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Research Article

A Second-Order Numerical Approximation for Volterra-Fredholm Integro-Differential Equations With Boundary Layer and an Integral Boundary Condition

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Abstract. This study introduces a novel second-order computational technique to effectively tackle Volterra-Fredholm integro-differential equations, which are characterized by integral conditions and boundary layers. Initially, some analytical properties of the solution are given. Then, the approach involves implementing a finite difference scheme on the piece-wise uniform mesh (Shishkin type mesh). It integrates a composite trapezoidal formula for the integral component and utilizes interpolating quadrature rules and linear exponential basis functions for the differential part. The analysis of the method demonstrates that both the numerical scheme and its convergence rate exhibit second-order accuracy, ensuring uniform convergence with respect to the small parameter in the discrete maximum norm. Finally, two test examples are given.

Keywords. Singular perturbation, Integro-differential equation, Finite difference methods, Piece-wise uniform mesh, Uniform convergent

Mathematics Subject Classification (2020). 65L11, 65L10, 65L12, 65L20, 65R20

1. Introduction

In this study, we are examining a numerical approach to solve Volterra-Fredholm integro-differential equations of the reaction-diffusion type, which include integral boundary conditions and singularly perturbed equations. These problems serve as mathematical models in various scientific fields, including chemical reaction modeling, biological systems, population dynamics, and nuclear reactor physics (Harrison [7], and Mei [11]). Reaction-diffusion equations are commonly used mathematical models in biology, physics, and chemistry, where the diffusion coefficients of substances are determined by parameters multiplied by their largest derivatives. Recently, reaction-diffusion systems have gained considerable attention as prototypes for pattern formation. It has been suggested that reaction-diffusion processes in biology (Harrison [7]) and may also play a role in the development of animal coats and skin pigmentation (Meinhardt [12], and Murray [14]). In a prior study, the authors explored a system of coupled singularly perturbed evolutionary reaction-diffusion equations and recommended solving such systems by applying the implicit Euler method to discretize the continuous problem in time (Munyakazi and Patidar [13]). The author introduced a second-order numerical method for solving Volterra-Fredholm integro-differential equations of the singularly perturbed reaction-diffusion type. A second-order convergent finite difference scheme was established on the Shishkin mesh. This scheme utilized composite trapezoidal formulas and interpolation quadrature rules for an integral part, along with linear exponential basis functions for the differential part. The convergence analysis of this scheme demonstrated its second-order convergence (Durmaz [5]). Cakir and Amiraliyev have introduced a uniform finite difference method for numerically solving nonlinear singularly perturbed convection-diffusion problems with nonlocal and third-type boundary conditions. The numerical method is implemented on a piece-wise uniform Shishkin-type mesh, and the convergence analysis of this method has been demonstrated (Cakir and Amiraliyev [3]).

Boundary layers typically arise in solving singularly perturbed differential equations, characterized by a small parameter that multiplies some or all of the highest-order terms in the equations. It is widely acknowledged that traditional numerical methods are unsuitable for such singularly perturbed cases (Matthews *et al.* [10]). Therefore, designing numerical approaches that are independent of the parameter value is crucial. Numerous techniques in the literature address the numerical solution of these problems (Cimen and Amiraliyev [4], Durmaz *et al.* [6], Kadalbajoo and Gupta [8]). Integro-differential equations are utilized in numerous scientific and technical fields as a means to approximate partial differential equations, which play a crucial role in describing a broad spectrum of continuous phenomena. Applications of these equations span various areas such as aerodynamics, aeroelastic phenomena, fluid dynamics, electrodynamics, different models of population growth, neural network modeling, epidemic models, mathematical modeling of the diffusion of discrete particles in turbulent fluid, the theory of population dynamics, compartmental systems, nuclear reactors, and the mathematical modeling of hereditary phenomena (Maleknejad *et al.* [9]).

The Volterra-Fredholm integral equations are derived from parabolic boundary value problems, and the mathematical modeling of the spatio-temporal development of an epidemic, and various physical and biological models (Wazwaz [16]). Such equations are prevalent in a broad spectrum of scientific and engineering domains, serving as crucial elements in modeling observed phenomena in other scientific subjects (Nemati [15]). The authors tackle the initial value problem of linear first-order Volterra-Fredholm integro-differential equations with a boundary layer, employing the finite difference method for their solution. Utilizing implicit difference rules and composite numerical quadrature rules, the proposed method is implemented on the Shishkin mesh. The effectiveness of this approach is examined through the solution of two test problems, with a comprehensive analysis of both stability and convergence (Cakir and Gunes [2]). The authors specifically address the linear first-order singularly perturbed Fredholm integro-differential equation, devising a fitted difference scheme on a Shishkin mesh to tackle this challenge. The method, grounded in the integral identities method and incorporating exponential basis functions and interpolating quadrature rules with weight and remainder terms in integral form, is shown to exhibit second-order convergence in the discrete maximum norm (Amiraliyev *et al.* [1]).

Consider the following singularly perturbed Volterra-Fredholm reaction-diffusion integro-differential equation with integral boundary condition

$$Lv := L_1 v + Vv + Fv = f(x), \quad x \in I = (0,l), \tag{1.1}$$

$$v(0) = A, \;\; v(l) = \int_0^l g(x)v(x)dx + B, \tag{1.2}$$

here

$$L_1 v = -\varepsilon^2 v''(x) + a(x)v(x), \;\; Vv = \lambda_1 \int_0^x \Lambda_1(x,t)v(t)dt, \;\; Fv = \lambda_2 \int_0^l \Lambda_2(x,t)v(t)dt,$$

where $0 < \varepsilon \ll 1$ is a perturbation parameter, λ, A and B are given constants. We suppose that $f(x), a(x) \ge \alpha > 0$ $(x \in [0,l])$, $\Lambda_1(x,t)$ and $\Lambda_2(x,t)$ $((x,t) \in [0,l]^2)$ are the sufficiently smooth functions. Under the assumptions accepted above, the problem (1.1)-(1.2) has a unique solution $v(x)$, which may exhibit boundary layers neighborhood of $x = 0$ and $x = l$.

