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Research Article

Inventory Optimization Model for Deteriorating Items under Inventory Follows Shortage (IFS) and Shortage Follows Inventory (SFI) Policies

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Abstract. This study focuses on optimizing inventory control for perishable commodities using preservation technologies, where payment delays and shortages are acceptable. Implementing correct preservation technology can help retailers reduce the negative impact of product deterioration on their earnings. We examine preservation technology as well as the permitted payment delay and aim to minimize the total cost under two different policies: SFI and IFS. The flow of inventory is quantitatively described for both policies using dynamic differential equations and the necessary boundary conditions. The decision variables' values are determined using the derivative method of calculus. Furthermore, the optimum values of the Total cost function satisfy the Hessian matrix requirement, confirming its convexity. Based on all the results of the study we compare both the models for optimum Total Cost and found out the most and least affecting parameters for this model. The key finding of this study is the comparison of these models under two different policies to identify which policy is better to minimize the total cost for the retailers. We observed that IFS performs well to optimize the total cost. Numerical examples are also provided to show the practical use of the proposed model. Given the presence of preservation technology, we observed that the total cost in IFS policy is 6.81% higher than that of the case when it is absent, whereas in SFI the total cost is 5.29% higher than when it is not there. A table showing the effect of various parameters on the total cost function is provided and subsequent insights that are beneficial for retailers are also drawn and some unanswered queries are the highlights of this problem.

Keywords. SFI policy, IFS policy, Allowed shortages, Complete backlog, Permissible delay in payment, Preservation technology

Mathematics Subject Classification (2020). 90B05

1. Introduction

For businesses to cut costs and enhance customer service, inventory management has become crucial as they compete in the global market space. Businesses that manufacture and provide services, whether they are of large or small scale handle inventory. The products in inventory serve to improve the efficiency of the company's day-to-day operations and the flow of commodities. Lack of inventory results in other severe issues like lost sales and unhappy customers. On the other hand, an excessive amount of inventory limits the capital available for use in other aspects of the company's activities. In the absence of a potential source, radioactive materials, photographic film, grain, etc., deteriorate over time. When foods are stored for an extended period of time, direct spoiling causes them to lose nutrients. Also, items like gasoline, oil, alcohol, etc., are very perishable. As a result, the study of the deterioration or decay of physical commodities kept in stock is very practical. In this field, numerous researchers have worked and are still continuing their studies. In 1963, Ghare and Schrader [6] created the first exponential decay inventory model. By creating an EOQ model for perishable products, Covert and Philip [4] have made a significant contribution to the field. According to Covert and Philip, the rate of decline is deterministic and follows the Weibull distribution. Philip [18] extended the EOQ model by including the deterioration rate as a variable with a three-parameter Weibull distribution. An order-level inventory model was created by Mandal and Phaujdar [15] for degrading goods with a constant rate of production and stock-dependent demand.

Retailers make enough preparations for supplying such products by obtaining and storing them with the use of the most recent technology and facilities, as clients are constantly looking for new and fresh products from the market. Many authors have created inventory models to deal with such scenarios, offering suitable methods and answers for the suppliers as well as the retailers. This is because they consider this to be a significant intervening factor. The rate of deterioration is currently being stopped by researchers using preservation technology intervention strategies. Dye [5] in his paper discussed the joint problem where they optimize the replenishment policy and preservation technology investment cost. Dye [5] developed an inventory model for non-instantaneous deteriorating items in which they optimize the replenishment and preservation technology investment cost strategies. Hsu *et al.* [10] studied a policy that restricts the deteriorating inventory when a store makes investments in preservation technology to slow down product deterioration. In their study, they optimize the cycle time, shortage period, order quantity, and preservation technology investment cost. Mishra *et al.* [17] created a model with stock- and selling-price-dependent demand rates for EOQ. They consider two cases, one in which shortages are completely backlogged and the other in which the backlog of shortages is partial. It is also considered beneficial to invest in preservation technology in order to reduce the deterioration of products and to maximize profits. The study by Rahman *et al.* [19] demonstrated a hybrid inventory system that can be used for perishable items and under partial backlogs in a fixed ratio. Advance payment with a discount facility and preservation investment are also introduced. With the adoption of credit period policy, preservation technology, and permitted shortages, Tayal *et al.* [25] established an integrated production-distribution framework for degrading items within a two-echelon supply chain.

Sarker *et al.* [21] suggested a model for degrading inventory by optimizing cycle time and time frames for the payment when a supplier offers the buyer a predetermined credit period for payment without penalty. An iterative search procedure is applied to solve the problem. Goyal [9] constructed an EOQ model for an item under the conditions of allowable payment delays. Khanra *et al.* [13] introduced an inventory model based on the EOQ for perishable items with quadratic time-dependent demand under different conditions: when the credit period is shorter than cycle time, and when it is longer than cycle time. Aggarwal and Jaggi [1] formulated a model for inventory control to determine the optimum order quantity of deteriorating items under permissible delay in payments. Kaushik [12] developed a model with a limited time horizon for objects that deteriorate. His model examined two scenarios: one with and one without an acceptable payment delay. He also covered the best replenishment strategy for maximizing profit while taking into account two distinct interest rates.

Ghosh *et al.* [7] established an EOQ model for perishable product with price-dependent demand, and partial backlogging. He discussed and compared the two inventory policies SFI and IFS. Bhunia and Shaikh [2] investigated a two-warehouse inventory problem under inflation with different types of deterioration rate in both warehouses. The main objective of this study was to determine the optimal lot size under two different inventory policies, i.e., SFI and IFS. Shaikh *et al.* [22] developed a fuzzy inventory model for deteriorating items with partial backlogging and demand is influenced by selling price and advertising frequency. They applied the SFI policy in their study and used fuzzy numbers' nearest interval approximation technique to solve the problem. Chen and Chen [3] and Iqbal *et al.* [11] also worked on the concept of SFI and IFS policies.

Demand plays a very important role in marketing. Market demand will certainly alter over time because it is constantly tied to a specific time frame. As a result, time is crucial in the management of inventory. In most inventory models, Giri and Chaudhuri [8], Min *et al.* [16], and Sarkar and Sarkar *et al.* [20] have shown that the amount of stock that is currently on hand has an impact on the demand rate. Demand for some things increases up to a certain point, reaches saturation, and then tends to stabilize. Ramp-type demand is the term used to describe this type of demand rate. Mandal and Pal [14], Sharma *et al.* [23], and Skouri *et al.* [24] have developed models in this direction. For instance, the demand for seasonal/fashionable goods like apparel, footwear, children's toys, and electronics, etc., exhibits this behavior. Having recognized this need, we have incorporated it into our model. We have made several significant additions to our model to make it distinctive and better suited to business requirements in contrast to the research models.

The following features of our model, which highlight its novelty are:

- To reduce the rate of deterioration and the effects of degradation, we have considered preservation technology.
- Also, we evaluated the model under two different policies, i.e., SFI and IFS, and analysed its effect on the model's overall cost. Additionally, we have permitted shortages in the model with an entirely backlogged case.
- We compare the total cost for these models and suggest which policy is better for the retailer under the given scenario of the model.
- Further, a comparison of the model under two different policies in the case of with and without preservation technology is also presented using graphs.

