In particular, it was claimed that the weakly connected domination number of the reduced graph obtained from removing some edges is greater than or equal to its weakly connected domination number and no more than the sum of the weakly connected domination number and one.

In this paper, we characterize the weakly connected 2-dominating sets in the complementary prisms of graphs and obtain the corresponding weakly connected 2-domination numbers of complementary prisms of some families of graphs. Also, we present in this paper inequalities for the weakly connected 2-domination numbers of graphs G and $G-e$, where $G-e$ is a connected subgraph of G obtained by deleting an edge e. Moreover, the parameter was examined for other unary operations such as vertex deletion and line graphs. For this work, we examine the graph G in terms of being finite, simple, undirected and connected. For other graph theoretic terms, the reader may refer to Chartrand *et al.* [2].

2. Preliminary Notes

Some definitions useful in this paper are presented in this section.

Definition 2.1 ([2])**.** The complement $\bar{G}$ of a graph G is a graph with vertex set $V(G)$ such that two vertices are adjacent in $\bar{G}$ if and only if these two vertices are not adjacent in G.

Definition 2.2 ([1])**.** For a graph $G = (V(G), E(G))$, the complementary prism, denoted by $G\bar{G}$ is formed from the disjoint union of G and its complement $\bar{G}$ by adding a perfect matching between corresponding vertices of G and $\bar{G}$.

For each $v \in V(G)$, let $\bar{v}$ denote the vertex corresponding to v in $\bar{G}$. Formally, the graph $G\bar{G}$ if formed from $G \cup \bar{G}$ by adding the edge $v\bar{v}$ for every $v \in V(G)$.

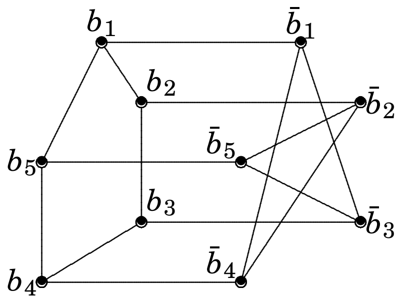

Figure 1. The Petersen graph is a complementary prism $C_5\bar{C}_5$, of a cycle graph C_5

Definition 2.3 ([2]). For a vertex v and an edge e in a nonempty graph $G = (V(G), E(G))$, the subgraph $G-v$, obtained by deleting v from G, is the induced subgraph $\langle V(G)-\{v\}\rangle$ of G and the subgraph $G-e$, obtained by deleting e from G, is the spanning subgraph of G with edge set $E(G)-e$.

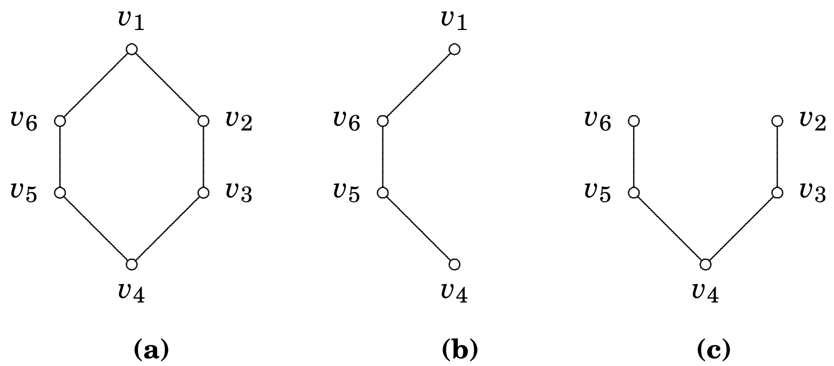

Figure 2. (a) The cycle graph $G = C_6$, (b) the subgraph of C_6 obtained by deleting v_2 and v_3, and (c) the subgraph of C_6 obtained by deleting edges v_1v_6 and v_1v_2

Definition 2.4 ([2]). The line graph $L(G)$ of a graph G is a graph whose vertices can be put in one-to-one correspondence with the edges of G in such a way that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent.

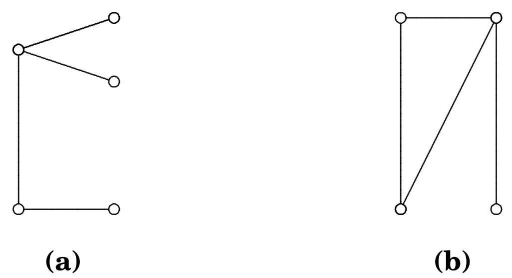

Figure 3. (a) A connected graph G and its (b) line graph $L(G)$

Definition 2.5 ([4]). A subset S of $V(G)$ is called a 2-*dominating set* of G if, for every vertex $v \in V(G) \setminus S$, there exists at least two vertices in S adjacent to v, that is, $|N_G(v) \cap S| \geq 2$. The 2-*domination number*, $\gamma_2(G)$, of G is the smallest cardinality of a 2-dominating set in G. Any dominating set in G whose cardinality is equal to $\gamma_2(G)$, is called a γ_2-set in G.

Definition 2.6 ([3]). Let G be a graph. A subset D of $V(G)$ is called *weakly connected* if the subgraph $\langle D\rangle_w = (N_G[D], E_w)$ weakly induced by D is connected, where E_w consists of edges in G with at least one of its end-vertices found in D and $N_G[D]$ is the closed neighborhood of D.