2. Asymptotic Estimate

We now present some properties of (1.1)-(1.2), which are needed in later sections for the analysis of the appropriate numerical solution.

Lemma 2.1. *Assume that* $a, f \in C^2[0,l]$, $\frac{\partial^m \Lambda_1}{\partial x^m}, \frac{\partial^m \Lambda_2}{\partial x^m} \in C[0,l]^2$ $(m = 0,1,2)$. *Furthermore*

$$\max_{0\le x\le l} \int_0^l |g(x)|dx + \alpha^{-1}\left(|\lambda_1| \max_{0\le x\le l} \int_0^x |\Lambda_1(x,t)|dt + |\lambda_2| \max_{0\le x\le l} \int_0^l |\Lambda_2(x,t)|dt\right) < 1. \tag{2.1}$$

Then, the solution $v(x)$ *of the problem* (1.1)-(1.2) *satisfies the bounds*

$$\|v\|_\infty \le C \quad (C = (1-\delta)^{-1}[|A| + |B| + \alpha^{-1}\|f\|_\infty]) \tag{2.2}$$

and

$$|v^{(k)}(x)| \le C\left\{1 + \varepsilon^{-k}\left(e^{-\frac{\sqrt{\alpha}x}{\varepsilon}} + e^{-\frac{\sqrt{\alpha}(l-x)}{\varepsilon}}\right)\right\}, \quad x \in [0,l], k = 1,2. \tag{2.3}$$

Proof. Initially, when applying the maximum principle to the $L(\cdot)$ operator, we get

$$|v(x)| \le |A| + |B| + \alpha^{-1} \max_{0\le x\le l} |f(x)| + \max_{0\le x\le l} \int_0^l |g(x)|\,|v(x)|dx$$
$$+ \alpha^{-1}|\lambda_1| \max_{0\le x\le l} \int_0^x |\Lambda_1(x,t)|\,|v(x)|dt + \alpha^{-1}|\lambda_1| \max_{0\le x\le l} \int_0^l |\Lambda_2(x,t)|\,|v(x)|dt.$$

Considering inequality (2.1) and δ transform together, we get

$$\|v\|_\infty \le (1-\delta)^{-1}[|A| + |B| + \alpha^{-1}\|f\|_\infty],$$

where

$$\delta = \max_{0\le x\le l}\int_0^l |g(x)|dx + \alpha^{-1}\left(|\lambda_1|\max_{0\le x\le l}\int_0^x |\Lambda_1(x,t)|dt + |\lambda_2|\max_{0\le x\le l}\int_0^l |\Lambda_2(x,t)|dt\right).$$

Here, inequality (2.2) is shown. Next, we will prove equation (2.3). We can rewrite (1.1) as follows

$$|v''| = \frac{1}{\varepsilon^2}\left|f(x) - a(x)v(x) - \lambda_1\int_0^x \Lambda_1(x,t)v(t)dt - \lambda_2\int_0^l \Lambda_2(x,t)v(t)dt\right|.$$

After taking into account (2.2), we get

$$|v''| \le \frac{C}{\varepsilon^2},\quad 0\le x\le l.$$

Then, the following differential formula is used to estimate both $|v'(0)|$ and $|v'(l)|$. We know that

$$g'(x) = g(\alpha_0;\alpha_1) - \int_{\alpha_0}^{\alpha_1} K_0(\xi,x)g''(\xi)d\xi,\quad g\in C^2[0,l],\ \alpha_0\le x\le\alpha_1, \tag{2.4}$$

where

$$K_0(\xi,x) = T_0(\xi-x) - (\alpha_1-\alpha_0)^{-1}(\xi-\alpha_0),\quad T_0(\lambda) = \begin{cases}1, & \lambda\ge 0,\\ 0, & \lambda<0.\end{cases}$$

Equality (2.4) with the values $g(x) = v(x)$, $x = 0$, $\alpha_0 = 0$, $\alpha_1 = \varepsilon$ yields

$$|v'(0)| \le \left|\frac{v(\varepsilon)-v(0)}{\varepsilon} - \int_0^\varepsilon K_0(\xi,x)v''(\xi)d\xi\right| \le \frac{C}{\varepsilon}. \tag{2.5}$$

Similarly, using (2.4) with the values $g(x) = v(x)$, $x = l$, $\alpha_0 = l-\varepsilon$, $\alpha_1 = \varepsilon$, we get

$$|v'(l)| \le \left|\frac{v(\varepsilon)-v(l-\varepsilon)}{\varepsilon} - \int_{l-\varepsilon}^\varepsilon K_0(\xi,x)v''(\xi)d\xi\right| \le \frac{C}{\varepsilon}. \tag{2.6}$$

Next, when we take the derivative of (1.1), we get

$$-\varepsilon^2 w''(x) + a(x)w'(x) = \phi(x). \tag{2.7}$$

Also, from inequalities (2.5) and (2.6), we get

$$w(0) = O\left(\frac{1}{\varepsilon}\right),\quad w(l) = O\left(\frac{1}{\varepsilon}\right), \tag{2.8}$$

with $v'(x) = w(x)$, and

$$\phi(x) = f'(x) - a'(x)v(x) - \lambda_1\int_0^x \frac{\partial}{\partial x}\Lambda_1(x,t)v(t)dt - \lambda_1\Lambda_1(x,x)v(x) - \lambda_2\int_0^l \frac{\partial}{\partial x}\Lambda_2(x,t)v(t)dt.$$

