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Research Article

Border Singularities as Solutions of an Ordinary Differential Equation

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Abstract. The border singularities of a sixth-degree smooth function will be examined in this article by using real analysis and catastrophe theory. Next that, we provide an application of an ordinary differential equation (ODE) together with its boundary conditions. Using the local Lyapunov-Schmidt approach, we demonstrate that this function is identical to the key function that corresponds to the functional of the ODE. The bifurcation analysis of the function has been investigated by border singularities. The parametric equation for the bifurcation set (caustic) and its geometric description together with the critical points' bifurcation spreading has been found.

Keywords. Boundary singularities, Lyapunov-Schmidt approach, Bifurcation solutions, Caustic

Mathematics Subject Classification (2020). 26B99, 58K35

1. Introduction

The nonlinear equations in the infinite dimension of the type:

$$g(x,\lambda)=b, \quad x\in O, b\in Y, \lambda\in\mathbb{R}^n, \tag{1}$$

can be convert to an equations of the type

$$\theta(\xi,\lambda)=\eta, \quad \xi\in\mathcal{A}, \eta\in\mathcal{B}, \tag{2}$$

in the finite dimension by Lyapunov-Schmidt (LS) method, so that the smooth Fredholm map g has an index of zero, Banach's spaces are X and Y, $O\subseteq X$ is open and, $\mathcal{A}$, $\mathcal{B}$ are smooth finite dimensional manifolds. The bifurcation diagram, multiplicity, and analytical and topological properties, etc. of equation (1) are all included in equation (2) (see Darinskii *et al.* [4], Loginov [10], Sandstede [13], Shveriova [15], Thompson and Stewart [16], Vainberg and Trenogin [17]). The research of smooth map singularities is necessary for the investigation of the bifurcation solutions of BVPs ([5]). The study of the singularities of smooth maps and its applications to VPB captured interest of the Sapronov group in many of their studies (e.g, see, Arnol'd [1], Danilova [3], Ishibashi [6], Kadhim and Hussain [7], Krasnosel'skii [8], Li and Qiao [9]).

The method (LS) assumes that $g:\Omega\subset E\to F$ is an index 0 smooth nonlinear Fredholm map. The map g has variational property, when there is a functional $V:\Omega\subset E\to\mathbb{R}$ such that $\frac{\partial V}{\partial x}(x,\lambda)h=\langle g(x,\lambda),h\rangle_H$, for all $x\in\Omega$, $h\in E$, where $\langle\cdot,\cdot\rangle_H$ is the scalar inner product in Hilbert space H and $E\subset F\subset H$. The solutions of equation $g(x,\lambda)=0$ are the own critical points of functional $V(x,\lambda)$. The method (LS) can reduce the problem $V(x,\lambda)\to extr$, $x\in E$, $\lambda\in\mathbb{R}^n$ into an equivalent problem $W(\xi,\lambda)\to extr$, $\xi\in\mathbb{R}^n$, where $W(\xi,\lambda)$ is called key function. The function W possesses the topological and analytical properties of the functional V (multiplicity, bifurcation diagram, etc.) (Sapronov [13]).

The study of functional V bifurcating solutions is identical to the study of key function W bifurcating solutions.

In this paper, we study the boundary singularities of the following real smooth function,

$$W(y,\lambda)=\frac{1}{3}v_1^6+\frac{1}{6}v_2^6+v_1^4v_2^2+v_1^2v_2^4+\epsilon v_1^2+\delta v_2^2, \tag{3}$$

$y=(v_1,v_2)$, $\lambda=(\epsilon,\delta)$ and ϵ,δ are parameters with considering the functional,

$$(z,\eta)=01\left(-\alpha\frac{(z')^2}{2}+\beta\frac{z^2}{2}+\frac{z^6}{6}\right)dy, \tag{4}$$

where $z=z(y)$, $\eta=(\alpha,\beta)$ and α,β are parameters.

2. Fredholm Functional's Border Singularities [5]

To study how a Fredholm functional behaves in a border singular point's neighborhood, the reducing to an equivalent extremes problem is employed:

$$W(x)\to extr,$$

where $x\in D$, $D=\{x=(x_1,x_2)^\top\in\mathbb{R}^2:x_1\geq 0\}$.