Definition 2.7 ([6]). Let G be a nontrivial connected graph. A subset $D \subseteq V(G)$ of a graph G, is called a *weakly connected* 2-*dominating set* (WC2D-set) in G if D is a 2-dominating set in G and the subgraph weakly induced by D is connected. The *weakly connected* 2-*domination number* of G, denoted by $\gamma_{2w}(G)$, is the smallest cardinality of a weakly connected 2-dominating set in G. A weakly connected 2-dominating set $D \subseteq V(G)$ with $|D| = \gamma_{2w}(G)$ is called a γ_{2w}-set in G.

Theorem 2.8 ([6])**.** (i) Let G be a connected graph of order at least 2. If D is a weakly connected 2-dominating set of G and x is a leaf of G, then $x \in D$.

(ii) Let P_n be a path graph of order $n \geq 2$. Then $\gamma_{2w}(P_n) = \lceil \frac{n+1}{2} \rceil$.

(iii) Let C_n be a cycle graph of order $n \geq 3$. Then $\gamma_{2w}(C_n) = \lceil \frac{n}{2} \rceil$.

(iv) Let G be a connected graph of order $n \geq 2$. Then $2 \leq \gamma_{2w}(G) \leq n$.

3. Main Results

This section presents the characterization for the weakly connected 2-dominating sets in the complementary prism of graphs, and the weakly connected 2-domination numbers of the complementary prism of some families of graphs. Some findings are also demonstrated on the weakly connected 2-domination numbers of graphs resulting from eliminating certain edges and vertices of a graph and some line graphs.

3.1 Complementary Prism

Theorem 3.1. Let G be any graph of order n. Then the complementary prism $G\bar{G}$ is a connected graph having $2n$ vertices.

Proof. If $n = 1$, then $G\bar{G} \cong K_2$ which is connected graph having two vertices. If $n \geq 2$, let $u, v \in V(G\bar{G})$ and consider the following cases:

Case 1. $u, v \in V(G)$. Then either $uv \in E(G)$ or there exist $\bar{u}, \bar{v} \in V(\bar{G})$ such that $[u, \bar{u}, \bar{v}, v]$ is an $u - v$ path in $G\bar{G}$.

Case 2. $u \in V(G)$ and $v \in V(\bar{G})$. Then $uv \in E(G\bar{G})$ whenever $v = \bar{u}$. Suppose that $v = \bar{w}$, for some $w \in V(G) \setminus \{u\}$. Then either $[u, \bar{u}, \bar{w} = v]$ is an $u - v$ path in $G\bar{G}$ or $[u, w, \bar{w} = v]$ is an $u - v$ path in $G\bar{G}$.

Case 3. $u, v \in V(\bar{G})$. Then $uv \in E(\bar{G})$ or there exist $a, b \in V(G)$ so that $[u, a, b, v]$ is an $u - v$ path in $G\bar{G}$ where a and b are the corresponding vertices of u and v in $\bar{G}$, respectively.

Consequently, the three cases show that $G\bar{G}$ is a connected graph. Moreover, by definition of complementary prism, $|V(G\bar{G})| = 2|V(G)| = 2n$. □

Theorem 3.1 guarantees that the complementary prism of any graph is connected. The next result is a direct consequence which says that the complementary prism $G\bar{G}$ of G always has a WC2D-set.

Corollary 3.2. Let G be any graph. Then $G\bar{G}$ admits a weakly connected 2-dominating set.

Theorem 3.3. Every isolated vertex in G (or $\bar{G}$) is contained in any weakly connected 2-dominating set in $G\bar{G}$.

Proof. Let $D \subseteq V(G\bar{G})$ be a WC2D-set in $G\bar{G}$. If x is an isolated vertex in G (or $\bar{G}$, respectively), then $\deg_G(x) = 0$ (or $\deg_{\bar{G}}(x) = 0$). Thus, $\deg_{G\bar{G}}(x) = 1$ (or $\deg_{\bar{G}G}(x) = 1$). Hence, x is a leaf in $G\bar{G}$. By Theorem 2.8(i), $x \in D$. We conclude that every isolated vertex in G (or $\bar{G}$) belongs to every WC2D-set in $G\bar{G}$. □

Corollary 3.4. Let G be any graph of order n and $v \in V(G)$. If $\deg_G(v) = n - 1$, then $\bar{v}$ in $\bar{G}$ belongs to any WC2D-set in $G\bar{G}$.

Proof. Let D be a WC2D-set in $G\bar{G}$. Let $v \in V(G)$. Suppose $\deg_G(v) = n-1$. Then $\deg_{\bar{G}}(\bar{v}) = 0$, where $\bar{v}$ is the corresponding vertex of v in $G\bar{G}$. Thus, $\bar{v}$ is an isolated vertex in $\bar{G}$. By Theorem 3.3, $\bar{v} \in D$. □

Theorem 3.5. Let $G\bar{G}$ be the complementary prism of graph G. If D is a WC2D-set in $G\bar{G}$, then D must contain some vertices in G and some vertices in $\bar{G}$.