2. Notations and Assumptions

2.1 Notations

θ	Deterioration rate ($0 < \theta < 1$)
ξ	Cost of Preservation Technology (PT), which lowers the pace of product's deterioration, $\xi \geq 0$
a, b, c	Parameters of price-dependent demand ($a > 0, b, c$ are non-zero and constant)
a_1, a_2, a_3	Parameters of Quadratic time-dependent demand (a_1, a_2, a_3 are positive constants)
$d(\xi)$	Decreased rate of deterioration as a function of ξ
$D_1(t)$, $D_2(t)$	Demand Rates
h	Carrying cost to hold a unit item (constant)
I_0	Initial stock level at the beginning of the inventory cycle
I_1	Earned rate of interest ($/week)
I_2	Rate applied to finance unsold inventory
K	Ordering Cost ($/order)
m	Permitted time to finish the prepayment (in week)
p	Per unit acquisition price ($/item)
P	Selling Price ($ per unit)
R	Maximum Backlogged quantity
s	Shortage cost ($/item)
T	Inventory Cycle Time (time unit)
t_1	Time at which shortage occurred (time unit)
t_d	Time at which deterioration starts decay commencing time
TC	Total cost per unit time for the inventory procedure
v_1	Unit cost incurred from the deterioration of one item in IFS condition ($/unit/unit time)
v_2	Unit cost incurred from the deterioration of one item in SFI condition ($/unit/unit time)

2.2 Assumptions

For the development of both models, the following common presumptions are made:

(i) The inventory system is limited to a single item.

(ii) Shortages are taken into account and are backlogged completely.

(iii) Permissible delays in payments are included.

(iv) There will be no interest charged after the shortages begin.

(v) Beyond the permissible hold-off periods, no interest is to be earned.

(vi) The deterioration rate is constant and is given by θ.

(vii) There is a planning period of infinite length.

3. Mathematical Model Formulation

3.1 IFS Model Derivation

According to the *Inventory Follows Shortages* (IFS) model:

(i) This is the first stage in the model where the linear form of the price-dependent and time-dependent functions has been taken. We have taken a quadratic function at the second stage for the time interval $[t_1, T]$ with an initial inventory level I_0, whereas the linear function has been taken at the first stage for the initial inventory level I_0.

(ii) The first stage of depreciating items involved evaluating them as shown in Figure 1 over the time period $[0, T_1)$. Because of shortages in inventory, depreciated goods were ignored in the second stage of the process as shown in Figure 1 during the time interval $[t_1, T]$.

(iii) The cost of a shortage has been calculated in the second step of Figure 1 throughout the time span $[t_1, T]$. It is zero, however, at the beginning since inventory exists.

(iv) To describe a drop in inventory and a shortage, respectively, negative signs are placed in front of the demand functions while solving the differential equation.

(v) Preservation technologies are used to reduce the deterioration rate.

(vi) Due to the occurrence of inventory stock-out, holding costs were taken into consideration in the first stage of Figure 1 but were avoided in the second stage.

(vii) We have considered a credit period for permitted payment delays. During the time period $[0, m]$, positive stock is present in Figure 1.

(viii) The stock which remains unsold over the interval $[m, t_1)$ in Figure 1, a rate of interest E_2 is applied to it, post to the credit period $[0, m]$.

To satisfy market demand, the inventory level $I(t)$ typically declines from I_0 due to demand, and the product's deterioration at any time t reaches zero at t_1. As a result, shortages build up over time $[t_1, T]$. Consequently, the governing differential equation can be used to represent the variation of inventory with respect to time:

$$\frac{dI(t)}{dt} + (\theta - d)t = -(a - bP + ct), \quad I(0) = I_0, \ 0 \le t < t_1. \tag{3.1}$$

Negative sign before the demand function in eq. (3.1) shows the decrement in inventory

$$I(t) = \frac{-(a-bP)}{(\theta-d)} - \frac{ct}{(\theta-d)} + \frac{c}{(\theta-d)^2} + \left(I_0 - \frac{c}{(\theta-d)^2} + \frac{a-bP}{(\theta-d)}\right)\exp^{-(\theta-d)t}. \tag{3.2}$$

Also, during the time interval $[t_1, T]$, we have

$$\frac{dI(t)}{dt} = -(a_1 + a_2 t + a_3 t^2), \quad I(T_1) = 0, \ t_1 \le t < T, \tag{3.3}$$

$$I(t) = a_1(t_1 - T) + \frac{a_2}{2}(t_1^2 - T^2) + \frac{a_3}{3}(t_1^3 - T^3). \tag{3.4}$$

Also, the total inventory from eq. (3.2) an eq. (3.4), we get

$$\begin{aligned} Q(t) = {} & \frac{-(a-bP)}{(\theta-d)} - \frac{ct}{(\theta-d)} + \frac{c}{(\theta-d)^2} + \left(I_0 - \frac{c}{(\theta-d)^2} + \frac{a-bP}{(\theta-d)}\right)\exp^{-(\theta-d)t} \\ & + a_1(t_1 - T) + \frac{a_2}{2}(t_1^2 - T^2) + \frac{a_3}{3}(t_1^3 - T^3). \end{aligned} \tag{3.5}$$

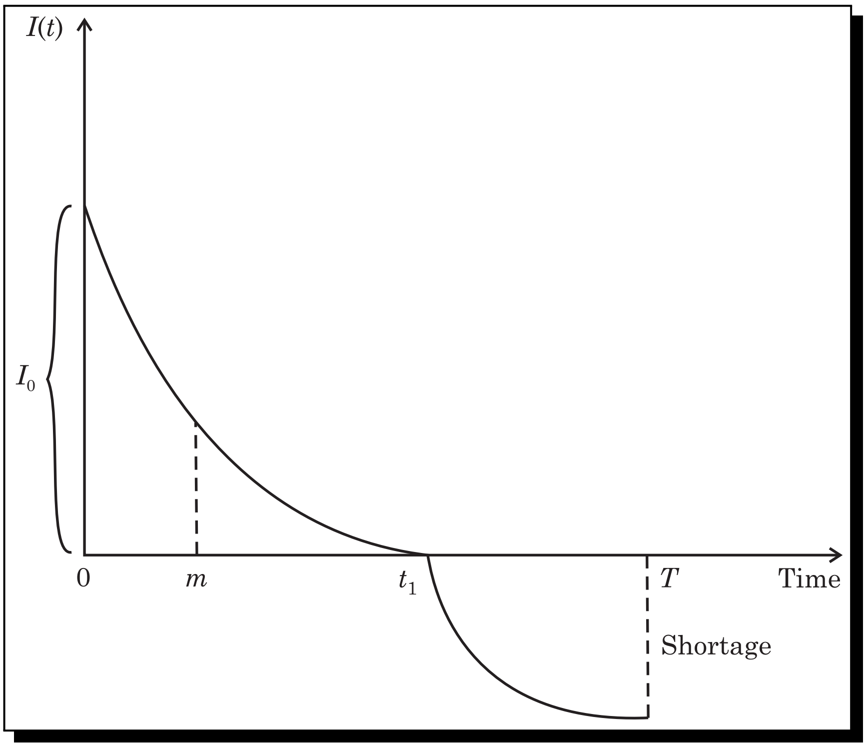

Figure 1. Flow of inventory [11] in IFS policy during the interval $[0,T]$

The total number of deteriorated items is given by

$$I_0 - \int_0^{t_1}(a - bP + c)dt = I_0 - \left((a-bP)t_1 - \frac{ct_1^2}{2}\right).$$