A point $a\in D$ is referred to as conditionally critical for a smooth function W in $\mathbb{R}^2$ if $grad\,W(a)$ (grad denotes gradient of W) is perpendicular to the least face of D containing a. The multiplicity of the conditionally critical point a (and its symbol is $\bar{\mu}$) is the quotient algebra's dimension where the quotient algebra denotes by, $\bar{Q}=\frac{\Gamma_a(\mathbb{R}^2)}{I}$, such that $\Gamma_a(\mathbb{R}^2)$ is the ring of germs of smooth functions on $\mathbb{R}^2$ at point a and $I=\left(x_1\frac{\partial W}{\partial x_1},\frac{\partial W}{\partial x_2}\right)$ is the corner Jacobi ideal in $\Gamma_a(\mathbb{R}^2)$. The multiplicity $\bar{\mu}$ of a conditionally critical point a is equal to the sum of multiplicities $\mu+\mu_0$, where μ is the (usual) multiplicity of W on $\mathbb{R}^2$, while μ_0 is the (usual) multiplicity of the restriction $W\mid\partial D$ (where ∂D is the boundary of the set D).

If a critical point is usual, then spreading of bifurcating extremes (bif-spreading) are represented by the row (r_0,r_1,r_2), where r_i is the number of critical points of the Morse index i. If we are dealing with a border critical point, then bif-spreading are represented by the following matrix of order 2×3:

$$\begin{pmatrix} r_0^1 & r_1^1 & r_2^1 \\ r_0 & r_1 & r_2 \end{pmatrix},$$

where r_i^j is the number of the border critical points of index i (for $j = 1$), while r_i is the number of usual (situated inside D) critical points of index i ($i = 0,1,2$).

3. Main Results

In this section, we take into account the function (3), which is defined in the first section. The function (3) has codimension twenty-four at the origin, hence it has multiplicity twenty-five. The main goals are to first determine the geometry bifurcation diagram of the caustic of the function (3), and secondly to determine the distribution of the critical points of this function. In order to use the border singularities method in studying the function (3), we make the following assumptions, $v_1^2 = u$, $v_2 = v$. Therefore, studying of the function (3) is equivalent to the studying of the following function:

$$W(y,\lambda) = \frac{1}{3}u^3 + \frac{1}{6}v^6 + u^2v^2 + uv^4 + \epsilon u + \delta v^2, \tag{5}$$

where $y = (u,v)$, $\lambda = (\epsilon,\delta)$, and $u \geq 0$.

Since, the germ (the principal part) of the function (5) is $W_0 = \frac{1}{3}u^3 + \frac{1}{6}v^6$, so, from the second section, we have $I = \left(u\frac{\partial W_0}{\partial u}, \frac{\partial W_0}{\partial v}\right) = (u^3, v^5)$, and $\bar{\mu} = 15$, where $\mu = 10$, $\mu_0 = 5$. Since multiplicity $\bar{\mu}$ is equal to the number of critical points (Berczi [2]), function (5) has fifteen critical points, five of which are on the border $u = 0$ and the remaining ten are in the interior. So, the following union of three sets is the caustic of function (5):

$$\Xi = \Xi_{0,1}^{int} \bigcup \Xi_{0,1}^{ext} \bigcup \Xi_{1,1},$$

where $\Xi_{0,1}^{int}$ and $\Xi_{0,1}^{ext}$ are the subsets (components) of the caustic that correspond to the border singularities degenerating along the border and along the normal, respectively, and $\Xi_{1,1}$ is the component that corresponds to the interior critical points degenerating (non-boundary).

Lemma 3.1 (Ddegeneration on the border $u = 0$ and its normal)**.** (a) *There is no parametric equation that describes the set* $\Xi_{0,1}^{int}$.