Proof. Let D be a weakly connected 2-dominating set in $G\bar{G}$. We need to show that $D \cap V(G) \neq \varnothing$ and $D \cap V(\bar{G}) \neq \varnothing$. Suppose $D \cap V(G) = \varnothing$. Then $D \subseteq V(\bar{G})$. Let $x \in V(G)$. Clearly, $x \notin V(G) \setminus D$ and $|N_{G\bar{G}}(x) \cap D| \leq 1$. The value 1 is attained if $\bar{x} \in D$. This contradicts the assumption that D is a 2-dominating set in $G\bar{G}$. Hence, $D \cap V(G) \neq \varnothing$. Similarly, if $D \cap V(\bar{G}) = \varnothing$ we would have a contradiction. Therefore, $D \cap V(\bar{G}) \neq \varnothing$. □

Theorem 3.6. For a graph G with complementary prism $G\bar{G}$, $\gamma_{2w}(G\bar{G}) = 2$ if and only if G is a trivial graph.

Proof. If $G = K_1$, then $G\bar{G} = P_2$ and $\gamma_{2w}(G\bar{G}) = \gamma_{2w}(P_2) = 2$. Next, assume that $\gamma_{2w}(G\bar{G}) = 2$. Let D be a $\gamma_{2w}(G\bar{G})$-set. By Theorem 3.5, $D = D_1 \cup D_2$, where $D_1 = D \cap V(G)$ and $D_2 = D \cap V(\bar{G})$. Let $|D_1| = |D_2| = 1$. Let $D_1 = \{v\}$. Then the vertex $\bar{v}$ must be in D_2 implying that $D_2 = \{\bar{v}\}$. It follows that $V(G) \setminus \{v\} = \varnothing$ because no vertex in $G\bar{G}$ can be 2-dominated by $D = \{v, \bar{v}\}$. Consequently, $G = K_1$. □

Corollary 3.7. For a graph G with complementary prism $G\bar{G}$, $\gamma_{2w}(G\bar{G}) = 2$ if and only if $G\bar{G} = K_2$.

Theorem 3.8. Let G be any graph. Then $D \subseteq V(G\bar{G})$ is a WC2D-set in $G\bar{G}$ if and only if one of the following holds:

(i) $D = V(G) \cup D'$, where D' is a dominating set in $\bar{G}$.

(ii) $D = D^* \cup V(\bar{G})$, D^* is a dominating set in G.

(iii) $D = D_1 \cup D_2$, where $D_1 = D \cap V(G)$ is a dominating set in G and $D_2 = D \cap V(\bar{G})$ is a dominating set in $\bar{G}$ such that either $\bar{v} \in D_2$ whenever $v \in V(G) \setminus D_1$ and $|N_{G\bar{G}}(v) \cap (D_1 \cup D_2)| = 1$; or $v \in D_1$ whenever $\bar{v} \in V(\bar{G}) \setminus D_2$ and $|N_{G\bar{G}}(v) \cap (D_1 \cup D_2)| = 1$. Moreover, $\bar{x} \in D_2$ whenever $x \notin D_1$ and $|N_{G\bar{G}}(\bar{x}) \cap (D_1 \cup D_2)| = 1$ or $x \in D_1$ whenever $\bar{x} \notin D_2$ and $|N_{G\bar{G}}(\bar{x}) \cap (D_1 \cup D_2)| = 1$.

Proof. Let D be a WC2D-set in $G\bar{G}$. By Theorem 3.6, $D \cap V(G) \neq \varnothing$ and $D \cap V(\bar{G}) \neq \varnothing$.
Consider the following cases:

Case 1. $D \cap V(G) = V(G)$. Then every vertex $x \in V(G\bar{G}) \setminus D$ is 1-dominated by vertex in G. Since D is a 2-dominating set there exists $y \in D \cap V(\bar{G})$ such that $xy \in E(G\bar{G})$. Since $x \in V(\bar{G})$, it follows that $D = V(G) \cup D'$, where $D' \subseteq V(\bar{G})$ and D' is a dominating set in $\bar{G}$.

Case 2. $D \cap V(\bar{G}) = V(\bar{G})$. Similarly to *Case* 1, we obtain $D = V(\bar{G}) \cup D^*$, where $D^* \subseteq V(G)$ and D^* is a dominating set in G.

Case 3. $D \cap V(G) \neq V(G)$ and $D \cap V(\bar{G}) \neq V(\bar{G})$. Suppose that at least one of $D \cap V(G)$ and $D \cap V(\bar{G})$ is not a dominating set, then there exists $x \in V(G\bar{G}) \setminus D$ such that $|N_{G\bar{G}}(x) \cap D| \leq 1$ which is a contradiction.

Now, let $v \in V(G) \setminus D_1$ and $|N_{G\bar{G}}(\bar{v}) \cap (D_1 \cup D_2)| = 1$. Since D is a 2-dominating set in $G\bar{G}$, $\bar{v}$ must be in D_2. Similarly, if $\bar{v} \in V(\bar{G}) \setminus D_2$ and $|N_{G\bar{G}}(v) \cap (D_1 \cup D_2)| = 1$, then v must be in D_1.