Also, the cost incurred in deterioration is as

$$\text{Deterioration Cost} = v_1\left(I_0 - \left((a-bP)t_1 - \frac{ct_1^2}{2}\right)\right). \tag{3.6}$$

With the help of eq. (3.3), The stock-out cost is calculated as

$$\begin{aligned}\text{Shortage Cost} &= -s\int_{t_1}^{T} I(t)dt \\ &= s\left(a_1\left(t_1T - \frac{T^2}{2}\right) - \frac{a_2}{2}\left(t_1^2T - \frac{T^3}{3} - \frac{2t_1^3}{3}\right) - \frac{a_3}{3}\left(t_1^3T - \frac{T^4}{4} - \frac{3t_1^4}{4}\right)\right).\end{aligned} \tag{3.7}$$

Now, the cost of maintaining items is given by

$$\begin{aligned}\text{Holding Cost} &= h\int_0^{t_1} I(t)dt \\ &= h\left(\frac{-(a-bP)}{(\theta-d)}t_1 - \frac{ct_1^2}{2(\theta-d)} + \frac{ct_1}{(\theta-d)^2} + \left(I_0 - \frac{c}{(\theta-d)^2}\right.\right. \\ &\qquad \left.\left. + \frac{a-bP}{(\theta-d)}\left(\frac{-1}{(\theta-d)}\right)\right)(\exp^{-(\theta-d)t_1} - 1)\right).\end{aligned} \tag{3.8}$$

Further, Preservation Technology Cost is given by

$$\text{Preservation Technology Cost} = \xi T.$$

With the sales income, a buyer can earn interest of E_1 in $[0,m]$ using I_1 as a rate of return by using the credit term authorized delay-payment throughout the time $[0,m]$. The whole-seller or distributor determines the allowable delay period m for the retailer's store or consumer.

$$\begin{aligned} E_1 &= pI_1 \int_0^m t(a-bP+ct)dt \\ &= pI_1\left((a-bP)\frac{m^2}{2}+\frac{m^3}{3}\right). \end{aligned} \tag{3.9}$$

After the credit period is over, the unsold stock is intended to be financed at a rate of I_2 during $[m,t_1]$ and is represented as E_2 which can be calculated as

$$\begin{aligned} E_2 &= pI_2 \int_m^{t_1} I(t)dt \\ &= pI_2 \int_m^{t_1} \frac{-(a-bP)}{(\theta-d)} - \frac{ct}{(\theta-d)} + \frac{c}{(\theta-d)^2} + \left(I_0 - \frac{c}{(\theta-d)^2} + \frac{a-bP}{(\theta-d)}\right)\exp^{-(\theta-d)t} dt \\ &= pI_2 \left[\frac{-(a-bP)}{(\theta-d)}(t_1-m) - \frac{c}{2(\theta-d)}(t_1^2-m^2) \right. \\ &\quad \left. + \frac{c}{(\theta-d)^2} + \left(I_0 - \frac{c}{(\theta-d)^2} + \frac{a-bP}{(\theta-d)}\right)\left(\frac{-1}{(\theta-d)}\right)\right](\exp^{-(\theta-d)t_1} - \exp^{-(\theta-d)m}). \end{aligned} \tag{3.10}$$

Using the following equation, we can calculate the total average cost per unit time

$$\begin{aligned} U(T,T_1) = \frac{1}{T}(&\text{Ordering Cost} + \text{Holding Cost} + \text{Shortage Cost} \\ &+ \text{Deterioration Cost} + E_2 - E_1 + \text{Preservation Technology Cost}). \end{aligned}$$

3.2 Related Theorem [11]

If a function $U(T,t_1) = \frac{1}{T}H(T,t_1)$, where $H(T,t_1)$ have continuous second order partial derivatives, and has minimum value at $t_1 = t_1^*$, $T = T^*$. Also, if each principal minors of the Hessian matrix are positive definite, i.e., if $\frac{\partial^2 H}{\partial T^2} > 0$, $\frac{\partial^2 H}{\partial t_1^2} > 0$, and

$$\begin{vmatrix} \frac{\partial^2 H}{\partial T^2} & \frac{\partial^2 H}{\partial T \partial t_1} \\ \frac{\partial^2 H}{\partial T \partial t_1} & \frac{\partial^2 H}{\partial t_1^2} \end{vmatrix} > 0. \tag{3.11}$$

By putting $\frac{\partial H}{\partial T} = 0$, $\frac{\partial H}{\partial t_1} = 0$, we get a system of equations in two variables. The solution of this system of simultaneous equations gives an optimal value of T and t_1, which fulfills the criteria of the second-order derivative and Hessian matrix given in eq. (3.11), proving the convexity of the total average cost function.

3.3 Derivation of SFI Model

According to the policy Shortage Follows Inventory (SFI) model, the inventory cycle is such as:

- Assuming the beginning inventory level is zero (accumulation of shortages), while the non-linear function, i.e., quadratic function has been considered at the second stage over the time interval $[t_1,T]$ and with I_0 inventory level after instant replenishment. During the stage first of the model over the time interval $[0,t_1)$ a linear form of price-dependent and time-dependent function has been taken as shown in Figure 2.