When we take into account (2.2) in the above equation, we get

$$|\phi(x)| \le C. \tag{2.9}$$

In order to estimate the solution for problem (2.7)-(2.8), we provide it in the form of $w(x) = w_0(x) + w_1(x)$, where $w_0(x)$ and $w_1(x)$ are the solutions of the following problems, respectively

$$Lw_0 = \phi(x),\quad 0<x<l, \tag{2.10}$$

$$w_0(0) = w_0(l) = 0, \tag{2.11}$$

$$Lw_1 = 0,\quad 0<x<l, \tag{2.12}$$

$$w_1(0) = O\left(\frac{1}{\varepsilon}\right),\ w_1(l) = O\left(\frac{1}{\varepsilon}\right). \tag{2.13}$$

For the problem (2.10)-(2.11), utilizing the maximum principle and take into account (2.9), one can write

$$|w_0(x)| \le \alpha^{-1}\|\phi\|_\infty \le C, \quad 0 \le x \le l. \tag{2.14}$$

When the comparison principle is applied to the problem (2.12)-(2.13), we get

$$|w_1(x)| \le |\nu(x)|, \tag{2.15}$$

here the function $\kappa(x)$ is the solution to the following problem

$$-\varepsilon^2\kappa''(x) + \alpha\kappa(x) = 0, \quad 0 < x < l, \tag{2.16}$$

$$\kappa(0) = |w_1(0)|, \ \kappa(l) = |w_1(l)|. \tag{2.17}$$

The solution of the problem (2.16)-(2.17) with constant coefficients is clearly

$$\kappa(x) = \frac{1}{\sinh\left(\frac{\sqrt{\alpha}l}{\varepsilon}\right)}\left\{\kappa(0)\sinh\left(\frac{\sqrt{\alpha}(l-x)}{\varepsilon}\right) + \kappa(l)\sinh\left(\frac{\sqrt{\alpha}l}{\varepsilon}\right)\right\}.$$

Therefore, considering (2.13), we get

$$\kappa(l) \le \frac{C}{\varepsilon}\{e^{-\frac{\sqrt{\alpha}l}{\varepsilon}} + e^{-\frac{\sqrt{\alpha}(l-x)}{\varepsilon}}\}. \tag{2.18}$$

When, we take into account the bounds (2.14)-(2.15) and (2.18) in the inequality

$$|v'(x)| \le |w_0(x)| + |w_1(x)|,$$

and this implies (2.3) inequality for $k = 1$. The proof of $k = 2$ can be shown similarly. Hence the lemma is proven. □

3. Discretization and Mesh

Let ω_N be any non-uniform mesh on $[0, l]$:

$$\omega_N = \{0 < x_1 < \ldots < x_{N-1} < l, h_i = x_i - x_{i-1}, i = 1, N-1,\}$$

and $\bar{\omega}_N = \omega_N \cup \{x_0 = 0, x_N = l\}$. We establish the difference scheme on point-wise mesh to solve the problem (1.1)-(1.2). For an even number N, we divide each of the subintervals $[0,\sigma]$, $[\sigma, l-\sigma]$ and $[l-\sigma, l]$ into $\frac{N}{4}$ equidistant subintervals. The transition point σ is determined as $\sigma = \min\{\frac{l}{4}, \frac{\varepsilon}{\sqrt{\alpha}\ln N}\}$. We use the notation $h^{(1)}$ and $h^{(2)}$. Hence, the mesh stepsize are $h^{(1)} = \frac{4\sigma}{N}$, $h^{(2)} = \frac{2(l-2\sigma)}{N}x_i$ and node points are specified as

$$x_i = \begin{cases} ih^{(1)}, & i = 0,1,\ldots,N/4, \\ \sigma + (i - N/4)h^{(2)}, & i = N/4+1,\ldots,3N/4, \\ l - \sigma + (i - 3N/4)h^{(1)}, & i = 3N/4+1,\ldots,N. \end{cases}$$

We proceed to establish a difference technique. To obtain a difference approach for the problem (1.1), we begin with the following identity:

$$\chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}(L_1v + Vv + Fv)\varphi_i(x)dx = \chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} f(x)\varphi_i(x)dx, \tag{3.1}$$

with the basis functions

$$\varphi_i(x) = \begin{cases} \varphi_i^{(1)}(x) \equiv \frac{\sinh\gamma_i(x-x_{i-1})}{\sinh\gamma_i h_i}, & x \in (x_{i-1}, x_i), \\ \varphi_i^{(2)}(x) \equiv \frac{\sinh\gamma_i(x_{i+1}-x)}{\sinh\gamma_i h_{i+1}}, & x \in (x_i, x_{i+1}), \\ 0, & x \notin (x_{i-1}, x_{i+1}). \end{cases}$$

Note that the functions $\varphi_i^{(1)}(x)$ and $\varphi_i^{(2)}(x)$ are the solutions of the following problems:

$$-\varepsilon^2\varphi''(x)+a_i\varphi(x)=0,\ \ x\in(x_{i-1},x_i),\qquad -\varepsilon^2\varphi''(x)+a_i\varphi(x)=0,\ \ x\in(x_i,x_{i+1}),$$
$$\varphi(x_{i-1})=0,\ \ \varphi(x_i)=1,\qquad \varphi(x_i)=1,\ \ \varphi(x_{i+1})=0,$$

that $\gamma_i=\frac{\sqrt{a(x_i)}}{\varepsilon}$, and

$$\chi_i=\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}\varphi_i(x)dx=\hbar_i^{-1}\gamma_i^{-1}\left(\tanh\left(\frac{\gamma_i h_i}{2}\right)+\tanh\left(\frac{\gamma_i h_{i+1}}{2}\right)\right).$$