Conversely, suppose first that $D = V(G) \cup D'$, where D' is a dominating set in $\bar{G}$. Let $u \in V(G\bar{G}) \setminus D$. Then $u \in V(\bar{G}) \setminus D$. By adjacency of vertices in $G\bar{G}$, there exists $u^* \in V(G) \cap D$ such that $uu^* \in E(G\bar{G})$. Since D' is a dominating set in $\bar{G}$, there exists $w \in D'$ such that $uw \in E(\bar{G})$. Thus, $\{u^*, w\} \subseteq N_{G\bar{G}}(u) \cap D$. Hence, $|N_{G\bar{G}}(u) \cap D| \geq 2$. The subgraph $\langle D \rangle_w$ is a spanning subgraph of $G\bar{G}$ is a connected subgraph. As a consequence, D is a WC2D-set.

Secondly, suppose that $D = D^* \cup V(\bar{G})$, where D^* is a dominating set in G. Then similar to *Case* 1, D is a WC2D-set in $G\bar{G}$.

Finally, suppose that (iii) holds. Let $v \in V(G\bar{G}) \setminus D$. Suppose $v \in V(G) \setminus D_1$. If $|N_{G\bar{G}}(v) \cap D| \geq 2$, then we are done. Suppose $|N_{G\bar{G}}(v) \cap (D_1 \cup D_2)| = 1$. Then by statement given in (iii), that is $\bar{v} \in D_2$, we have $|N_{G\bar{G}}(v) \cap (D_1 \cup D_2)| \geq 2$. Therefore, D is a 2-dominating set in $G\bar{G}$. Similarly, if $u \in V(\bar{G}) \setminus D_2$, then D is a 2-dominating set in $G\bar{G}$.

Now, let $u, w \in D$. Consider the following cases:

Case 1. $u, w \in D_1$. Then either $uw \in E(G)$; or there exist $\bar{u}, \overline{w} \in V(\bar{G})$ such that $[u, \bar{u}, \bar{w}, w]$ is an $u - w$ path in $G\bar{G}$ where $\bar{w} \in D_2$; or there exist $\bar{u}, \bar{w} \in V(\bar{G}) \setminus D_2$ and $\bar{z} \in D_2$ such that $[u, \bar{u}, \bar{z}, \bar{w}, w]$ is an $u - w$ path in $G\bar{G}$.

Case 2. $u \in D_1$ and $w \in D_2$. Then either $uw \in E(G\bar{G})$ or there exist $\bar{u} \in V(\bar{G})$ such that $[u, \bar{u}, w]$ is an $u - w$ path in $G\bar{G}$.

Case 3. $u, w \in D_2$. This case is similar to *Case* 1. The three cases show that $\langle D \rangle_w$ is connected. Therefore, D is a WC2D-set in $G\bar{G}$.

This completes the proof. □

Corollary 3.9. Let G be any graph of order n. Then $\gamma_{2w}(G\bar{G}) \leq \min\{n + \gamma(G), n + \gamma(\bar{G})\}$.

Proof. Let D be a WC2D-set in $G\bar{G}$. By Theorem 3.8(i) and (ii), $\gamma_{2w}(G\bar{G}) \leq |D| = \min\{|V(G)| + \gamma(\bar{G}), |V(\bar{G})| + \gamma(G)\} = \min\{n + \gamma(\bar{G}), n + \gamma(G)\}$. □

Example 3.10. Consider the graph of a complementary prism of the complete graph K_5 in Figure 4(a). It can be checked that $\gamma_{2w}(K_5\bar{K}_5) = 6$. The figure shows that the upper bound in Corollary 3.9 is sharp. Strict inequality in Corollary 3.9 is attained when $G = C_5$. The graph of $C_5\bar{C}_5$ in Figure 4(b) is the Petersen graph and it can be checked that $\gamma_{2w}(C_5\bar{C}_5) = 4$.

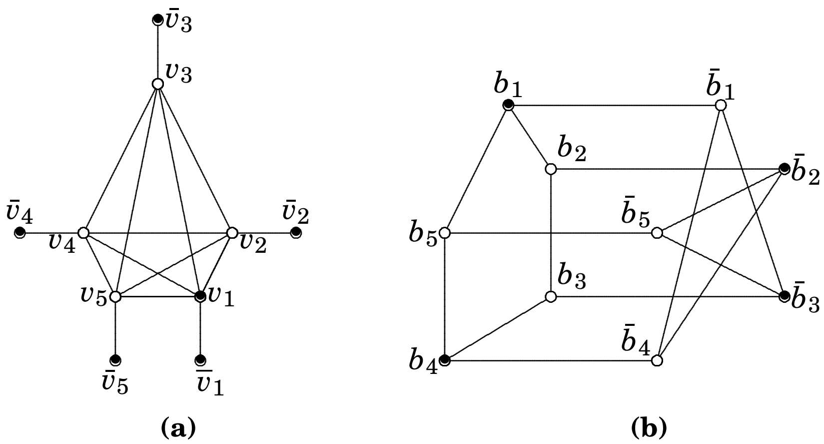

Figure 4. (a) The complementary prism $K_5\bar{K}_5$ of the complete graph K_5 and (b) the complementary prism $C_5\bar{C}_5$ of the cycle C_5 with darkened vertices in some γ_{2w}-sets

Theorem 3.11. If G is a star graph $K_{1,n-1}$ with $n \geq 3$, then $\gamma_{2w}(G\bar{G}) = n+1$.