(b) *The parametric equation that describes the set* $\Xi_{0,1}^{ext}$ *has the following structure:*

$$\delta\epsilon^2(2\delta - \epsilon) = 0.$$

Proof. (a) All points $(0,v,\epsilon,\delta)$ that satisfy the following relations are represented by the set $\Xi_{0,1}^{int}$: $\frac{\partial W(0,v,\epsilon,\delta)}{\partial v} = \frac{\partial^2 W(0,v,\epsilon,\delta)}{\partial v^2} = 0$. From these relations, we have $v^5 + 2\delta v = 2v = 0$. In case, $v^5 + 2\delta v = 0$ or $v(v^4 + 2\delta) = 0$, such that $v = 0$ and $v^4 + 2\delta \neq 0$, we have $\delta \neq 0$. This is, there is not a parametric equation which represents the set $\Xi_{0,1}^{int}$ for all points $(0,v)$. Hence, the parametric equation which represents the set $\Xi_{0,1}^{int}$ is not exist.

(b) All points $(0,v,\epsilon,\delta)$ that fulfill the following relations are represented by the set $\Xi_{0,1}^{ext}$: $\frac{\partial W(0,v,\epsilon,\delta)}{\partial v} = \frac{\partial W(0,v,\epsilon,\delta)}{\partial u} = 0$, this implies $v^5 + 2\delta v = v^4 + \epsilon = 0$. Then, we have the following system:

$$v(v^4+2\delta)=0, \tag{6}$$

$$v^4+\epsilon=0. \tag{7}$$

The analysis of the equation (6) is divided into three cases:

Case 1: If $v=(v^4+2\delta)=0$, then we have from this and the equation (7) the following equation

$$\delta\epsilon=0. \tag{8}$$

Case 2: If $v=0$, $(v^4+2\delta)\neq 0$, then put $v=0$ in the equation (7), to get

$$\epsilon=0. \tag{9}$$

Case 3: If $(v^4+2\delta)=0$, $v\neq 0$, then we can eliminate v from the equation $v^4+2\delta=0$ and the equation (7) to obtain

$$2\delta-\epsilon=0. \tag{10}$$

From the equations (8), (9) and (10) we get the equation with parameters that represents the set $\Xi_{0,1}^{\text{ext}}: \delta\epsilon^2(2\delta-\epsilon)=0$. □

Lemma 3.2 (Interior (non-boundary) degeneration)**.** *The equation with parameters that represents the set $\Xi_{1,1}$ is given by the equation:*

$$\delta-\epsilon=0.$$

Proof. The set $\Xi_{1,1}$ consists of all points (ϵ,δ) that satisfy the following relations:

$$\frac{\partial W(u,v,\epsilon,\delta)}{\partial u}=\frac{\partial W(u,v,\epsilon,\delta)}{\partial v}=0, \quad u>0.$$

These relations imply

$$v^4+2uv^2+u^2+\epsilon=0, \tag{11}$$

$$v^5+4uv^3+2u^2v+2\delta v=0. \tag{12}$$

To obtain the degenerate critical points, we make the determinate of the Hessian matrix of the function (5) equal to zero as follows:

$$-6v^6+2uv^4+12u^2v^2+4\delta v^2+4u^3+4\delta u=0. \tag{13}$$

Simplifying and taking the common factor for the equation (13), one gets

$$2(v^2+u)(-3v^4+4uv^2+2u^2+2\delta)=0,$$

but $u>0$, thus we have $(v^2+u)\neq 0$ and

$$-3v^4+4uv^2+2u^2+2\delta=0. \tag{14}$$

Taking the common factor for the equation (12), we have

$$v(v^4+4uv^2+2u^2+2\delta)=0.$$

Then, we have three cases:

Case 1: if $v=0$ and $(v^4+4uv^2+2u^2+2\delta)\neq 0$, then, put $v=0$ in the equation (14) to get $u^2+\delta=0$, but $u^2+\delta\neq 0$ (by supposing), this is a contradiction.

Case 2: if $v=0$ and $(v^4+4uv^2+2u^2+2\delta)=0$, then put $v=0$ in the equation (11) and in the equation: $v^4+4uv^2+2u^2+2\delta=0$, respectively to get $u^2+\epsilon=0$ and $u^2+\delta=0$, and eliminating the variable u from the last two equations, we get $\delta-\epsilon=0$.