Using the suitable quadrature formulas with the remainder term in integral form for the first term in the left side of (3.1), we obtain

$$\chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}(-\varepsilon^2 v''(x)+a(x)v(x))\varphi_i(x)dx$$
$$=-\varepsilon^2\rho^2 v_i+a_iv_i+\chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}[a(x)-a(x_i)]v(x)\varphi_i(x)dx, \tag{3.2}$$

where

$$\rho^2 v_i=\frac{\rho_i^{(2)}v_{x,i}-\rho_i^{(1)}v_{\bar{x},i}}{\hbar_i} \tag{3.3}$$

and

$$\rho_i^{(1)}=\frac{a_ih_i\hbar_i}{\varepsilon^2\sinh(\gamma_ih_i)\left[\tanh\left(\frac{\gamma_ih_i}{2}\right)+\tanh\left(\frac{\gamma_ih_{i+1}}{2}\right)\right]},$$
$$\rho_i^{(2)}=\frac{a_ih_{i+1}\hbar_i}{\varepsilon^2\sinh(\gamma_ih_{i+1})\left[\tanh\left(\frac{\gamma_ih_i}{2}\right)+\tanh\left(\frac{\gamma_ih_{i+1}}{2}\right)\right]}.$$

Next, with Newton's interpolation formula for the last term on the right side of (3.2), we get $a(x)=a(x_i)+(x-x_i)a_{x,i}+\frac{1}{2}(x-x_i)(x-x_{i+1})a''(\xi_i(x))$ and also using $v(x)=v(x_i)+\int_{x_i}^{x}v'(t)dt$, we can express as

$$\chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}[a(x)-a(x_i)]v(x)\varphi_i(x)dx=a_{x,i}\chi_i^{-1}\eta_iv_i+R_i^{(1)},$$

where

$$\eta_i=\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}(x-x_i)\varphi_i(x)dx=\hbar_i^{-1}\gamma_i^{-1}\left(\frac{h_i}{\sinh(\gamma_ih_i)}-\frac{h_{i+1}}{\sinh(\gamma_ih_{i+1})}\right),$$
$$R_i^{(1)}=\frac{1}{2}\chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}(x-x_i)(x-x_{i+1})a''(\xi_i(x))v(x)\varphi_i(x)dx$$
$$+\chi_i^{-1}\hbar_i^{-1}a_{x,i}\int_{x_{i-1}}^{x_{i+1}}(x-x_i)\varphi_i(x)\left(\int_{x_i}^{x}v'(t)dt\right)dx. \tag{3.4}$$

From here, (3.2) is obtained as follows:

$$\chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}L_1v\varphi_i(x)dx=-\varepsilon^2\rho^2v_i+(a_i+a_{x,i}\chi_i^{-1}\eta_i)v_i+R_i^{(1)}. \tag{3.5}$$

Similarly, we derive

$$\chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}f(x)\varphi_i(x)dx=f_i+f_{x,i}\chi_i^{-1}\eta_i+R_i^{(2)}=\widetilde{f}_i+R_i^{(2)}, \tag{3.6}$$

where

$$\widetilde{f}_i=f_i+f_{x,i}\chi_i^{-1}\eta_i, \tag{3.7}$$

$$R_i^{(2)} = \frac{1}{2}\chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}(x-x_i)(x-x_{i+1})f''(\xi_i(x))v(x)\varphi_i(x)dx. \tag{3.8}$$

Next, when Taylor expansion is applied to find the approximation of the terms Λ_1 and Λ_2, we get

$$\Lambda_1(x,t) = \Lambda_1(x_i,t)+(x-x_i)\frac{\partial}{\partial x}\Lambda_1(x_i,t)+\frac{(x-x_i)^2}{2}\frac{\partial^2}{\partial x^2}\Lambda_1(\xi_i(x),t)$$

and

$$\Lambda_2(x,t) = \Lambda_2(x_i,t)+(x-x_i)\frac{\partial}{\partial x}\Lambda_2(x_i,t)+\frac{(x-x_i)^2}{2}\frac{\partial^2}{\partial x^2}\Lambda_2(\xi_i(x),t),$$

therefore

$$\chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}\left(\lambda_1\int_0^x\Lambda_1(x,t)v(t)dt\right)\varphi_i(x)dx+\chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}\left(\lambda_2\int_0^l\Lambda_2(x,t)v(t)dt\right)\varphi_i(x)dx$$

$$= \lambda_1\int_0^{x_i}\bar{\Lambda}_1(x_i,t)v(t)dt+\hbar_i\eta_i\lambda_1\Lambda_1(x_i,x_i)v_i+\lambda_2\int_0^l\bar{\Lambda}_2(x_i,t)v(t)dt+R_i^{(3)}+R_i^{(4)}, \tag{3.9}$$

where

$$\bar{\Lambda}_1(x_i,t) = \Lambda_1(x_i,t)+\hbar_i\eta_i\frac{\partial}{\partial x}\Lambda_1(x_i,t),\quad \bar{\Lambda}_2(x_i,t) = \Lambda_2(x_i,t)+\hbar_i\eta_i\frac{\partial}{\partial x}\Lambda_2(x_i,t), \tag{3.10}$$

$$R_i^{(3)} = \chi_i^{-1}\hbar_i^{-1}\lambda_1\int_{x_{i-1}}^{x_{i+1}}dx\varphi_i(x)\int_{x_{i-1}}^{x_{i+1}}\frac{d^2}{d\xi^2}\left(\int_0^\xi\Lambda_1(\xi_i(x),t)v(t)dt\right)T_1(\xi-t)d\xi, \tag{3.11}$$

where $T_s(\lambda)$ is defined in [3].

$$R_i^{(4)} = \frac{1}{2}\chi_i^{-1}\hbar_i^{-1}\lambda_2\int_{x_{i-1}}^{x_{i+1}}(x-x_i)^2\varphi_i(x)\left(\int_0^l\frac{d^2}{dx^2}\Lambda_2(\xi_i(x),t)v(t)dt\right)dx. \tag{3.12}$$