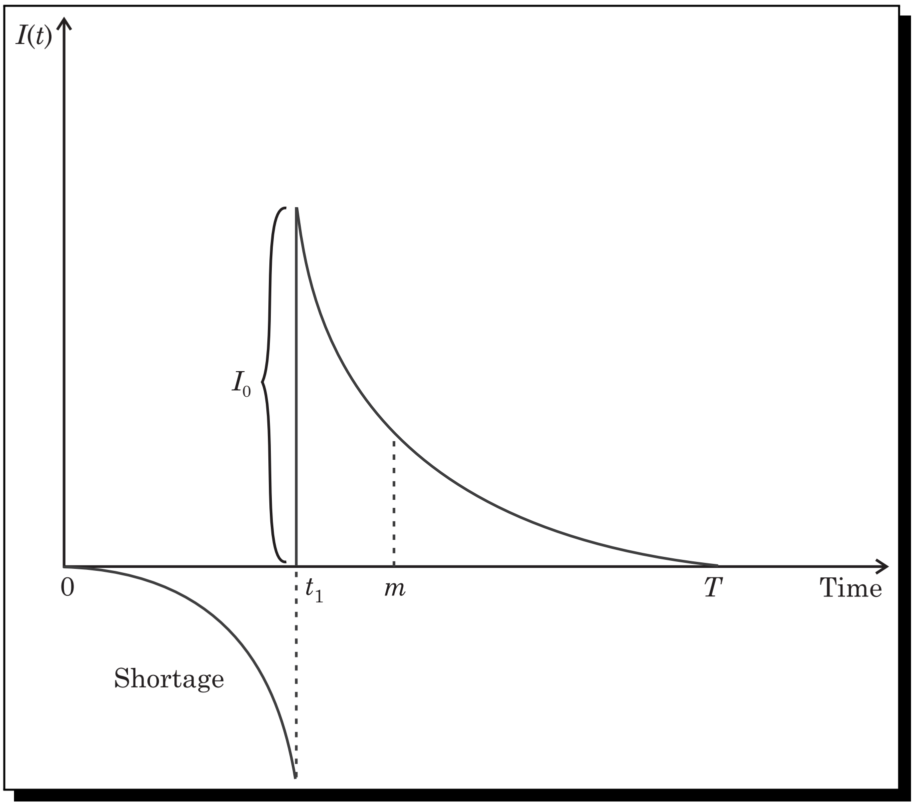

Figure 2. Flow of inventory [11] in SFI policy during the interval $[0,T]$

- In Figure 2, degraded items were evaluated in the second stage across the period $[t_1,T]$, whereas they were ignored in the first stage.
- In Figure 2, we calculated the stock-out cost over the initial time interval $[0,t_1]$. As inventory exists, it cannot be applied in the second stage as there is no stock in the storage space.
- To describe the shortage or decrease in inventory, demand functions in differential equations are preceded by negative signs. The cost of holding inventory was considered in the second stage of Figure 2 but avoided in the first stage due to inventory limitations.
- Preservation technologies are used to reduce the deterioration rate.
- In Figure 2, the credit term for allowable payment delays has been considered over the period $[t_1,m]$. Over the period $[m,T]$ in Figure 2 that is past the credit period $[t_1,m]$, the unsold stock is financed at an annual rate of I_2,

$$\frac{dI(t)}{dt} = -(a-bP+ct), \quad I(0)=0,\ 0\le t<t_1, \tag{3.12}$$

$$\frac{dI(t)}{dt} + (\theta-d)I(t) = -(a_1+a_2t+a_3t^2), \quad t_1\le t\le T \text{ and } I(t_1)=I_0. \tag{3.13}$$

From eq. (3.12), we get the value of $I(t)$ as

$$I(t) = -(a-bP)t - \left(\frac{ct^2}{2}\right) + c_1. \tag{3.14}$$

A negative sign here indicates a shortage of inventory and a negative sign in eq. (3.13) indicates a decrease in inventory at any instant of time t. Using the boundary conditions as mentioned in eq. (3.12) we get

$$I(t) = -(a-bP)t - \frac{ct^2}{2}. \tag{3.15}$$

From eq. (3.13), we get the value of $I(t)$ as

$$I(t) = \exp^{(\theta-d)(t_1-t)}\left(I_0 + \frac{a_1}{(\theta-d)} + \frac{a_2}{(\theta-d)}t_1 - \frac{a_2}{(\theta-d)^2} + \frac{a_3}{(\theta-d)}t_1^2 - \frac{2a_3t_1}{(\theta-d)^2} + \frac{2a_3}{(\theta-d)^3}\right)$$
$$- \frac{a_1}{(\theta-d)} - \frac{a_2}{(\theta-d)}t + \frac{a_2}{(\theta-d)^2} - \frac{a_3}{(\theta-d)}t^2 + \frac{2a_3t}{(\theta-d)^2} - \frac{2a_3}{(\theta-d)^3}. \tag{3.16}$$

Therefore, the total inventory from eq. (3.15) and solution of eq. (3.13) is given by

$$Q(t) = -(a-bP)t - \frac{ct^2}{2}$$
$$+\exp^{(\theta-d)(t_1-t)}\left(I_0 + \frac{a_1}{(\theta-d)} + \frac{a_2}{(\theta-d)}t_1 - \frac{a_2}{(\theta-d)^2} + \frac{a_3}{(\theta-d)}t_1^2 - \frac{2a_3t_1}{(\theta-d)^2} + \frac{2a_3}{(\theta-d)^3}\right)$$
$$- \frac{a_1}{(\theta-d)} - \frac{a_2}{(\theta-d)}t + \frac{a_2}{(\theta-d)^2} - \frac{a_3}{(\theta-d)}t^2 + \frac{2a_3t}{(\theta-d)^2} - \frac{2a_3}{(\theta-d)^3}. \tag{3.17}$$

The total quantity of damaged articles is provided by

$$I_0 - \int_{t_1}^{T}(a_1+a_2t+a_3t^2)dt = I_0 - \left(a_1(T-t_1) + \frac{a_2(T^2-t_1^2)}{2} + \frac{a_3(T^3-t_1^3)}{3}\right). \tag{3.18}$$

Using the eq. (3.12) and eq. (3.15), we have

$$\text{Shortage Cost} = -s\int_0^{t_1} I(t)dt = s\left(\frac{(a-bP)t_1^2}{2} + \frac{ct_1^3}{6}\right). \tag{3.19}$$

Also, Preservation Technology Cost is given by

$$\text{Preservation Technology Cost} = \xi T.$$

For holding cost during the interval $[T_1, T]$, we have

Holding Cost

$$= h\int_{t_1}^{T} I(t)dt$$
$$= h\int_{t_1}^{T}\exp^{(\theta-d)(t_1-t)}\left(I_0 + \frac{a_1}{\theta-d} + \frac{a_2}{\theta-d}t_1 - \frac{a_2}{(\theta-d)^2} + \frac{a_3}{(\theta-d)}t_1^2 - \frac{2a_3t_1}{(\theta-d)^2} + \frac{2a_3}{(\theta-d)^3}\right)$$
$$- \frac{a_1}{(\theta-d)} - \frac{a_2}{(\theta-d)}t + \frac{a_2}{(\theta-d)^2} - \frac{a_3}{(\theta-d)}t^2 + \frac{2a_3t}{(\theta-d)^2} - \frac{2a_3}{(\theta-d)^3}dt$$
$$= h\left(\frac{a_1}{\theta-d}(t_1-T) + \frac{a_2(t_1^2-T^2)}{2(\theta-d)} + \frac{a_3(t_1^3-T^3)}{3(\theta-d)} + \frac{2a_3(t_1-T)}{(\theta-d)^3} + \frac{a_2(T-t_1)}{(\theta-d)^2} + \frac{a_3(T^2-t_1^2)}{(\theta-d)^2}\right)$$
$$+ \frac{1}{(\theta-d)}\left(I_0 + \frac{a_1}{(\theta-d)} + \frac{a_2t_1}{(\theta-d)} - \frac{a_2}{(\theta-d)^2} + \frac{a_3t_1^2}{(\theta-d)} - \frac{2a_3t_1}{(\theta-d)^2} + \frac{2a_3}{(\theta-d)^3}\right)$$
$$- \frac{1}{(\theta-d)}\left(I_0 + \frac{a_1}{(\theta-d)} + \frac{a_2t_1}{\theta-d} - \frac{a_2}{(\theta-d)^2} + \frac{a_3t_1^2}{(\theta-d)} - \frac{2a_3t_1}{(\theta-d)^2} + \frac{2a_3}{(\theta-d)^3}\right)(\exp^{(\theta-d)(t_1-T)}). \tag{3.20}$$

With the help of eq. (3.18), we can find the deterioration cost as

$$\text{Deterioration Cost} = v_2\left(I_0 - \left(a_1(T-t_1) + \frac{a_2(T^2-t_1^2)}{2} + \frac{a_3(T^3-t_1^3)}{3}\right)\right). \tag{3.21}$$