Case 3: if $v^4+4uv^2+2u^2+2\delta=0$ and $v\neq 0$, then, solving the equation $v^4+4uv^2+2u^2+2\delta=0$, and the equation (14) simultaneously, yields $v=0$, this is a contradiction.

This completes the proof. □

Theorem 3.1. *The formula that follows, represents the parametric equation of bifurcation set* (*the caustic*) *of the function* (5)*:*

$$\delta\epsilon^2(2\delta-\epsilon)(\delta-\epsilon)=0.$$

Proof. Considering that the caustic of the function (5) is the union of the following three sets:

$$\Xi=\Xi_{0,1}^{int}\bigcup\Xi_{0,1}^{ext}\bigcup\Xi_{1,1},$$

the left sides of all the equations for the caustic components will therefore be multiplied by one another and set to zero to create the parametric equation for the caustic. We know the equations of the caustic components have been found in Lemmas 3.1 and 3.2. Hence the following equation,

$$\delta\epsilon^2(2\delta-\epsilon)(\delta-\epsilon)=0,$$

represents the parametric equation of the bifurcating set (caustic) of the function (5).

Proposition 3.1. (a) *If $\epsilon<0$ and $\delta<0$, then one inside point and three border points make up the function* (5) *has four non-degenerate critical points.*

(b) *If $\epsilon<0$ and $\delta>0$, then the function* (5) *has two real non-degenerate critical points* (*one interior point and one border point*)*.*

(c) *If $\epsilon>0$ and $\delta>0$, then the function* (5) *has one real non-degenerate border critical point.*

(d) *If $\epsilon>0$ and $\delta<0$, then the function* (5) *has three real non-degenerate border critical points.*

Proof. The following equations system represents the critical points of the function (5):

$$v^4+2uv^2+u^2+\epsilon=0, \tag{15}$$

$$v^5+4uv^3+2u^2v+2\delta v=0. \tag{16}$$

Taking the common factor for equation (16), we get $v(v^4+4uv^2+2u^2+2\delta)=0$, this equation implies $v=0$ or $v^4+4uv^2+2u^2+2\delta=0$. Then, we have three cases:

Case 1: If $v\neq 0$ and $v^4+4uv^2+2u^2+2\delta=0$, then we get the following equation:

$$v^4+4uv^2+2u^2+2\delta=0. \tag{17}$$

Subtracting the equation (15) from the equation (17) and solving for v, we have $v=\mp\frac{1}{2}\frac{\sqrt{-2u(u^2+2\delta-\epsilon)}}{u}$, substituting the value of v in the equation (15) and solving for u, we have $u=\mp\sqrt{2\delta-3\epsilon+2\sqrt{-2\delta\epsilon+2\epsilon^2}}$. Since, $u>0$, thus we get $u=\sqrt{2\delta-3\epsilon+2\sqrt{-2\delta\epsilon+2\epsilon^2}}$.

In order get a real interior critical point, we must set the equation $2\delta-3\epsilon+2\sqrt{-2\delta\epsilon+2\epsilon^2}>0$ and $-2\delta\epsilon+2\epsilon^2\geq 0$, from this, we get $\epsilon<0$ and $\delta=\epsilon$, but this implies the real interior critical point is degenerate (see Theorem 3.1).

Case 2: If $v = 0$ and $v^4 + 4uv^2 + 2u^2 + 2\delta = 0$, then, from the equations (15) and (17) we get $\delta = \epsilon$, so we have a real degenerate interior critical point.

Case 3: If $v = 0$ and $v^4 + 4uv^2 + 2u^2 + 2\delta \neq 0$. Put $v = 0$ in the equation (15) to get $u^2 + \epsilon = 0$, and solving for u, we have $u = \mp\sqrt{-\epsilon}$. Since, $u > 0$, we have $\epsilon < 0$. This is, we have one real non-degenerate interior critical point ($u = \sqrt{-\epsilon}$, with $\epsilon < 0$).