Proof. Let D be a $\gamma_{2w}(G\bar{G})$-set. Since G is a star, then the support vertex v in G is an isolated vertex $\bar{v}$ in $\bar{G}$ and a leaf in $G\bar{G}$. By Theorem 2.8(i), $\bar{v} \in D$. Denote the set of $n-1$ leaves of G by $\{u_1, u_2, \ldots, u_{n-1}\}$. Since G is a star, $\gamma(G) = 1$ specifically, $\gamma(G)$-set is $\{v\}$. Thus, the leaves of G are dominated once by v. The corresponding leaf vertices in G will form a complete graph on $n-1$ vertices in $\bar{G}$. Consider the set $S = \{\bar{u}_1, \bar{u}_2, \bar{u}_3, \ldots, \bar{u}_{n-1}\} \cup \{v\} \cup \{\bar{v}\}$. This set forms a WC2D-set in $G\bar{G}$ since $\langle S \rangle_w = G\bar{G}$ and hence, $\gamma_{2w}(G\bar{G}) \leq |S| = (n-1)+1+1 = n+1$. Now, let D' be a γ_{2w}-set in $G\bar{G}$. The corresponding vertex $\bar{v}$ in $\bar{G}$ of the central vertex v in G is a leaf in $G\bar{G}$. Thus, by Theorem 2.8(i), $\bar{v} \in D'$. The $n-2$ vertices in $\bar{G}$ and the central vertex $\{v\}$ must be 2-dominated. We must also take the $n-1$ leaves of G and a vertex from a complete graph of $n-1$ vertices from $\bar{G}$. Therefore, $\gamma_{2w}(G\bar{G}) = |D'| \geq 1 + (n-1) + 1 = n+1$. Consequently, $\gamma_{2w}(G\bar{G}) = n+1$. □

Theorem 3.12. For complete graph K_n, where $n \geq 2$, $\gamma_{2w}(K_n\bar{K}_n) = n+1$.

Proof. The complementary prism of K_n is isomorphic to the corona of a complete graph of n vertices and the trivial graph K_1, that is, $K_n\bar{K}_n \cong K_n \circ K_1$. Consider $S = \{v_r, \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{n-1}, \bar{v}_n\} \subseteq V(K_n\bar{K}_n)$, where $\bar{v}_i$, $1 \leq i \leq n$ are the corresponding vertices of K_n in $K_n\bar{K}_n$ and $v_r \in V(K_n)$ for some r, $1 \leq r \leq n$. It follows that $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{n-1}, \bar{v}_n\} = V(\bar{K}_n)$. Let $u \in V(K_n\bar{K}_n) \setminus S$, then $u \in V(K_n)$. Thus, $uv_r \in E(K_n)$ and so, $uv_r \in E(K_n\bar{K}_n)$. By adjacency of vertices in $K_n\bar{K}_n$, there exist $\bar{v}_i = \bar{u} \in S$ for some i such that $u\bar{u} \in E(K_n\bar{K}_n)$. Thus, $|N_{K_n\bar{K}_n}(u) \cap S| \geq 2$ since $\{v_r, \bar{u}\} \subseteq N_{K_n\bar{K}_n}(u)$. Hence, S is a 2-dominating set in $K_n\bar{K}_n$. Every pair of vertices in S is joined by a path in $\langle S \rangle_w$, and so $\langle S \rangle_w$ is connected. Therefore, S is a WC2D-set and consequently, $\gamma_{2w}(K_n\bar{K}_n) \leq |S| = n+1$.

To prove the other direction of the inequality, let D^* be a γ_{2w}-set in $K_n\bar{K}_n$. The vertices in $\bar{K}_n$ are leaves of $K_n\bar{K}_n$. Thus, $V(\bar{K}_n) \subseteq D^*$. Since D^* is weakly connected $D \cap V(K_n) \neq \varnothing$. Hence, $|D^* \cap V(K_n)| \geq 1$. Therefore, $\gamma_{2w}(K_n\bar{K}_n) \geq |D^*| + 1 \geq n+1$. Consequently, $\gamma_{2w}(K_n\bar{K}_n) = n+1$. □

The next result is a consequence of Theorem 3.12 since it is easily observed that $\bar{\bar{G}}$ is G.

Corollary 3.13. If $G = \bar{K}_n$, $n \geq 2$, then $\gamma_{2w}(G\bar{G}) = n+1$.

Theorem 3.14. For any graph G of order at least 2, $\gamma_{2w}(G\bar{G}) \geq 3$.

Proof. Let G be a nontrivial graph. Let D be a $\gamma_{2w}(G\bar{G})$-set. Suppose $\gamma_{2w}(G\bar{G}) = 2$. By Theorem 3.5, $D \cap V(G) \neq \varnothing$ and $D \cap V(\bar{G}) \neq \varnothing$. Let $D_1 = D \cap V(G)$ and $D_2 = D \cap V(\bar{G})$, where $\{|D_1|, |D_2|\} = \{1,1\}$. Let $D_1 = \{u\}$ and $D_2 = \{v\}$. Then either $uv \in E(G\bar{G})$ or $uv \notin E(G\bar{G})$. If $uv \in E(G\bar{G})$, then there exists a vertex in $V(\bar{G}) \setminus \{u\}$ that is not 2-dominated by vertices in D. If $uv \notin E(G\bar{G})$, then there exist a vertex in $V(\bar{G}) \setminus \{v\}$ that is not 2-dominated by vertices in D. Either case leads to a contradiction. Therefore, $\gamma_{2w}(G\bar{G}) \geq 3$. □

Theorem 3.15. Let $G\bar{G}$ be the complementary prism of graph G. Then $\gamma_{2w}(G\bar{G}) = 3$ if and only if G is either a path of order 2 or the complement of a complete graph of two vertices.