With the sales income, a buyer can earn interest of E_1 in $[t_1, m]$ using I_1 as a rate of return by using the credit term authorized delay-payment throughout the period of time $[t_1, m]$. The whole-seller or distributor determines the allowable delay period m for the retailer's store or consumer,

$$\begin{aligned} E_1 &= pI_1\int_{t_1}^{m} t(a_1 + a_2 t + a_3 t^2)dt \\ &= pI_1\left(\frac{1}{2}a_1(m^2 - t_1^2) + \frac{1}{3}a_2(m^3 - t_1^3) + \frac{1}{4}a_3(m^4 - t_1^4)\right). \end{aligned} \tag{3.22}$$

After the credit period expires, the stock that remains unsold is intended to be financed at an annual rate of I_2 after the credit period is over during $[m, T]$ and is represented as E_2 and can be calculated as follows

$$\begin{aligned} E_2 &= pI_2\int_m^T I(t)dt \\ &= pI_2\int_m^T \exp^{(\theta-d)(t_1-t)}\left(I_0 + \frac{a_1}{(\theta-d)} + \frac{a_2}{(\theta-d)}t_1 - \frac{a_2}{(\theta-d)^2} + \frac{a_3}{(\theta-d)}t_1^2 - \frac{2a_3t_1}{(\theta-d)^2} + \frac{2a_3}{(\theta-d)^3}\right) \\ &\quad - \frac{a_1}{(\theta-d)} - \frac{a_2}{(\theta-d)}t + \frac{a_2}{(\theta-d)^2} - \frac{a_3}{(\theta-d)}t^2 + \frac{2a_3t}{(\theta-d)^2} - \frac{2a_3}{(\theta-d)^3}dt \\ &= pI_2\left(\frac{a_1}{(\theta-d)}(m-T) + \frac{a_2}{2(\theta-d)}(m^2-T^2) + \frac{a_3}{3(\theta-d)}(m^3-T^3) + \frac{2a_3}{(\theta-d)^3}(m-T)\right. \\ &\quad \left. + \frac{a_2}{(\theta-d)^2}(T-m) + \frac{a_3}{(\theta-d)^2}(T^2-m^2)\right) \\ &\quad - pI_2\frac{1}{(\theta-d)}\left(I_0 + \frac{a_1}{(\theta-d)} + \frac{a_2}{(\theta-d)}t_1 - \frac{a_2}{(\theta-d)^2} + \frac{a_3}{(\theta-d)}t_1^2 - \frac{2a_3t_1}{(\theta-d)^2} + \frac{2a_3}{(\theta-d)^3}\right) \\ &\quad + \exp^{(\theta-d)(T-m)} pI_2\left(\frac{a_1}{(\theta-d)}(m-T) + \frac{a_2}{2(\theta-d)}(m^2-T^2) + \frac{a_3}{3(\theta-d)}(m^3-T^3)\right. \\ &\quad \left. + \frac{2a_3}{(\theta-d)^3}(m-T) + \frac{a_2}{(\theta-d)^2}(T-m) + \frac{a_3}{(\theta-d)^2}(T^2-m^2)\right) \\ &\quad - pI_2\frac{1}{(\theta-d)}\left(I_0 + \frac{a_1}{(\theta-d)} + \frac{a_2}{(\theta-d)}t_1 - \frac{a_2}{(\theta-d)^2} + \frac{a_3}{(\theta-d)}t_1^2 - \frac{2a_3t_1}{(\theta-d)^2} + \frac{2a_3}{(\theta-d)^3}\right). \end{aligned} \tag{3.23}$$

Hence, the average total cost per unit of time is provided by

$$\begin{aligned} U(T, t_1) = \frac{1}{T}(&\text{Ordering Cost} + \text{Holding Cost} + \text{Shortage Cost} + \text{Deterioration Cost} \\ &+ E_2 - E_1 + \text{Preservation Technology Cost}). \end{aligned}$$

Theorem in Section 3.2 can also used here for demonstrating the minimality of the overall costs per unit of time.

4. Results and Discussion

A mathematical solution has been found for the specified model in this study, and various crucial parameter values are also investigated to validate the model. Among the models presented, sensitivity analysis was conducted to determine the best.

4.1 Numerical Illustrations

Here are some numerical examples with values for the parameters to illustrate the proposed inventory model numerically. The model parameter values taken into account were not chosen from any real-world case study, but are realistic to address the issues in various scenarios related to each example. We utilize MATHEMATICA 11.3 software to identify the best solution to this inventory problem.

Numerical 4.1 (Inventory Model with IFS Policy (Employing Preservation Technology)). The following numerical values to the parameters have been taken into account for the entire computation of the IFS inventory policy $d = 0.2$, $\xi = 3$, $K = 500$, $a_1 = 30$, $a_2 = 10$, $a_3 = 15$, $I_1 = 0.05$, $I_2 = 0.03$, $s = 6.5$, $v_1 = 5.5$, $h = 2$, $a = 100$, $b = 0.40$, $\theta = 0.4$, $c = 4$, $p = 5$, $m = 0.3$ and $P = 8$. The ideal time for the shortage is supplied by $t_1^* = 1.016$ week, while the ideal value for the inventory cycle time is $T^* = 1.595$ week. The average cost function's ideal value, which corresponds to these ideal values, is given by $TC^* = 507.9\$$.

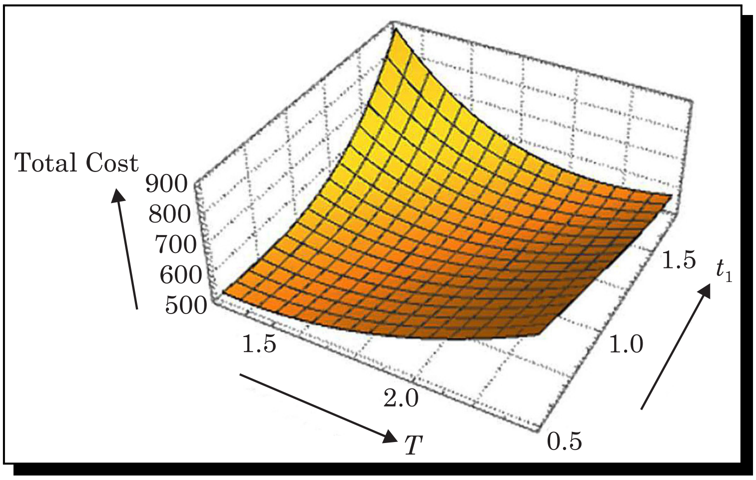

Figure 3. Convex optimization of Total cost function in IFS policy

Numerical 4.2 (Inventory Model with IFS Policy (Without any Preservation Technology)). Considering the same input parameter as in Numerical 4.1 with $\xi = 0$, we get $t_1^* = 0.6054$ week and $T^* = 1.439$ week. The average cost function's ideal value is given by $TC^* = 542.5\$$.

Numerical 4.3 (Inventory Model with SFI Policy (Employing Preservation Technology)). Now, for the SFI inventory policy, the following numerical values have been taken into account for the entire computation given by $d = 0.2$, $\xi = 3$, $K = 500$, $a_1 = 100$, $a_2 = 10$, $a_3 = 15$, $I_1 = 0.05$, $I_2 = 0.03$, $s = 5.5$, $v_2 = 3.5$, $h = 2$, $a = 30$, $b = 0.40$, $\theta = 0.4$, $c = 4$, $p = 5$, $m = 0.3$ and $P = 8$. The ideal time for the shortage is supplied by $t_1^* = 1.036$ week, while the ideal value for the inventory cycle time is $T^* = 1.711$ week. The average cost function's ideal value, corresponding to these ideal values, is given by $TC^* = 421.5\$$.

