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Generalized Hilbert-type Operator on Hardy Space

Research Article

S. Naik* and P.K. Nath

Department of Applied Sciences, Gauhati University, Guwahati 781014, Assam, India
***Corresponding author:** spn20@yahoo.com

Abstract. If f be an analytic function on the unit disc $\mathbb{D}$ with Taylor series expansion $f(z)=\sum_{n=0}^{\infty} a_n z^n$, we consider the generalized Hilbert-type operator defined by

$$\mathscr{H}_{a,b}(f)(z)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{\Gamma(n+a+1)\Gamma(n+k+1)}{\Gamma(n+1)\Gamma(n+k+b+2)}a_k\right)z^n,$$

where Γ denotes the Gamma function and $a,b\in\mathbb{C}$. We find an upper bound for the norm of the generalized Hilbert-type operator on Hardy space.

Keywords. Generalized Hilbert-type operator; Hardy spaces

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1. Introduction

We denote $\mathbb{D}$ as unit disc in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. If $1<p<\infty$, then for every sequences $a=\{a_n\}$ in the sequence spaces l^p, the classical Hilbert inequality

$$\left(\sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{a_k}{n+k+1}\right|^p\right)^{\frac{1}{p}}\leq\frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{k=0}^{\infty}|a_k|^p\right)^{\frac{1}{p}} \tag{1}$$

is valid. The constant $\frac{\pi}{\sin(\frac{\pi}{p})}$ is best possible [10].

Thus the Hilbert matrix

$$\mathscr{H} = \left(\frac{1}{i+j+1}\right), \quad i,j = 0,1,2\dots$$

can be viewed as an operator on spaces of analytic functions, called Hilbert operators, by its action on the Taylor coefficients:

$$a_n \to \sum_{k=0}^{\infty} \frac{a_k}{n+k+1}, \quad n = 0,1,2,\dots$$

that is, if $f \in H(\mathbb{D})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then so-called Hilbert operator (see [7]) $\mathscr{H}$ on $H(\mathbb{D})$ defined by

$$\mathscr{H}(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1}\right) z^n. \tag{2}$$

Hardy's inequality (see [9]) guarantees that the transformed power series (2) converges on $\mathbb{D}$ and defines there an analytic function $\mathscr{H}(f)$ whenever $f \in H^1$. In other words, $\mathscr{H}(f)$ is a well defined analytic function for every $f \in H^1$.

A calculation shows that

$$\mathscr{H}(f)(z) = \int_0^1 f(t)\frac{1}{1-tz}\,dt,$$

where the convergence of the integral is guaranteed by the Fejer-Riesz inequality ([9, p. 46]) and the fact that $\frac{1}{(1-tz)}$ is bounded in t for each $z \in \mathbb{D}$.

For $0<r<1$, $0<p<\infty$ and $f \in H(\mathbb{D})$, the integral mean $M_p(f,r)$ is defined by

$$M_p(f,r) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\,d\theta\right)^{\frac{1}{p}}.$$

The Hardy space H^p is defined as follows:

$$H^p = \left\{f : f \in H(\mathbb{D}), \|f\|_{H^p} = \sup_{0\le r<1} M_p(f,r) < \infty\right\}$$

and

$$\|f\|_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)|.$$

Additional properties of Hardy space can be found in [9].

The resulting Hilbert operator $\mathscr{H}$ is bounded from H^p to H^p, whenever $1<p<\infty$ but $\mathscr{H}$ is not bounded on H^1 ([7, Theorem 1.1]). In [8] the norm of $\mathscr{H}$ acting on Hardy spaces was computed. This is known and quick way to see this is to view $\mathscr{H}$ as a Hankel operator. In fact $\mathscr{H}$ is a prototype for Hankel operators see [13] for details.