The real border critical points can get by the equation $2\delta v + v^5 = 0$, where its solution is as follows $v = 0$; for all δ and $v = \pm\sqrt[4]{-2\delta}$, where $\delta \leq 0$. The non-degenerate border critical points can be obtained by setting $\delta \neq 0$. From this and *Case* 3 of this proof, we get the wanted result. □

Theorem 3.2. *The matrices of bif-spreading of the critical points of the function* (5) *are as follow:*

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 3 & 0\\ 1 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 3 & 0\\ 0 & 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 3 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{18}$$

Proof. The caustic equation has been established by Theorem 3.1, and we can infer its geometric form from this equation, which is depicted in Figure 1.

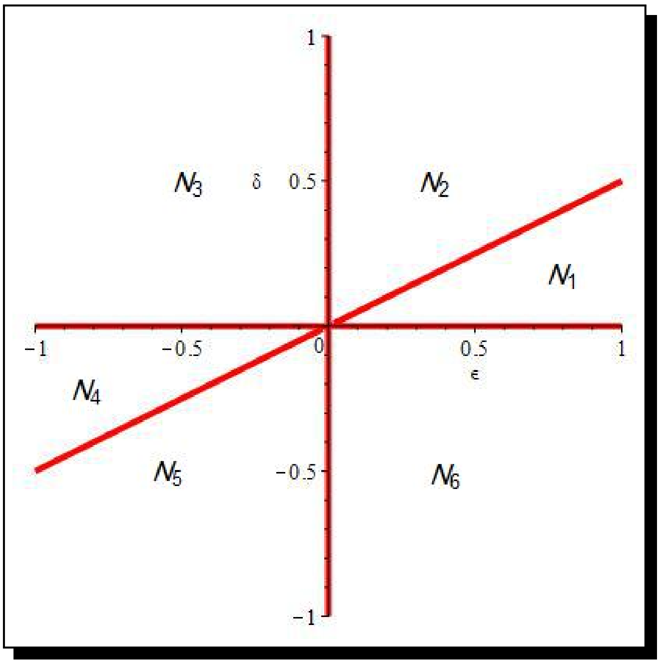

Figure 1. The caustic of the function 5 in $\epsilon\delta$-plane

The plane of parameters can be divided into six parts (regions) N_i, $i = 1,2,3,4,5,6$ using this figure. There are a definite number of non-degenerate real critical points in each region. There are two types of these points: internal and border points. The internal points' and border points' quality can be determined using the second derivative test (with the help of MATHEMATICA program in classification the critical points). Hence, the spreading of the critical points is as follows:

1. if the parameters pair (ϵ, δ) belongs to N_1 or N_2, then there is one minimal border critical point on border $u = 0$.

2. if the parameters pair (ϵ,δ) belongs to N_3, then there are two critical points (one minimal border point and one minimal interior point).
3. if the parameters pair (ϵ,δ) belongs to N_4, then there are four critical points (three saddle border points and one minimum point in the interior).
4. if the parameters pair (ϵ,δ) belongs to N_5, then there are four critical points (three saddle border points and one saddle point in the interior).
5. if the parameters pair (ϵ,δ) belongs to N_6 then there are three saddle border critical points.

The matrices of bif-spreading are obtained from the above points, as indicated in (18). □

Parts (a), (b), (c), (d), and (e) of Figure 2 illustrate the locations of contour lines with regard to the domain borders of the function (5), as well as the number and kind of critical points corresponding to all regions in caustic of the function (5).