Proof. If $G \in \{P_2, \bar{K}_2\}$, then by Theorem 3.14, $\gamma_{2w}(G\bar{G}) \geq 3$. Now, let $V(P_2) = \{u, v\}$ and $V(\bar{P}_2) = \{\bar{u}, \bar{v}\}$. Let $D = \{u, \bar{u}, \bar{v}\}$. Let D be a WC2D-set in the complementary prism $P_2\bar{P}_2$.

Thus, $\gamma_{2w}(G\bar{G}) = \gamma_{2w}(P_2\bar{P}_2) \le |D| = 3$. Also, let $V(\bar{K}_2) = \{a,b\}$ and $V(\bar{\bar{K}}_2) = V(K_2) = \{\bar{a},\bar{b}\}$. Take $D' = \{a,b,\bar{a}\}$. Then D' is a WC2D-set in $\bar{K}_2\bar{\bar{K}}_2$. Then $\gamma_{2w}(\bar{K}_2\bar{\bar{K}}_2) = \gamma_{2w}(\bar{K}_2K_2) \le |D'| = 3$. Thus, if $G \in \{P_2, \bar{K}_2\}$, then $\gamma_{2w}(G\bar{G}) \le 3$.

For the converse, let G be a graph such that $\gamma_{2w}(G\bar{G}) = 3$. Let D^* be a $\gamma_{2w}(G\bar{G})$-set. Let $D_1 = D^* \cap V(G)$ and $D_2 = D^* \cap V(\bar{G})$. Note that D_1 and D_2 are nonempty sets by Theorem 3.5. Since $|D_1 \cup D_2| = 3$, this means that either $|D_1| = 1$ and $|D_2| = 2$; or $|D_1| = 2$ and $|D_2| = 1$. Suppose first that $|D_1| = 1$ and $|D_2| = 2$. Let $D_1 = \{u\}$, then $\bar{u} \in D_2$ or u will not be 2-dominated. Now, let $\bar{v}$ be the other element in D_2. Notice that vertex v must be adjacent to u to be 2-dominated. Therefore, $V(G) = \{u,v\}$ and so $G = P_2$. Next, suppose that $|D_1| = 2$ and $|D_2| = 1$. Take $D_1 = \{x,y\}$, then either $\bar{x}$ or $\bar{y}$ belongs to D_2. Suppose $D_2 = \{\bar{x}\}$. Then $\bar{x}$ must be adjacent to x to be 2-dominated. Then $\bar{y}$ must be 2-dominated by y and $\bar{x}$. Thus, $\bar{y}$ is not adjacent to x. Since $\bar{x}\,\bar{y}$ is an edge in $\bar{G}$, it follows that $xy \notin E(G)$. Hence, $V(G) = \{x,y\}$ and so $G = 2K_1 \cong \bar{K}_2$. □

Theorem 3.16. Let $G\bar{G}$ of order $2n$ be the complementary prism of graph G. Then $G\bar{G}$ is bipartite if and only if $\gamma_{2w}(G\bar{G}) = 2$ or $\gamma_{2w}(G\bar{G}) = 3$.

Proof. Let $G\bar{G}$ be the complementary prism of G. Then $\gamma_{2w}(G\overline{G}) = 2$ if and only if $G = K_1$. Also, $\gamma_{2w}(G\bar{G}) = 3$ if and only if $G = K_2$ or $G = \bar{K}_2$. Thus, $\gamma_{2w}(G\bar{G}) = 2$ if and only if $G\bar{G} = K_2$ and $\gamma_{2w}(G\bar{G}) = 3$ if and only if $G\bar{G} = P_4$. Clearly, the two graphs K_2 and P_4 have no odd cycles. Hence, K_2 and P_4 are bipartite graphs.

Conversely, suppose that $G\bar{G}$ is bipartite, $G\bar{G}$ should have no odd cycles. This means that any two nonadjacent vertices in G or $(\bar{G})$ can not have a common neighbor. To see this, suppose that u and v are two nonadjacent vertices in G and x is a common neighbor of u and v. Then $\bar{u}\,\bar{v} \in E(\bar{G})$. Thus, $[u,x,v,\bar{v},\bar{u},u]$ is a 5-cycle in $G\bar{G}$. Also, G (or $\bar{G}$) must have at least two adjacent vertices, that is, G (or $\bar{G}$) must not be an empty graph since if it is the case $\bar{G}$ (or $\bar{\bar{G}} = G$) is a complete graph of order $n \ge 3$. So, $G\bar{G}$ contains an odd cycle. Thus, $|V(G)| \le 2$. It follows that $G\bar{G} = K_2$ or $G\bar{G} = P_4$. Consequently, $\gamma_{2w}(G\bar{G}) = 2$ or $\gamma_{2w}(G\bar{G}) = 3$. □

3.2 Edge and Vertex Deletion and Line Graphs

Theorem 3.17. Let G be a connected graph of order at least 3. If e is an edge in G with $G-e$ connected, then $2 \le \gamma_{2w}(G) \le \gamma_{2w}(G-e) \le \gamma_{2w}(G)+1$.