In this article we shall be dealing with certain generalized Hilbert operators. If $f \in H(\mathbb{D})$ and given $a, b \in \mathbb{C}$, we consider the generalized Hilbert type operator $\mathscr{H}_{a,b}$ defined by

$$\mathscr{H}_{a,b}(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{\Gamma(n+a+1)\Gamma(n+k+1)}{\Gamma(n+1)\Gamma(n+k+b+2)}a_k\right)z^n.$$

It is easy to see that the above mentioned series defines an analytic function on $\mathbb{D}$. In the notation of Li [11], we have

$$\mathscr{H}_{\beta,\beta} = \mathscr{H}_{\beta} \quad (\beta \geq 0)$$

and, in particular, for $\beta = 0$

$$\mathscr{H}_{0,0} = \mathscr{H},$$

where $\mathscr{H}_{\beta}$ is called generalized Hilbert operator (see [11]). If $\beta = 0$ is the classical Hilbert operator.

We find the integral representation of $\mathscr{H}_{a,b}(f)(z)$ by simple steps of calculations as following

$$\mathscr{H}_{a,b}(f)(z) = \frac{\Gamma(a+1)}{\Gamma(b+1)}\int_0^1 \frac{f(t)(1-t)^b}{(1-tz)^{a+1}}dt, \tag{3}$$

which is an integral type operator. Recently there has been a huge interest in studying integral type operators on analytic function spaces. For some results in this area, mostly in the settings of the unit ball and the unit polydisk see ([2, 3, 4, 5, 6, 7, 12]). In particular if $a = b = \beta$, we find that the representation for generalized Hilbert operator is given by

$$\mathscr{H}_{\beta}(f)(z) = \mathscr{H}_{\beta,\beta} = \int_0^1 \frac{f(t)(1-t)^{\beta}}{(1-tz)^{\beta+1}}dt$$

as in see ([11]).

Now for the representation of $\mathscr{H}_{a,b}$ in terms of composition operators, for $z \in \mathbb{D}$, we choose the path

$$\zeta(t) = \zeta_z(t) = \frac{t}{(t-1)z+1}, \quad 0 \leq t \leq 1$$

i.e a circular arc in $\mathbb{D}$ joining 0 to 1. Change of variable in (3) gives the following:

$$\mathscr{H}_{a,b}(f)(z) = \frac{\Gamma(a+1)}{\Gamma(b+1)}\int_0^1 f\left(\frac{t}{(t-1)z+1}\right)\frac{(1-t)^b(1-z)^{b-a}}{(1+(t-1)z)^{b-a+1}}dt.$$

If $0 \leq t \leq 1$, set

$$\phi_t(z) = \frac{1}{(t-1)z+1} \quad \text{and} \quad \omega_t(z) = \frac{\Gamma(a+1)}{\Gamma(b+1)}\frac{(1-t)^b(1-z)^{b-a}}{(1+(t-1)z)^{b-a+1}}, \quad z \in \mathbb{D}.$$

It is easy to see that ϕ_t is a self map of $\mathbb{D}$ and if $b \geq a$, $\omega_t(z) \in H(\mathbb{D})$. Thus the transformation $\mathscr{H}_{a,b}$ is an average

$$\mathscr{H}_{a,b}(f)(z) = \int_0^1 T_t(f)(z)dt$$

of the weighted composition operators

$$T_t(f)(z) = \omega_t(z) f(\phi_t(z)).$$

In [7], the authors proved an analogue of the inequality (1) on Hardy space. More precisely the authors proved the following result:

Theorem 1. *The following inequality are valid*

(a) *If $2 \le p < \infty$, then*

$$\|\mathscr{H}(f)\|_{H^p} \le \frac{\pi}{\sin\frac{\pi}{p}} \|f\|_{H^p}$$

for each $f \in H^p$.

(b) *If $1 < p < 2$, then*

$$\|\mathscr{H}(f)\|_{H^p} \le \frac{\pi}{\sin\frac{\pi}{p}} \|f\|_{H^p}$$

for each $f \in H^p$ with $f(0) = 0$.