Finally, if the composite trapezoidal formula is used to compute the first and third components on the right side of (3.9) and the remainder term is found in the integral form, we get

$$\lambda_1\int_0^{x_i}\bar{\Lambda}_1(x_i,t)v(t)dt = \lambda_1\sum_{j=0}^{i}\hbar_j\bar{\Lambda}_{1,ij}v_j+R_i^{(5)}, \tag{3.13}$$

where

$$R_i^{(5)} = \frac{1}{2}\lambda_1\sum_{j=1}^{i}\int_{x_{j-1}}^{x_j}(x_j-\xi)(x_{j-1}-\xi)\frac{d^2}{d\xi^2}(\bar{\Lambda}_1(x_i,\xi)v(\xi))d\xi \tag{3.14}$$

and

$$\lambda_2\int_0^l\bar{\Lambda}_2(x_i,t)v(t)dt = \lambda_2\sum_{j=0}^{N}\hbar_j\bar{\Lambda}_{2,ij}v_j+R_i^{(6)}, \tag{3.15}$$

where

$$R_i^{(6)} = \frac{1}{2}\lambda_2\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}(x_j-\xi)(x_{j-1}-\xi)\frac{d^2}{d\xi^2}(\bar{\Lambda}_2(x_i,\xi)v(\xi))d\xi. \tag{3.16}$$

To approximate the boundary condition (1.2), we once again employ the composite trapezoidal integration rule and thereby obtain:

$$v(l) = \sum_{i=1}^{N}\hbar_i g(x_i)v_i+r_i+B, \tag{3.17}$$

where

$$r_i = \frac{1}{2}\sum_{j=1}^{i}\int_{x_{j-1}}^{x_j}(x_j-\xi)(x_{j-1}-\xi)\frac{d^2(g(\xi)v(\xi))}{d\xi^2}d\xi. \tag{3.18}$$

Then we assembling (3.5)-(3.6), (3.9), (3.13), (3.15) in (3.1) and we obtain following difference relation

$$L_N v_i := -\varepsilon^2\rho^2 v_i + \widetilde{a}_i v_i + V v_i + F v_i = \widetilde{f}_i - R_i, \tag{3.19}$$

where

$$\widetilde{a}_i = a_i + (a_{x,i}\chi_i^{-1} + \hbar_i\lambda_1\Lambda_{1,ii})\eta_i,\ \ V v_i = \lambda_1\sum_{j=0}^{i}\hbar_j\bar{\Lambda}_{1,ij}v_j,\ \ F v_i = \lambda_2\sum_{j=0}^{N}\hbar_j\ \bar{\Lambda}_{2,ij}v_j, \tag{3.20}$$

with remainder term

$$R_i = R_i^{(1)} - R_i^{(2)} + R_i^{(3)} + R_i^{(4)} + R_i^{(5)} + R_i^{(6)}, \tag{3.21}$$

where $R_i^{(k)}$ are defined by (3.4), (3.8), (3.11), (3.12), (3.14) and (3.16), respectively. By disregarding the error term in (3.19) the following difference scheme is submitted for the approximate solution

$$L_N y_i := -\varepsilon^2\rho^2 y_i + \widetilde{a}_i y_i + V y_i + F y_i = \widetilde{f}_i, \tag{3.22}$$

$$y(0) = A,\ \ y(l) = \sum_{i=1}^{N}\hbar_i g(x_i)y_i + B, \tag{3.23}$$

where

$$V y_i = \lambda_1\sum_{j=0}^{i}\hbar_j\bar{\Lambda}_{1,ij}y_j,\quad F y_i = \lambda_2\sum_{j=0}^{N}\hbar_j\bar{\Lambda}_{2,ij}y_j.$$

4. Error Estimates

In this section, the convergence analysis of the suggested method is investigated. Note that the error function $z_i = y_i - v_i$, $i = \overline{0,N}$ is the solution of the following problem:

$$L_N z_i := -\varepsilon^2\rho^2 z_i + \widetilde{a}_i z_i + V z_i + F z_i = R_i, \tag{4.1}$$

$$z_0 = 0,\ \ z_N = \sum_{i=1}^{N}\hbar_i g(x_i)z_i - r. \tag{4.2}$$

Lemma 4.1. *Let* $a, f \in C^2[0,l]$ *and* $\frac{\partial^m\Lambda_1}{\partial x^m}, \frac{\partial^m\Lambda_2}{\partial x^m} \in C[0,l]^2$, $(m = 0,1,2)$. *Furthermore*

$$\alpha^{-1}\left(|\lambda_1|\max_{1\le i\le l}\sum_{j=1}^{i}\hbar_j|\bar{\Lambda}_{1,ij}| + |\lambda_2|\max_{1\le i\le l}\sum_{j=1}^{N}\hbar_j|\bar{\Lambda}_{2,ij}| + \max_{1\le i\le l}\sum_{j=1}^{N}\hbar_j|g_j|\right) < 1. \tag{4.3}$$

Then, for the solution z of the difference problem (4.1)-(4.2)*, we can establish the following estimate*

$$\|z\|_{\infty,\bar{\omega}_N} \le (1-\bar{\delta})^{-1}\alpha^{-1}\|R\|_{\infty,\omega_N}. \tag{4.4}$$