Numerical 4.4 (Inventory Model with SFI Policy (Without any Preservation Technology)). Substituting $\xi = 0$ in Numerical 4.3, we get the ideal time for the shortage is supplied by $t_1^* = 1.019$ week, while the ideal value for the inventory cycle time is $T^* = 1.59$ week. The average cost function's ideal value, corresponding to these ideal values, is given by $TC^* = 443.8\$$.

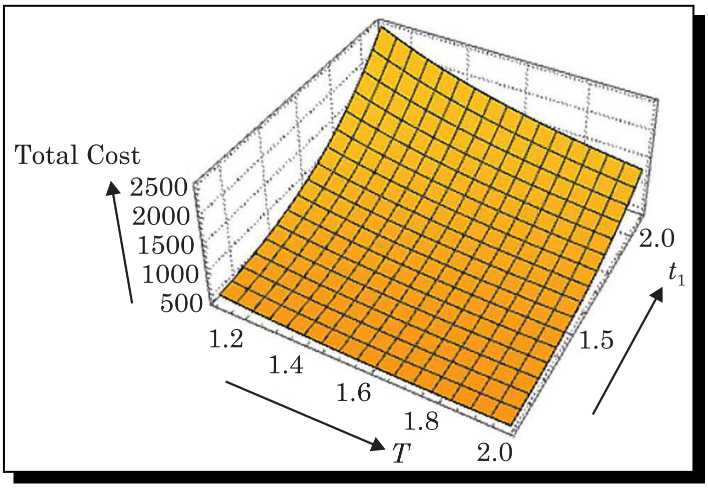

Figure 4. Convex optimization of Total cost function in SFI policy

Comparative Study of Total Cost in Case of With and Without Preservation Technology Investment

Figure 5 depicts a comparison of model outcomes in both scenarios with and without the implementation of preservation technology costs in the case of an IFS policy. It can be shown that the costs paid in each condition where preservation technology is absent are 6.81% greater than the case when it is present. This is because without proper preservation technology, businesses may encounter higher levels of product spoilage, resulting in more waste and shorter shelf life for items. Also, when preservation technology is not applied, we observed that the SFI policy costs 5.29% more than in the case when it is present. Furthermore, without preservation technologies, retailers' product offerings may be restricted, reducing market competitiveness. Overall, the lack of preservation technology may lead to increased pricing and reduced efficiency for businesses.

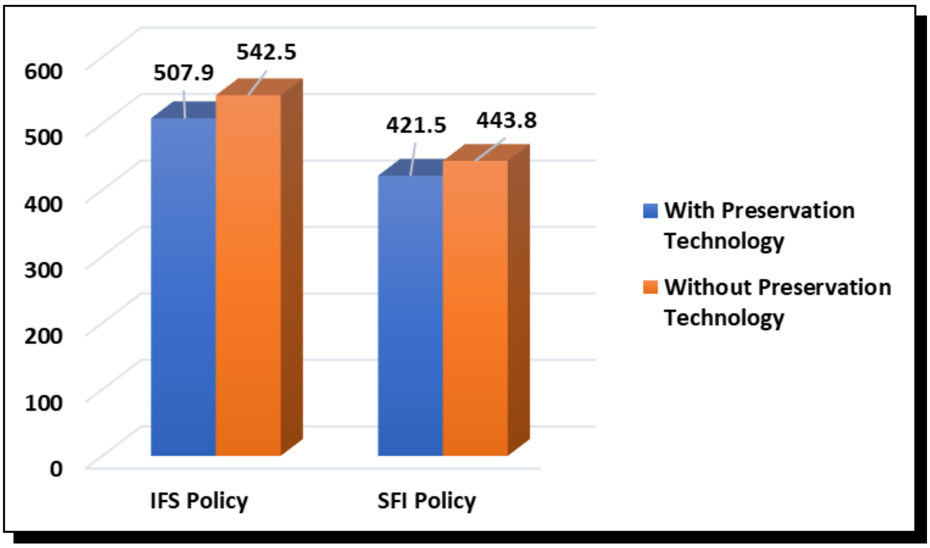

Figure 5. Comparison of Total cost for both policies with and without preservation Technology

4.2 Comparative Sensitivity Analysis

The holding cost per unit time h, the ordering cost K, the parameters of the quadratic function, and the total cost per unit time TC are all taken into consideration while performing the sensitivity analysis. For each of the above parameters, a change in standard value of -20%, -10%, 10%, or 20% was considered. All the remaining parameters are held constant while the sensitivity analysis is conducted on the aforementioned parameters keeping all other remaining parameters constant. Figures 6-9 shows the comparison of numerical results based on the impacts of the parameter on the total cost per unit time. A percentage change in parameters is represented on the x-axis, while the y-axis represents the total cost per unit time. Figures 6 and 8 show the sensitivity of the parameters K, h, ξ, and m with IFS and SFI models' total costs per unit time. Also, Figure 7 and 9 represent the variation of different demand parameters with total costs per unit time in IFS and SFI models for the demand function of both quadratic and price-dependent functions.

Table 1. Sensitivity effect of various parameters on t_1, T and Total Cost in IFS Policy

Parameter	Original Values	Percentage Change	t_1	T	Total Cost per unit time
$K = 500$	400.00	– 20	0.8895	1.466	442.6000
	450.00	– 10	0.9536	1.533	475.9159
	550.00	10	1.0760	1.655	538.6571
	600.00	20	1.1340	1.711	568.3669
$a_1 = 30$	24.00	– 20	0.9658	1.586	503.4870
	27.00	– 10	0.9923	1.591	505.7640
	33.00	10	1.0370	1.599	509.8744
	36.00	20	1.0560	1.602	511.7461
$a_2 = 10$	8.00	– 20	0.9923	1.597	504.4563
	9.00	– 10	1.0040	1.596	506.2022
	11.00	10	1.0270	1.594	509.5109
	12.00	20	1.0370	1.593	511.1000
$a_3 = 15$	12.00	– 20	0.9892	1.629	498.0000
	13.50	– 10	1.0020	1.610	503.1381
	16.50	10	1.0300	1.582	512.3000
	18.00	20	1.0440	1.571	516.3000
$a = 100$	80.00	– 20	1.2610	1.719	481.3000
	90.00	– 10	1.1300	1.650	495.8000
	110.00	10	0.9157	1.551	517.8300
	120.00	20	0.8291	1.516	526.1000
$b = 0.40$	0.32	– 20	1.0090	1.592	508.6000
	0.36	– 10	1.0120	1.594	508.2000
	0.44	10	1.0190	1.597	507.5300
	0.48	20	1.0230	1.598	507.1800

Table continued

Parameter	Original Values	Percentage Change	t_1	T	Total Cost per unit time
$c = 4$	3.20	– 20	1.0250	1.600	507.28
	3.60	– 10	1.0200	1.597	507.58
	4.40	10	1.0110	1.593	508.20
	4.80	20	1.0070	1.591	508.50
$h = 2$	1.60	– 20	1.1640	1.668	492.20
	1.80	– 10	1.0870	1.629	500.50
	2.20	10	0.9510	1.566	514.40
	2.40	20	0.8921	1.541	520.20
$\xi = 3$	2.40	– 20	1.0160	1.595	507.30
	2.70	– 10	1.0160	1.595	507.60
	3.30	10	1.0160	1.595	508.20
	3.60	20	1.0160	1.595	508.50
$m = 0.3$	0.24	– 20	1.0130	1.594	508.60
	0.27	– 10	1.0150	1.595	508.20
	0.33	10	1.0170	1.595	507.50
	0.36	20	1.0180	1.596	507.20