Proof. Let D be a WC2D-set in $G-e$. Let $v \in V(G-e)\setminus D$. Then $v \in V(G)\setminus D$. Since D is a 2-dominating set in $G-e$, $|N_{G-e}(v) \cap D| \ge 2$. Since $N_{G-e}(v) \subseteq N_G(v)$ it follows that $N_{G-e}(v) \cap D \subseteq N_G(v) \cap D$. Thus, $|N_G(v) \cap D| \ge |N_{G-e}(v) \cap D| \ge 2$. It follows that D is a 2-dominating set in G. Now, $D \subseteq V(G-e)$ and $E(G) = E(G-e) \cup \{e\}$. Since $\langle D_{G-e}\rangle_w$ is connected and all edges in $G-e$ are edges in G, $\langle D_G\rangle_w$ is connected. This means that D is a WC2D-set in G. Hence, $\{S : S$ is a WC2D-set in $G-e\} \subseteq \{T : T$ is a WC2D-set in $G\}$. It follows that $\min\{|T| : T$ is a WC2D-set in $G\} \le \min\{|S| : S$ is a WC2D-set in $G-e\}$. As a consequence, $\gamma_{2w}(G) \le \gamma_{2w}(G-e)$.

Next, let C be a γ_{2w}-set in G. That is, $\gamma_{2w}(G) = |C|$. Let $e = xy$. Consider the following cases:

Case 1. $x,y \in C$. Then $x,y \in V(G-e)$. Since $C \subseteq V(G-e)$ and C is weakly connected in G, subgraph $\langle C_{G-e}\rangle_w$ is connected. Since C is 2-dominating set in G and $C \subseteq V(G-e)$ it follows that $|N_{G-e}(z) \cap C| \ge 2$ for all $z \in V(G-e)\setminus C$. Thus, C is a 2-dominating set in $G-e$. Hence, C is a WC2D-set in $G-e$. Therefore, $\gamma_{2w}(G-e) \le |C| = \gamma_{2w}(G) \le \gamma_{2w}(G)+1$.

Case 2. $x \in C$ and $y \notin C$. Let $z \in V(G-e)\setminus C$. Consider the following cases:

Subcase 1. $|N_{G-e}(y) \cap C| \geq 2$. Since C is a 2-dominating set in G, $|N_G(y) \cap C| \geq 2$. In fact, $|N_{G-e}(z) \cap C| \geq 2$ for all $z \in V(G-e)\setminus C$. Thus, C is a 2-dominating set in $G-e$. Since $C \subseteq V(G-e)$ and $\langle C_G \rangle_w$ is connected, we have $\langle C_{G-e} \rangle_w$ is connected. Hence, C is a WC2D-set in $G-e$. Therefore, $\gamma_{2w}(G-e) \leq |C| = \gamma_{2w}(G) \leq \gamma_{2w}(G)+1$.

Subcase 2. $|N_G(y) \cap C| \geq 2$ but $|N_{G-e}(y) \cap C| = 1$. Since C is a 2-dominating set in G, $|N_G(z) \cap C| \geq 2$ for all $z \in V(G)\setminus C$. Since $C \subseteq V(G-e)$, $|N_{G-e}(z) \cap C| \geq 2$, for all $z \neq y$. Thus, $|N_{G-e}(z) \cap (C \cup \{y\})| \geq 2$ for all $z \in V(G-e)\setminus C$. It follows that $C \cup \{y\}$ is a 2-dominating set in $G-e$. Since $C \subseteq V(G-e)$ and C is weakly connected in G, $\langle C_{G-e} \rangle_w$ is connected. Hence, $C \cup \{y\}$ is a WC2D-set in $G-e$. Therefore, $\gamma_{2w}(G-e) \leq |C \cup \{y\}| = |C|+1 = \gamma_{2w}(G)+1$.

Case 3. $x \notin C$ and $y \in C$. This case is analogous to *Case* 2. Hence, we omit the proof.

Case 4. $x, y \notin C$. Let $w \in V(G)\setminus C$. Since C is a 2-dominating set in G.$|N_G(w) \cap C| \geq 2$ for all $w \in V(G)\setminus C$. Since $C \subseteq V(G-e) = V(G)$ and $x, y \notin C$, $|N_{G-e}(z) \cap C| \geq 2$ for all $z \in V(G-e)\setminus C$. Thus, C is a 2-dominating set in $G-e$.

The subgraph $\langle C_{G-e} \rangle_w$ weakly induced by C in $G-e$ is connected. It follows that C is a WC2D-set in $G-e$. Hence, $\gamma_{2w}(G-e) \leq |C| = \gamma_{2w}(G) \leq \gamma_{2w}(G)+1$. The four cases above show that $\gamma_{2w}(G) \leq \gamma_{2w}(G)+1$. Consequently, by Theorem 2.8(iv), $2 \leq \gamma_{2w}(G) \leq \gamma_{2w}(G-e) \leq \gamma_{2w}(G)+1$. □

Theorem 3.18. Let e be an edge of a connected graph G of order at least 3 with $G-e$ connected. If $\deg_G(v) = 2$ and v is incident to e, then v is contained in any WC2D-set in $G-e$.