Proof. By applying the discrete maximum principle to (4.1)-(4.2) difference problem, one can write

$$\|z\|_{\infty,\bar{\omega}_N} \le \alpha^{-1}\|R\|_{\infty,\omega_N} + |\lambda_1|\max_{1\le i\le l}\sum_{j=1}^{i}\hbar_j|\bar{\Lambda}_{1,ij}|\|z\|_{\infty,\bar{\omega}_N}$$

$$+|\lambda_2|\max_{1\le i\le l}\sum_{j=1}^{N}\hbar_j|\bar{\Lambda}_{2,ij}|\,\|z\|_{\infty,\omega_N}+\max_{1\le i\le l}\sum_{j=1}^{i}\hbar_j|g_j|\|z\|_{\infty,\bar{\omega}_N}$$
$$\le(1-\bar{\delta})^{-1}\alpha^{-1}\|R\|_{\infty,\omega_N},$$

we obtain

$$\|z\|_{\infty,\bar{\omega}_N}\le(1-\bar{\delta})^{-1}\alpha^{-1}\|R\|_{\infty,\omega_N},$$

where

$$\bar{\delta}=\alpha^{-1}\left(|\lambda_1|\max_{1\le i\le l}\sum_{j=1}^{i}\hbar_j|\bar{\Lambda}_{1,ij}|+|\lambda_2|\max_{1\le i\le l}\sum_{j=1}^{N}\hbar_j|\bar{\Lambda}_{2,ij}|+\max_{1\le i\le l}\sum_{j=1}^{i}\hbar_j|g_j|\right).$$

Hence, we arrive at (4.4). □

Lemma 4.2. *Assume that $a,f,g\in C^2[0,l]$ and $\frac{\partial^m\Lambda_1}{\partial x^m},\frac{\partial^m\Lambda_2}{\partial x^m},\frac{\partial^m\Lambda_1}{\partial t\partial x^m},\frac{\partial^m\Lambda_2}{\partial t\partial x^m}\in C^2[0,l]^2$ $(m=0,1,2)$. Following that, we can express the estimates the remainder terms R_i and $r_i$satisfy the following inequalities*

$$\|R\|_{\infty,\omega_N}\le CN^{-2}\ln N, \tag{4.5}$$
$$|r|\le CN^{-2}\ln N. \tag{4.6}$$

Proof. We will estimate the error terms $R_i^{(k)}$ $(k=1,\dots,6)$ and r_i, one by one only for the $\sigma=\frac{\varepsilon}{\sqrt{\alpha}}\ln N$ case. The case of $\sigma=\frac{l}{4}$ can be easily demonstrated by classical technique. Next, start by estimating the error term R_i^1 for the interval $[0,\sigma]$. Let $a\in C^2[0,l]$, $|x-x_i|\le\max(h_i,h_{i+1})$, $|x-x_{i+1}|\le 2\hbar_i$, $h^{(1)},h^{(2)}\le CN^{-1}$, $|v(x)|\le C$ and from (3.4), we get

$$|R_i^{(1)}|\le CN^{-2}+CN^{-1}\int_{x_{i-1}}^{x_{i+1}}\varepsilon^{-1}\left(e^{-\frac{\sqrt{\alpha}x}{\varepsilon}}+e^{-\frac{\sqrt{\alpha}(l-x)}{\varepsilon}}\right)dx. \tag{4.7}$$

From (4.7), we can write

$$|R_i^{(1)}|\le CN^{-2}+CN^{-1}4h^{(1)}\varepsilon^{-1}\le CN^{-2}\ln N,\quad 1\le i\le\frac{N}{4}-1. \tag{4.8}$$

It can be shown that in a similar way for $[l-\sigma,l]$. For $\frac{N}{4}+1\le i\le\frac{3N}{4}-1$ from (4.7), we get

$$|R_i^{(1)}|\le CN^{-2}+CN^{-1}\alpha^{-1}\left(e^{-\frac{\sqrt{\alpha}x_{N/4}}{\varepsilon}}+e^{-\frac{\sqrt{\alpha}(l-x_{3N/4})}{\varepsilon}}\right)\le CN^{-2}. \tag{4.9}$$

Here $\frac{N}{4}+1\le i\le\frac{3N}{4}-1$. For the $i=\frac{N}{4}$ inequality (4.7), we get

$$|R_i^{(\frac{N}{4})}|\le CN^{-2}+CN^{-1}\alpha^{-1}\left(e^{-\frac{4\ln N}{N}}e^{-\frac{\sqrt{\alpha}x_{N/4}}{\varepsilon}}+e^{-\frac{\sqrt{\alpha}\sigma}{\varepsilon}}\right)\le CN^{-2}. \tag{4.10}$$

We can show that similarly to the $i=\frac{3N}{4}$ inequality (4.7). From (4.8) and (4.10), one can write

$$|R_i^{(1)}|\le CN^{-2}\ln N,\quad i=\overline{1,N-1}. \tag{4.11}$$

Next, we estimate $R_i^{(2)}$, let $f\in C^2[0,l]$, $|x-x_i|\le\max(h_i,h_{i+1})$, $|x-x_{i+1}|\le 2\hbar_i$, $h^{(1)},h^{(2)}\le CN^{-1}$ and from (3.8), we get

$$|R_i^{(2)}|\le C\chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}|(x-x_i)(x-x_{i+1})|\varphi_i(x)dx\le CN^{-2}. \tag{4.12}$$

Then, using the inequality (2.2) and the boundedness of $\frac{\partial_1^2\Lambda}{\partial x^2}$, from (3.11)

$$|R_i^{(3)}|\le C\{\max(h_i,h_{i+1})\}^2\le CN^{-2}. \tag{4.13}$$

Similar to above considering of (2.2) and the boundedness of $\frac{\partial_2^2\Lambda}{\partial x^2}$, from (3.12), we get

$$|R_i^{(4)}| \le C\{\max(h_i,h_{i+1})\}^2 \le CN^{-2}. \tag{4.14}$$

Next, we estimate the remainder term $R_i^{(6)}$ for this from (3.16), taking into account (2.3), one can write

$$|R_i^{(6)}| \le C\left[\sum_{j=1}^{N} h_j^3 + \sum_{j=1}^{N}\int_{x_{j-1}}^{x_j} \varepsilon^{-1}(x_j-\xi)(x_{j-1}-\xi)\left(e^{-\frac{\sqrt{\alpha}\xi}{\varepsilon}} + e^{-\frac{\sqrt{\alpha}(l-\xi)}{\varepsilon}}\right)d\xi\right]. \tag{4.15}$$