Table 2. Sensitivity effect of various parameter on t_1, T and Total Cost in SFI Policy

Parameter	Original Values	Percentage Change	t_1	T	Total Cost per unit time
$K = 500$	400.00	– 20	0.9971	1.601	361.10
	450.00	– 10	1.0180	1.658	391.80
	550.00	10	1.0530	1.760	450.50
	600.00	20	1.0690	1.805	478.50
$a_1 = 100$	80.00	– 20	1.0250	1.757	408.50
	90.00	– 10	1.0310	1.733	414.80
	110.00	10	1.0410	1.689	427.90
	120.00	20	1.0450	1.669	433.69
$a_2 = 10$	8.00	– 20	1.0360	1.720	419.70
	9.00	– 10	1.0360	1.715	420.60
	11.00	10	1.0370	1.706	422.30
	12.00	20	1.0370	1.702	423.20
$a_3 = 15$	12.00	– 20	1.0380	1.738	417.60
	13.50	– 10	1.0370	1.724	419.60
	16.50	10	1.0360	1.699	423.30
	18.00	20	1.0350	1.687	425.00
$a = 30$	24.00	– 20	1.0950	1.741	414.30
	27.00	– 10	1.0630	1.725	417.50
	33.00	10	1.0130	1.699	425.10
	36.00	20	0.9919	1.689	428.40

Table continued

Parameter	Original Values	Percentage Change	t_1	T	Total Cost per unit time
$b = 0.40$	0.32	– 20	1.0310	1.708	422.30
	0.36	– 10	1.0340	1.710	421.90
	0.44	10	1.0390	1.712	421.10
	0.48	20	1.0420	1.714	420.60
$c = 4$	3.20	– 20	1.0870	1.737	413.80
	3.60	– 10	1.0600	1.723	417.80
	4.40	10	1.0150	1.700	424.70
	4.80	20	0.9961	1.691	427.70
$h = 2$	1.60	– 20	1.0220	1.756	412.50
	1.80	– 10	1.0300	1.732	417.20
	2.20	10	1.0420	1.691	425.50
	2.40	20	1.0480	1.673	429.20
$\xi = 3$	2.40	– 20	1.0360	1.711	420.90
	2.70	– 10	1.0360	1.711	421.20
	3.00	0	1.0360	1.711	421.50
	3.30	10	1.0360	1.711	421.80
	3.60	20	1.0360	1.711	422.10
$m = 0.3$	0.24	– 20	1.0370	1.708	423.50
	0.27	– 10	1.0370	1.710	422.50
	0.33	10	1.0360	1.712	420.40
	0.36	20	1.0360	1.713	419.40