In this article we prove an analogue of Theorem 1. The following theorem is our main result of this article. More precisely we show that

Theorem 2. *The following inequality are true*

(a) *If $2 \le p < \infty$ and $Re(b) \ge Re(a)$, then*

$$\|\mathscr{H}_{a,b}(f)\|_{H^p} \le 2^{b-a} \frac{\Gamma(a+1)}{\Gamma(b+1)} B\left(\frac{1}{p} - b + a, a + 1 - \frac{1}{p}\right) \|f\|_{H^p}$$

for each $f \in H^p(\mathbb{D})$.

(b) *If $1 < p < 2$ and $Re(b) \ge Re(a)$, then*

$$\|\mathscr{H}_{a,b}(f)\|_{H^p} \le 2^{b-a} \frac{\Gamma(a+1)}{\Gamma(b+1)} B\left(\frac{1}{p} - b + a, a + 1 - \frac{1}{p}\right) \|f\|_{H^p}$$

for each $f \in H^p(\mathbb{D})$ with $f(0) = 0$.

2. Proof of the Main Result

In this section, we will give the proof of our main result in this paper.

Recall that by $H^p(\pi_R)$, where π_R denote the right half plane $\{z \in \mathbb{C} | Re(z) > 0\}$ is the Hardy space on π_R. i.e the set of all analytic functions $f : \pi_R \to \mathbb{C}$ such that

$$\|f\|^p_{H^p(\pi_R)} = \sup_{0<x<\infty} \int_{-\infty}^{\infty} |f(x+iy)|^p dy < \infty$$

In order to prove Theorem 1, we need the following lemma.

Lemma 1. *Let $Re(b) \geq Re(a)$ and $0 < t < 1$. Suppose $p \geq 2$, then for each $f \in H^p(\mathbb{D})$,*

$$\|T_t(f)\| \leq \frac{\Gamma(a+1)}{\Gamma(b+1)} \frac{t^{\frac{1}{p}+a-b-1}}{(1-t)^{\frac{1}{p}-b}} \|f\|_{H^p}.$$

The above inequality is also true for each $f \in H^p$ with $f(0) = 0$, when $1 < p < 2$.

Proof. We give the proof when a, b are real. For the proof of the complex case, we simply required to note the following for $t \in (0,1)$

$$|(1-t)^{b-a}| = (1-t)^{Re(b-a)}.$$

Let $p \geq 2$. Suppose $\mu(z) = \frac{1+z}{1-z}$ be the one-one analytic from $\mathbb{D}$ onto π_R with inverse $\mu^{-1}(z) = \frac{z-1}{z+1}$ and let

$$V(f)(z) = \frac{(4\pi)^{\frac{1}{p}}}{(1-z)^{\frac{2}{p}}} f(\mu(z)), \quad f \in H^p(\pi_R).$$

It is easy to check that the map is a linear isometry from $H^p(\pi_R)$ to H^p with inverse given by,

$$V^{-1}(g)(z) = \frac{1}{\pi^{\frac{1}{p}}(1+z)^{\frac{2}{p}}} g(\mu^{-1}(z)), \quad g \in H^p.$$

Let $\widetilde{T_t} : H^p(\pi_R) \to H^p(\pi_R)$ be the operators defined by $\widetilde{T_t} = V^{-1} T_t V$ and suppose $h \in H^p(\pi_R)$. A calculation shows that $\widetilde{T_t}$ are weighted composition operators given by

$$\widetilde{T_t} = \frac{\Gamma(a+1)}{\Gamma(b+1)} \frac{(1-\mu^{-1}(z))^{b-a}}{(1-t)^{\frac{2}{p}-b}} \left(\frac{1}{(t-1)\mu^{-1}(z)+1}\right)^{b-a+1-\frac{2}{p}} h(\Phi_t(z)), \quad 0 < t < 1,$$

where

$$\Phi_t(z) = \mu \circ \phi_t \circ \mu^{-1}(z) = \frac{t}{1-t} z + \frac{1}{1-t}$$

is an analytic function mapping π_R into itself. By an elementary argument we see that if $z \in \pi_R$ then $|(t-1)\mu^{-1}(z)+1| \geq t$ and since $b - a + 1 - \frac{2}{p} \geq 0$ we have

$$|\widetilde{T_t}(h)(z)| \leq \frac{\Gamma(a+1)}{\Gamma(b+1)} \frac{t^{\frac{2}{p}+a-b-1}}{(1-t)^{\frac{2}{p}-b}} |h(\Phi_t(z))||1-\mu^{-1}(z)|^{b-a}.$$