Now, we estimate of inequality (4.15). Firstly, for the first term in the right-side of (4.15), one can write

$$\sum_{j=1}^{N} h_j^3 = \frac{N}{2}|h^{(1)}|^3 + \frac{N}{2}|h^{(2)}|^3 \le CN^{-2}. \tag{4.16}$$

For the second term in the right-side of (4.15), we get $\vartheta(x) = e^{-\frac{\sqrt{\alpha}x}{\varepsilon}} + e^{-\frac{\sqrt{\alpha}(l-x)}{\varepsilon}}$,

$$\begin{aligned}\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j} \varepsilon^{-1}(x_j-\xi)(x_{j-1}-\xi)\vartheta(\xi)d\xi &= \varepsilon^{-1}\sum_{j=1}^{\frac{N}{4}}\int_{x_{j-1}}^{x_j}(x_j-\xi)(x_{j-1}-\xi)\vartheta(\xi)d\xi \\ &\quad + \varepsilon^{-1}\sum_{j=\frac{N}{4}+1}^{\frac{3N}{4}}\int_{x_{j-1}}^{x_j}(x_j-\xi)(x_{j-1}-\xi)\vartheta(\xi)d\xi \\ &\quad + \varepsilon^{-1}\sum_{j=\frac{3N}{4}+1}^{N}\int_{x_{j-1}}^{x_j}(x_j-\xi)(x_{j-1}-\xi)\vartheta(\xi)d\xi.\end{aligned} \tag{4.17}$$

Next, consider the terms in the above equation one by one. Firstly

$$\varepsilon^{-1}\sum_{j=1}^{\frac{N}{4}}\int_{x_{j-1}}^{x_j}(x_j-\xi)(x_{j-1}-\xi)\vartheta(\xi)d\xi \le CN^{-2}\ln N. \tag{4.18}$$

Secondly,

$$\varepsilon^{-1}\sum_{j=\frac{N}{4}+1}^{\frac{3N}{4}}\int_{x_{j-1}}^{x_j}(x_j-\xi)(x_{j-1}-\xi)\vartheta(\xi)d\xi \le 4\alpha^{-2}h^{(2)}N^{-1} \le CN^{-2}. \tag{4.19}$$

Considering the (4.16) and (4.19) together, we can write

$$|R_i^{(6)}| \le CN^{-2}\ln N, \quad i = 1,\overline{N-1}. \tag{4.20}$$

Similar to above, considering the (2.3) inequality in (3.14), we can write

$$|R_i^{(5)}| \le CN^{-2}\ln N, \quad i = \overline{1,N-1}. \tag{4.21}$$

The inequalities (4.11)-(4.14), (4.20) and (4.21), we can write

$$\|R\|_{\infty,\omega_N} \le CN^{-2}\ln N. \tag{4.22}$$

Finally, estimating the remaining term r. Under the condition of Lemma 4.1 and (3.18), we can write

$$|r| \le CN^{-2}\ln N. \tag{4.23}$$

Thus so, due to (4.22) and (4.23), the proof of the lemma is finished. □

Theorem 4.1. *Let v be the solution of problem* (1.1)-(1.2)*, and y be the solution of discrete problem* (3.22)-(3.23)*. So, the following estimate is satisfied*

$$\|y-v\|_{\infty,\bar{\omega}_N} \leq CN^{-2}\ln N.$$

Proof. Considering Lemma 4.1 and Lemma 4.2 together, one can deduce the proof of the theorem. □

5. Algorithm and Examples

The Thomas algorithm, also known as the Tridiagonal Matrix Algorithm (TDMA), is employed as a robust numerical technique system of linear equations characterized by a tridiagonal coefficient matrix, which commonly arises in the discretization of differential equations. The core concept behind Thomas' algorithm lies in its ability to efficiently solve systems of equations in which the matrix of coefficients exhibits a tridiagonal structure. In the context of numerical analysis, this is often encountered when numerically approximating integral-differential equations. One of the key advantages of Thomas' algorithm is its computational efficiency. With a time complexity of $O(n)$, where n is the size of the system, the algorithm stands out as a favorable choice for solving tridiagonal systems when compared to more general Gaussian elimination methods. This efficiency is particularly advantageous in numerical simulations and computations, making Thomas' algorithm a widely adopted tool in the realm of numerical analysis. In the context of the current study, the application of Thomas' algorithm to the solution of integral-differential equations is instrumental in obtaining accurate and timely results. The algorithm's ability to handle tridiagonal systems ensures that the numerical approach aligns seamlessly with the inherent structure of the equations under consideration. The numerical results presented in this section thus showcase not only the effectiveness of Thomas' algorithm in solving the specific integral-differential equations but also highlight its efficiency in the broader context of numerical analysis and simulations. In here

$$A_i y_{i-1}^{(n)} - C_i y_i^{(n)} + B_i y_{i+1}^{(n)} = F_i^{(n)}, \quad i=\overline{1,N-1},\ n=1,2,\dots,$$

$$y_0^{(n)} = A,\ \ y_N = \sum_{i=1}^{N} \hbar_i g(x_i) y_i + B,\ \ y_i^{(0)} = \frac{A+B+\sum\limits_{i=1}^{N} \hbar_i g(x_i) y_i}{2}, \quad i=\overline{1,N-1},$$

where

$$A_i = \frac{\varepsilon\rho_i^{(1)}}{\hbar_i h_i},\ B_i = \frac{\varepsilon\rho_i^{(2)}}{\hbar_i h_{i+1}},\ C_i = \frac{\varepsilon\rho_i^{(2)}}{\hbar_i h_{i+1}} + \frac{\varepsilon\rho_i^{(1)}}{\hbar_i h_i},\ F_i^{(n)} = \widetilde{f}_i - \lambda_1 \sum_{j=0}^{i} \hbar_j \bar{\Lambda}_{1,ij} y_j^{(n)} - \lambda_2 \sum_{j=0}^{N} \hbar_j \bar{\Lambda}_{2,ij} y_j^{(n)},$$

$$\alpha_1 = 0,\ \alpha_{i+1} = \frac{B_i}{C_i - \alpha_i A_i},\ \beta_1 = 1,\ \beta_{i+1}^{(n)} = \frac{A_i\beta_i - F_i^{(n)}}{C_i - \alpha_i A_i}$$

and

$$y_i^{(n)} = \alpha_{i+1} y_{i+1}^{(n)} + \beta_{i+1}^{(n)}, \quad i=N-1,N-2,\dots,1.$$