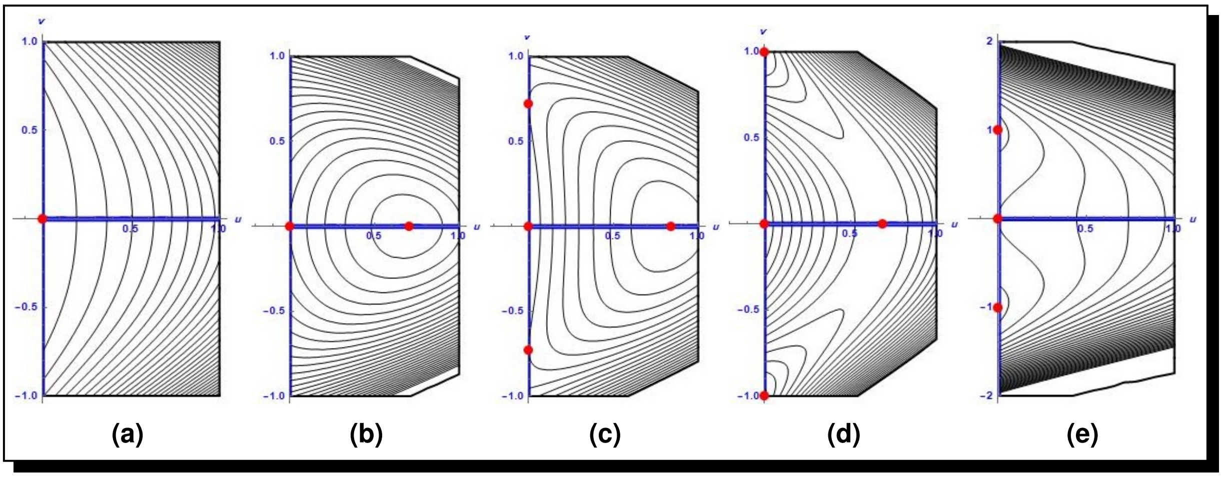

Figure 2

4. An Application

The generalized Korteweg-de Vries equation (KdV) is given by Sandstede [12]:

$$u_t + u^m u_x + u_{xxx} = 0, \tag{19}$$

where m is a positive parameter.

In this paper, we study equation (19) with $m = 4$ as an application of our work, i.e., we study the following equation:

$$u_t + u^4 u_x + u_{xxx} = 0. \tag{20}$$

The following transformation $u(x,t) = z(y)$, $y = kx - \lambda t$, can convert the equation (20) into the ordinary differential equation (ODE):

$$\alpha z'' + \beta z + z^5 = 0, \tag{21}$$

where $\alpha = 5k^2$, $\beta = -5\frac{\lambda}{k}$, $k \neq 0$, $z = z(y)$, $y \in [0,1]$ and $' = \frac{d}{\mathrm{dy}}$.

The equation (21) with the boundary conditions forms the following problem:

$$\left.\begin{aligned} &f(z,\eta)=\alpha z''+\beta z+z^5=0,\\ &z(0)=z(1)=0, \end{aligned}\right\} \tag{22}$$

where $\eta=(\alpha,\beta)$, and $f:E\to F$ is a not linear Fredholm operator with an index of zero, where $E=C^2([0,1],\mathbb{R})$ is the space of all continuous functions that have derivative of order at most two, $F=C^0([0,1],\mathbb{R})$ is the space of all continuous functions.

The purpose of the paper is to find the bifurcating solution areas of the equation (22) where each bifurcating solution of the equation (22) equals a critical point of the functional (4) and each critical point of the functional (4) coincides a critical point of the key function of the functional (4) (Darinskii *et al.* [4]). Therefore, we shall show that the function (3) is tantamount to the key function of the functional (4), that is, the study of the equation (22)'s bifurcating solutions is equivalent to the study of bifurcating solutions of the function (3). Hence, we have interested studying the bifurcating solutions of the function (3).

In the following theorem, we show that the function (3) is equivalent to the key function of the functional (4).

Theorem 4.1. *The key function's normal form* W_1*, which corresponds to the functional* (4)*, is*

$$W_1(y,\lambda)=\frac{1}{3}v_1^6+\frac{1}{6}v_2^6+v_1^4v_2^2+v_1^2v_2^4+\epsilon v_1^2+\delta v_2^2,$$

where $y=(v_1,v_2)$*,* $\lambda=(\epsilon,\delta)$*.*

Proof. The linearized equation that corresponds to equation (22) at point $(0,\lambda)$ using the Lyapunov-Schmidt scheme has the following structure:

$$Dh=0,\quad h\in E,$$

$$h(0)=h(1)=0,$$

where $D=\alpha\frac{d^2}{dx^2}+\beta$.