Since $|1-\mu^{-1}(z)| \leq 1 + |\mu^{-1}(z)| < 2$ as $|\mu^{-1}(z)| < 1$ and $b \geq a$, putting the change of variables $X = \frac{t}{1-t}x + \frac{1}{1-t}$ and $Y = \frac{t}{1-t}y$, and integrating for the norm we have

$$\begin{aligned}\|\widetilde{T_t}(h)\|_{H^p(\pi_R)} &= \sup_{0<x<\infty} \left(\int_{-\infty}^{\infty} |\widetilde{T_t}(h)(z)|^p dy\right)^{\frac{1}{p}} \\ &\leq M(a,b,p) \sup_{0<x<\infty} \left(\int_{-\infty}^{\infty} \left|h\left(\frac{t}{1-t}(x+iy) + \frac{1}{1-t}\right)\right|^p dy\right)^{\frac{1}{p}}\end{aligned}$$

$$= M(a,b,p) \sup_{\frac{1}{1-t}<X<\infty} \left(\int_{-\infty}^{\infty} |h(X+iY)|^p \frac{1-t}{t} dY \right)^{\frac{1}{p}}$$

$$\leq M(a,b,p) \sup_{0<X<\infty} \left(\int_{-\infty}^{\infty} |h(X+iY)|^p dY \right)^{\frac{1}{p}}$$

$$= M(a,b,p) \|h\|_{H^p(\pi_R)},$$

where $M(a,b,p) = 2^{b-a} \frac{\Gamma(a+1)}{\Gamma(b+1)} \frac{t^{\frac{2}{p}+a-b-1}}{(1-t)^{\frac{2}{p}-b}}$. □

For $1 < p < 2$, the proof is similar to the case $p \geq 2$, hence we emphasize only essential differences. Let $f \in H^p$ with $f(0) = 0$. Then $f(z) = z f_0(z)$ with $\|f\|_{H^p} = \|f_0\|_{H^p}$ and $\mathscr{H}_{a,b}$ can be written in the following form, that is,

$$\mathscr{H}_{a,b}(f)(z) = \int_0^1 T_t(f_0)(z) dt,$$

where

$$T_t(g)(z) = \frac{\Gamma(a+1)}{\Gamma(b+1)} \frac{t(1-t)^b (1-z)^{b-a}}{((t-1)z+1)^2} g\left(\frac{t}{(t-1)z+1} \right).$$

Now by following the proof of previous case (with the same notation) to estimate the norms of T_t. Letting $\widetilde{T_t} : H^p(\pi_R) \to H^p(\pi_R)$ we find

$$\widetilde{T_t} = \frac{\Gamma(a+1)}{\Gamma(b+1)} \frac{t(1-\mu^{-1}(z))^{b-a}}{(1-t)^{\frac{2}{p}-b}} \left(\frac{1}{(t-1)\mu^{-1}(z)+1} \right)^{b-a+2-\frac{2}{p}} h(\Phi_t(z)), \quad 0 < t < 1,$$

for each $h \in H^p(\pi_R)$. Since $p > 1$, we have $b - a + 2 - \frac{2}{p} > 0$, hence by proceeding similar to the previous case we conclude the result.