Example 5.1. Consider the following problem

$$-\varepsilon^2 v''(x) + (2-e^{-x})v(x) + \int_0^x (e^{-\sin(\pi x t)})dt + \int_0^1 (e^{-x\sin(\frac{\pi x}{4})})dt = \frac{1}{1+x^2} + \varepsilon x(\varepsilon - x) - 2,$$

$$v(0)=0,\ v(1)=2+\int_0^1 \sinh(x)v(x)dx,$$

where $0<x<1$. There is no exact solution to the above problem. Therefore, we apply the double-mesh method. Here, the maximum errors and computed ε-uniform maximum point-wise errors are defined

$$e_\varepsilon^N=\max_i |y_i^{\varepsilon,N}-\widetilde{y}_{2i}^{\varepsilon,2N}|_{\infty,\bar{\omega}_N},\quad e^N=\max_\varepsilon e_\varepsilon^N,$$

where $\widetilde{y}_{2i}^{\varepsilon,2N}$ is the approximate solution for the matching method

$$\widetilde{\omega}_{2N}=\{x_{i/2}: i=0,1,\dots,2N\},$$

with $x_{i/2}=\frac{x_i+x_{i+1}}{2}$, $i=0,1,\dots,N-1$. The convergence rates are specified as follows

$$p^N=\frac{\ln(e^N/e^{2N})}{\ln 2}.$$

The results of the first example considered are given in Table 1.

Table 1. ε-uniform maximum point-wise error e^N and convergence rate p^N

ε	$N=128$	$N=256$	$N=512$	$N=1024$	$N=2048$
2^{-8}	0.0492095594 1.9708	0.0125539134 1.1509	0.0056537235 1.9405	0.0014728998 2.0459	0.0003566971
2^{-9}	0.1236404262 2.2465	0.0260548775 2.0452	0.0063126355 1.9511	0.0016325579 2.0862	0.0003844621
2^{-10}	0.2616609842 2.0162	0.0646812718 2.2651	0.0134564356 2.0880	0.0031649987 2.0803	0.0007484156
2^{-11}	0.4834287697 1.7909	0.1396993457 2.0757	0.0331405260 2.2748	0.0068482458 2.0297	0.0016771539
2^{-12}	0.7788200315 1.5100	0.2734518054 1.9193	0.0722949041 2.1066	0.0167866548 1.9798	0.0042557912
2^{-13}	0.9648169105 1.1131	0.4460292738 1.7441	0.1331502783 1.9886	0.0335508904 2.0422	0.0081456517
2^{-14}	1.1729339836 0.5946	0.7767388124 1.5757	0.2605869498 1.7266	0.0787415220 1.8031	0.0225647811

Example 5.2. Consider the problem

$$-\varepsilon^2 v''(x)+(3-e^{-x})v(x)+\frac{1}{2}\int_0^x (te^{-\sin(\pi x)})dt+\frac{1}{2}\int_0^1 \left(e^{-t\sin(\frac{\pi x}{8})}\right)dt=\sin(\pi x)+\varepsilon^2 x-2,$$

$$v(0)=0,\quad v(1)=1+\int_0^1 \sin(\pi x)v(x)dx,$$

where $0<x<1$. Due to the unavailability of an exact solution for this problem, we utilize the double-mesh technique. Similar to the first example, the computed results are presented in Table 2.

Table 2. ε-uniform maximum point-wise error e^N and convergence rate p^N

ε	$N = 128$	$N = 256$	$N = 512$	$N = 1024$	$N = 2048$
2^{-8}	0.0319050695 2.2620	0.0066512409 2.2044	0.0014431185 1.9036	0.0003857189 2.0114	0.0000956687
2^{-9}	0.0751449323 2.1384	0.0170675637 2.2710	0.0035360891 1.8587	0.0009749820 1.8995	0.0002613347
2^{-10}	0.1492903899 1.8995	0.0400132117 2.1743	0.00886478430 2.2763	0.00182989203 2.2963	0.0003725269
2^{-11}	0.2597135141 1.6583	0.0822757492 1.9908	0.0206999699 2.1932	0.0045262907 1.9387	0.0011806891
2^{-12}	0.3874135235 1.3244	0.1546916884 1.8360	0.0433296461 2.0393	0.0105416515 2.0780	0.0024966380
2^{-13}	0.4783126312 0.8655	0.2625138235 1.6248	0.0851232834 1.9354	0.0222553192 1.9429	0.0057882604
2^{-14}	0.4775257937 0.3014	0.3874883326 1.3065	0.1566577206 1.8077	0.0447501448 1.8871	0.0120980662

6. Conclusion

We have introduced a second-order numerical approximation for solving singularly perturbed reaction-diffusion type Volterra-Fredholm integro differential equations with integral boundary conditions. Firstly, for the analysis of the numerical solution, we imposed certain constraints on the behavior and derivatives of the exact solution. Subsequently, we derived a difference approximation by employing suitable quadrature formulas for the differential component. Following the application of quadrature rules to both the integral component and the boundary condition, we employed the trapezoidal integral form to formulate a second-order approximate numerical method. The difference method was proven to be almost convergent to the second order. We also provided two examples. The results from these examples demonstrate that the computed values are in agreement with the theoretical findings, as presented in the accompanying tables.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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