The linearized equation solution that fulfills the boundary conditions is given by $e_q(y)=c_q\sin(q\pi y)$, $q=1,2,\dots$, and the characteristic equation to which this solution relates is $-\alpha(q\pi)^2+\beta=0$. This equation yields characteristic lines ℓ_q in 2-space. The characteristic lines ℓ_q are made up of the points (α,β) for which the linearized equation has non-zero solutions (Sapronov [14]). The bifurcation point $(\alpha,\beta)=(0,0)$ for the equation (22) is the point of intersection of the characteristic lines in $\alpha\beta$-plane. The parameters α, and β are localized as follows, $\alpha=0+\delta_1$, $\beta=0+\delta_2$, δ_1, and δ_2 are small parameters, lead to bifurcation along the modes, $e_1(y)=c_1\sin(\pi y)$, and $e_2(y)=c_2\sin(2\pi y)$. Since, $\|e_1\|=\|e_2\|=1$, then we have $c_1=c_2=\sqrt{2}$.

Let $N=ker(D)=span\{e_1,e_2\}$, then the space E can be decomposed in direct sum of two subspaces, N and the orthogonal complement to N,

$$E=N\oplus N^{\perp},\ N^{\perp}=\left\{v\in E:\int_0^1 ve_k\,dy=0,k=1,2\right\}.$$

There exist two projections $P : E \to N$ and $I - P : E \to N^{\perp}$ such that $(Pm) = \omega$ and $(I-P)m = v$, (I is the identity operator). Hence every vector $m \in E$ can be written in the form, $m = \omega + v$, where $\omega = v_1 e_1 + v_2 e_2 \in N$, $v \in N^{\perp}$, $v_i = \langle m, e_i \rangle$.

Thus, by the implicit function theorem, there exists a smooth map, $\Theta : N \to N^{\perp}$ such that $\widetilde{W}(\zeta, \vartheta) = V(\Theta(\omega, \vartheta), \vartheta)$, $\zeta = (v_1, v_2)$, $\vartheta = (\delta_1, \delta_2)$, the key function $\widetilde{W}$ can therefore be written in the way,

$$\begin{aligned}\widetilde{W}(\zeta,\vartheta) &= V(v_1 e_1 + v_2 e_2 + \Theta(v_1 e_1 + v_2 e_2, \vartheta), \vartheta)\\ &= W_2(\zeta,\vartheta) + o(|\zeta|^6) + O(|\zeta|^6) O(\vartheta),\end{aligned}$$

where

$$W_2(\zeta,\vartheta) = \frac{5{v_1}^6}{12} + \frac{5{v_2}^6}{12} + \frac{25{v_1}^4{v_2}^2}{8} + \frac{15{v_1}^2{v_2}^4}{4}(-1/2\pi^2\alpha + \beta/2)v_1^2 + (-2\pi^2\alpha + \beta/2)v_2^2.$$

The geometrical form of bifurcations of critical points and the first asymptotic of branches of bifurcating for the function $\widetilde{W}$ are completely determined by its principal part W_2. If, we replace v_1 by $\sqrt[6]{\frac{4}{5}}v_1$ and v_2 by $\sqrt[6]{\frac{2}{5}}v_2$ in the function W_2, then W_1 and W_2 are contact equivalence, since in this case they have the same germ(the same principal part), $W_0 = \frac{1}{3}{v_1}^6 + \frac{1}{6}{v_2}^6$, and the unfolding.

As a result, the caustic of the function W_2 corresponds with the caustic of the function W_1 (Marsden and Hughes [11]). Thus, the function W_1 has all of the functional (4)'s topological and analytical features. As a conclusion, studying the bifurcation analysis of equation (22) is identical to studying the bifurcation analysis of the function W_1. This demonstrates that studying the bifurcating solutions of equation (22) is similar to studying the bifurcating solutions of function (3). □

5. Conclusion

In this paper, we found the functional (4) that satisfies the variational property for operator (22). We found the key function corresponding to functional (4) in Theorem 4.1. We proved that function (3) of the sixth degree is equivalent to the key function. We found the bifurcation solution regions of equation (21), which are the critical points of function (3) spread in the branching diagram (caustic). Also, the parametric equation was found. The critical points were classified and their regions of existence in the diagram were found. Finally, an application of this work was given.

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Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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