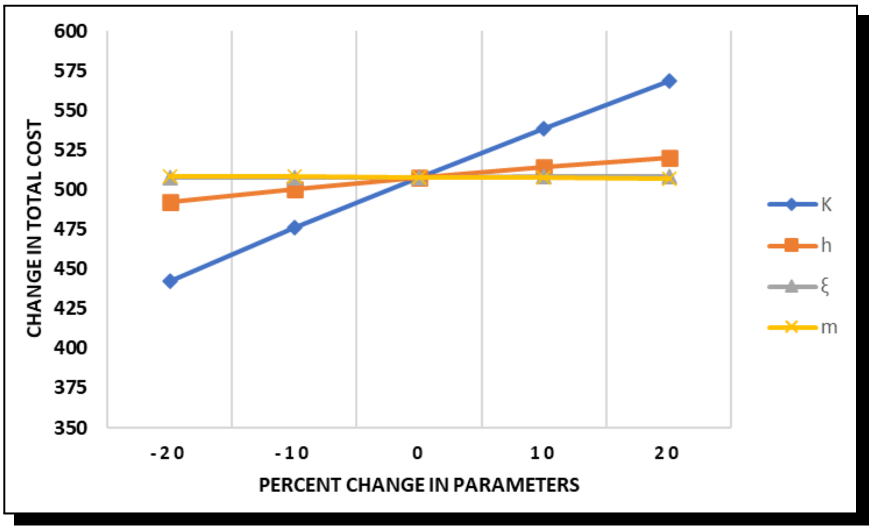

Figure 6. The impact of change in parameters on IFS's Total Cost

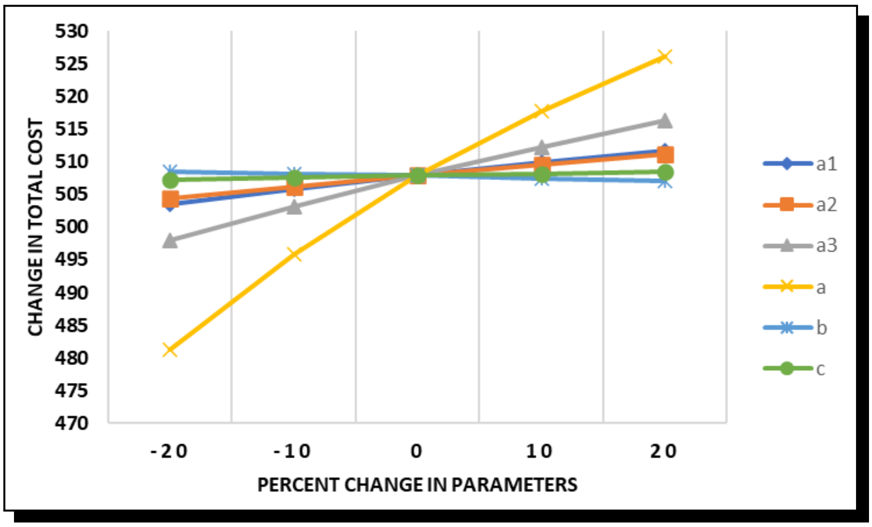

Figure 7. The impact of change in parameters on IFS's Total Cost

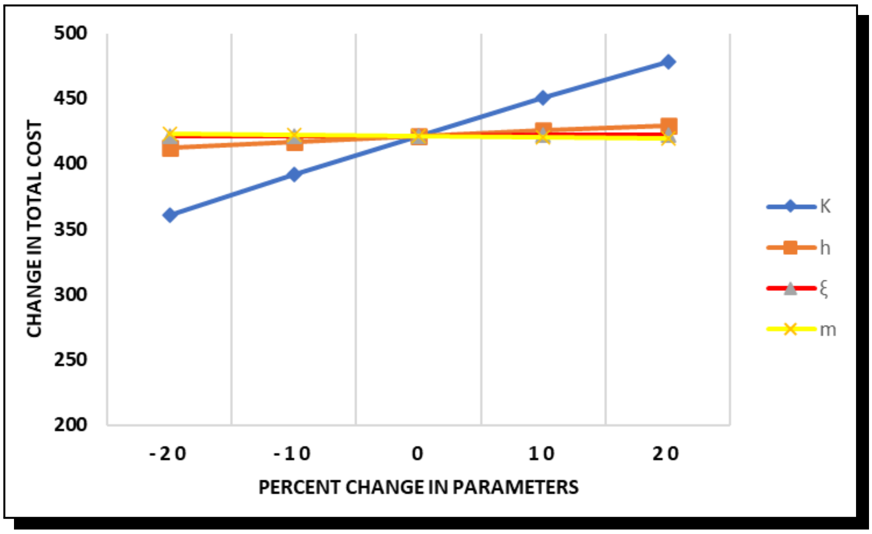

Figure 8. The impact of change in parameters on SFI's Total Cost

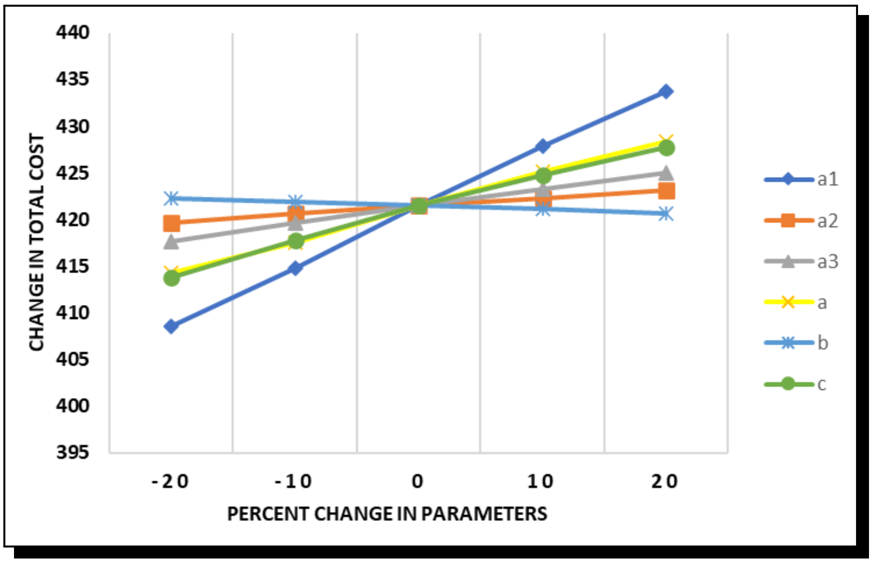

Figure 9. The impact of change in parameters on SFI's Total Cost

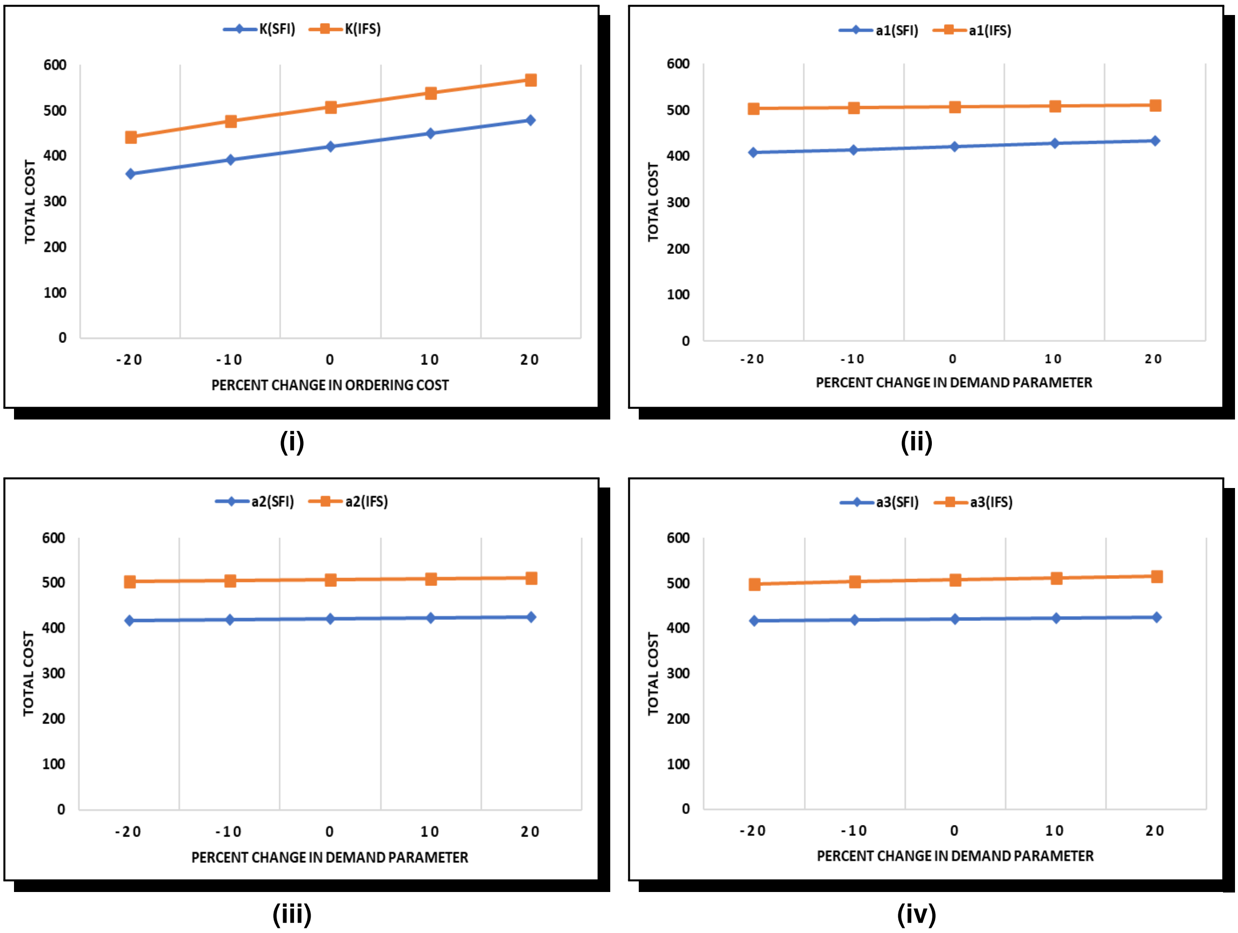

Figure 10. (i)-(iv) shows the comparison of sensitivity results of different parameters in SFI and IFS

5. Conclusion

In this study, an inventory model with time and price-dependent demand under two different policies has been suggested for a retailer with permissible delay in payment and allowed shortages. To manage deterioration, preservation technology has been successfully utilized. When a retailer has a large number of products in stock and the demand for those products fluctuates, the typical approaches cannot be employed; instead, they must be modified to better suit the situation to support the company's inventory management. *Inventory Follows Shortages* (IFS) and *Shortages Follows Inventory* (SFI), both with deterioration rates and allowable payment delays, have been affirmed as two sorts of practical-oriented conditions. This model provides some managerial insights for the retailer. Finally, a comparison of the SFI model versus the IFS model using the optimum total cost per unit of time shows that the SFI model is better and more cost-effective due to a minimal cost for each parameter adjustment. When a preservation investment under two different policies is implemented simultaneously with the influence of permissible delay in payments and price, time-dependent demand, some significant results have been produced that help the retailer in significant ways. Our study also offers some theoretical analysis to support the model's numerical sensitivity analysis of important parameters.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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