Proof. Let D be a WC2D-set in $G-e$ and let $v \in V(G)$ such that $\deg_G(v) = 2$ and v is incident to e. Then $\deg_{G-e}(v) = 1$. It means that v is a leaf in $G-e$. By Theorem 2.8(i), $v \in D$. □

Theorem 3.19. For positive integer $n \geq 3$ and for an edge e in a complete graph K_n and cycle graph C_n, then

(i) $\gamma_{2w}(K_n - e) = 2$.

(ii) $\gamma_{2w}(C_n - e) = \lceil \frac{n+1}{2} \rceil$.

Proof. (i) Let $e = xy$ be an edge in K_n and $D = \{x, y\}$. Then $|N_{K_n - e}(v) \cap D| \geq 2$ for every $v \in V(K_n - e)\setminus D$. By adjacency of vertices in $V(K_n - e)$, $[x, v, y]$ is an $x-y$ path. Thus, $\langle D \rangle_w$ is connected. It follows that D is a WC2D-set in $K_n - e$. Hence, $\gamma_{2w}(K_n - e) \leq |D| = 2$. Therefore, $\gamma_{2w}(K_n - e) = 2$.

(ii) $C_n - e \cong P_n$ where P_n is a path of order n. By Theorem 2.8(ii), $\gamma_{2w}(C_n - e) = \gamma_{2w}(P_n) = \lceil \frac{n+1}{2} \rceil$. □

Theorem 3.20. Let G be a connected graph of order at least 3. Construct H from G by joining the end-vertex v in G by a set of isolated vertices $\{x_1, x_2, \ldots, x_{r-1}, x_r\}$, $r \geq 2$. Then $\gamma_{2w}(H) \leq \gamma_{2w}(G)+r$.

Proof. Let D be the minimum WC2D-set in G. Then $\gamma_{2w}(G) = |D|$. Clearly, $D \cup \{x_1, x_2, \ldots, x_{r-1}, x_r\}$ is a WC2D-set in H. Then $\gamma_{2w}(H) \leq |D \cup \{x_1, x_2, \ldots, x_{r-1}, x_r\}| = |D| + |\{x_1, x_2, \ldots, x_{r-1}, x_r\}| = |D| + r = \gamma_{2w}(G)+r$. □

Corollary 3.21. Let G be a connected graph of order at least 4. If v is a leaf in G, then $\gamma_{2w}(G-v)=\gamma_{2w}(G)-1$.

Theorem 3.22. Let $n \geq 3$ be an integer and let K_n and P_n be complete and path graphs. Then

(i) for any vertex v, $\gamma_{2w}(K_n - v) = 2$;

(ii) for a leaf vertex v, $\gamma_{2w}(P_n - v) = \lceil \frac{n}{2} \rceil$, where v is a leaf of P_n.

Proof. To prove (i), we have $K_n - v \cong K_{n-1}$. Thus, $\gamma_{2w}(K_n - v) = \gamma_{2w}(K_{n-1}) = 2$. For (ii), we have $P_n - v \cong P_{n-1}$. Thus, by Theorem 2.8(ii), $\gamma_{2w}(P_n - v) = \gamma_{2w}(P_{n-1}) = \lceil \frac{(n-1)+1}{2} \rceil = \lceil \frac{n}{2} \rceil$. □

Theorem 3.23. The weakly connected 2-domination numbers of corresponding line graphs of two isomorphic graphs are equal. But the converse is not necessarily true.

Proof. Let G_1 and G_2 be two isomorphic graphs. Then $L(G_1) \cong L(G_2)$. Clearly, $\gamma_{2w}(L(G_1)) = \gamma_{2w}(L(G_2))$. But the converse is not true. To see this, consider the two graphs $K_{1,3}$ and K_3, the star graph of order 4, and the complete graph of order 3, respectively. The two graphs are not isomorphic. Moreover, $\gamma_{2w}(L(K_{1,3})) = 2 = \gamma_{2w}(L(K_3))$. □

Theorem 3.24. For integer $n \geq 3$ and for line graphs $L(P_n)$, $L(C_n)$, and $L(K_{1,n})$ of path P_n, cycle C_n and star $K_{1,n}$, respectively, then

(i) $\gamma_{2w}(L(P_n)) = \lceil \frac{n}{2} \rceil$;

(ii) $\gamma_{2w}(L(C_n)) = \lceil \frac{n}{2} \rceil$;

(iii) $\gamma_{2w}(L(K_{1,n})) = 2$.

Proof. For (i), $L(P_n) \cong P_{n-1}$. Thus, by Theorem 2.8(ii), $\gamma_{2w}(L(P_n)) = \gamma_{2w}(P_{n-1}) = \lceil \frac{n}{2} \rceil$. To prove (ii), $L(C_n) \cong C_n$. Thus, by Theorem 2.8(iii), $\gamma_{2w}(L(C_n)) = \gamma_{2w}(C_n) = \lceil \frac{n}{2} \rceil$ and finally for (iii), $L(K_{1,n}) \cong K_n$. Thus, $\gamma_{2w}L((K_{1,n})) = \gamma_{2w}(K_n) = 2$. □

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

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