Using this we can proof the Theorem 2 as follows:

Proof. Suppose $f \in H^p$, $p \geq 2$. Using Lemma 1 and Minkowski's inequality, we have

$$\|\mathscr{H}_{a,b}(f)\|_{H^p} \leq \sup_{0 \leq r < 1} \left(\int_0^{2\pi} |\mathscr{H}_{a,b}(f)(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{\frac{1}{p}}$$

$$= \sup_{0 \leq r < 1} \left(\int_0^{2\pi} \left| \int_0^1 T_t(f)(re^{i\theta} dt) \right|^p \frac{d\theta}{2\pi} \right)^{\frac{1}{p}}$$

$$\leq \int_0^1 \sup_{0 \leq r < 1} \left(\int_0^{2\pi} |T_t(f)(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{\frac{1}{p}} dt$$

$$\leq \|f\|_{H^p} \int_0^1 \left(2^{b-a} \frac{\Gamma(a+1)}{\Gamma(b+1)} \frac{t^{\frac{1}{p}-b+a-1}}{(1-t)^{\frac{1}{p}-b}} \right) dt$$

$$= 2^{b-a} \frac{\Gamma(a+1)}{\Gamma(b+1)} B\left(\frac{1}{p} - b + a, b + 1 - \frac{1}{p} \right) \|f\|_{H^p}.$$

The case $1 < p < 2$ is also similar. □

For $a = b = \beta$, we have the following result.

Corollary 3. (a) *Let* $\beta \ge 0$. *If* $2 \le p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\|\mathscr{H}_\beta(f)\|_{H^p} \le \frac{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1}{q}+\beta\right)}{\Gamma(1+\beta)}\|f\|_{H^p}$$

for each $f \in H^p(\mathbb{D})$.

(b) *Let* $\beta \ge 0$. *If* $1 < p < 2$, *then*

$$\|\mathscr{H}(f)\|_{H^p} \le \frac{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1}{q}+\beta\right)}{\Gamma(1+\beta)}\|f\|_{H^p}$$

for each $f \in H^p(\mathbb{D})$ *with* $f(0) = 0$.

3. Remark

For H^∞, since $(1-t) < |1-tz|$ and $a > 0$, if $z \in \mathbb{D}$, then from (3), we have

$$|\mathscr{H}_{a,b}f(z)| \le \frac{\Gamma(a+1)}{\Gamma(b+1)}\int_0^1 f(t)(1-t)^{b-a-1}dt.$$

The above inequality gives, if $f \in H^\infty$ then $\mathscr{H}_{a,b}f \in H^\infty$ provided $b > a$. A simple calculation show that for the constant function 1

$$\mathscr{H}_{a,b}(1)(z) = \frac{\Gamma(a+1)}{\Gamma(b+1)}F(1,a+1;b+2;tz), \tag{4}$$

where ${}_2F_1(a,b;c;z)$ denotes the classical/Gaussian hypergeometric function is defined by the power series expansion

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{z^n}{n!}, \quad (|z| < 1).$$

Here a,b,c are complex numbers such that $c \neq -m$, $m = 0,1,2,3,\cdots$ and (a,n) is the Pochhammer's symbol/shifted factorial defined by Appel's symbol

$$(a,n) := a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n \in \mathbb{N}$$

and $(a,0) = 1$ for $a \neq 0$ and Γ is the gamma function given by $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$. Obviously, $F(a,b;c;z)$ is analytic on the unit disc $\mathbb{D}$. Many properties of the hypergeometric series including the Gauss and Euler transformations are found in standard textbooks such as ([1], [14]).

Since ${}_2F_1(a,b;c;z)$ is bounded on $\mathbb{D}$ if $c > a+b$, if $b \le a$ then equation (4) gives $\mathscr{H}_{a,b}(1)$ is not a bounded function. Thus $\mathscr{H}_{a,b}$ is bounded on H^∞ for $b > a$ and unbounded for $b \le a$. Also in the proof of the Theorem 1 if we assume $b > a$, then $\mathscr{H}_{a,b}$ is bounded on H^1. Therefore, it is interesting to know whether $\mathscr{H}_{a,b}$ is bounded on Hardy space H^p for $1 \le p < \infty$ whenever $b < a$.

4. Conclusion

It would be interesting to know whether $\mathscr{H}_{a,b}$ is bounded on Dirichlet-type spaces S^p $(0 < p < 2)$ and on Bergman spaces A^p $(2 < p < \infty)$ and other function spaces so that it generalize few results of S. Li and E. Diamantopolous ([6], [11]).

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

Both the authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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